11 Perturbation Theory and Feynman Diagrams

We now turn our attention to interacting quantum field theories. All of the results that we will derive in this section apply equally to both relativistic and non-relativistic theories with only minor changes. Here we will use the path integrals approach we developed in previous chapters.

The properties of any field theory can be understood if the N-point Green functions are known

$$G_N(x_1, \ldots, x_N) = \langle 0|T\phi(x_1) \ldots \phi(x_N)|0\rangle \tag{1}$$

Much of what we will do below can be adapted to any field theory of interest. We will discuss in detail the simplest case, the relativistic self-interacting scalar field theory. It is straightforward to generalize this to other theories of interest. We will only give a summary of results for the other cases.

11.1 The Generating Functional in Perturbation Theory

The N-point function of a scalar field theory,

$$G_N(x_1, \ldots, x_N) = \langle 0|T\phi(x_1) \ldots \phi(x_N)|0\rangle, \tag{2}$$

can be computed from the generating functional $Z[J]$

$$Z[J] = \langle 0|T e^{i \int d^D x J(x)\phi(x)}|0\rangle \tag{3}$$

In $D = d + 1$-dimensional Minkowski space-time $Z[J]$ is given by the path integral

$$Z[J] = \int \mathcal{D}\phi \; e^{iS[\phi] + i \int d^D x J(x)\phi(x)} \tag{4}$$

where the action $S[\phi]$ is the action for a relativistic scalar field. The N-point function, Eq.(1), is obtained by functional differentiation, *i.e.,*

$$G_N(x_1, \ldots, x_N) = (-i)^N \; \frac{1}{Z[J]} \; \frac{\delta^N}{\delta J(x_1) \ldots \delta J(x_N)} Z[J]\Big|_{J=0} \tag{5}$$

Similarly, the Feynman propagator $G_F(x_1 - x_2)$, which is essentially the 2-point function, is given by

$$G_F(x_1 - x_2) = -i\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = i \; \frac{1}{Z[J]} \; \frac{\delta^2}{\delta J(x_1)\delta J(x_2)} Z[J]\Big|_{J=0} \tag{6}$$

Thus, all we need to find is to compute $Z[J]$.

We will derive an expression for $Z[J]$ in the simplest theory, the relativistic real scalar field with a ϕ^4 interaction, but the methods are very general. We will work in Euclidean space-time (*i.e.,* in imaginary time) where the generating function takes the form

$$Z[J] = \int \mathcal{D}\phi \; e^{-S[\phi] + \int d^D x J(x)\phi(x)} \tag{7}$$

where $S[\phi]$ now is

$$S[\phi] = \int d^D x \left[\frac{1}{2} \left(\partial\phi\right)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \right] \tag{8}$$

In the Euclidean theory the N-point functions are

$$G_N(x_1, \ldots, x_N) = \langle \phi(x_1) \ldots \phi(x_N) \rangle = \frac{1}{Z[J]} \; \frac{\delta^N}{\delta J(x_1) \ldots \delta J(x_N)} Z[J] \Big|_{J=0} \tag{9}$$

Let us denote by $Z_0[J]$ the generating action for the free scalar field, with action $S_0[\phi]$. Then

$$\begin{aligned} Z_0[J] &= \int \mathcal{D}\phi \; e^{-S_0[\phi] + \int d^D x J(x)\phi(x)} \\ &= \left[\text{Det}\left(-\partial^2 + m^2\right) \right]^{-1/2} \; e^{\frac{1}{2} \int d^D x \int d^D y J(x) G_0(x-y) J(y)} \end{aligned} \tag{10}$$

where ∂^2 is the Laplacian operator in D-dimensional Euclidean space, and $G_0(x-y)$ is the free field Euclidean propagator (*i.e.,* the Green function)

$$G_0(x-y) = \langle \phi(x)\phi(y) \rangle_0 = \langle x | \frac{1}{-\partial^2 + m^2} | y \rangle \tag{11}$$

where the sub-index label 0 denotes a free field expectation value.

We can write the full generating function $Z[J]$ in terms of the free field generating function $Z_0[J]$ by noting that the interaction part of the action contributes with a weight of the path-integral that, upon expanding in powers of the coupling constant λ takes the form

$$\begin{aligned} e^{-S_{\text{int}}[\phi]} &= e^{-\int d^D x \frac{\lambda}{4!} \phi^4(x)} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D x_1 \ldots \int d^D x_n \; \phi^4(x_1) \ldots \phi^4(x_n) \end{aligned} \tag{12}$$

Hence, upon an expansion in powers of λ, the generating function $Z[J]$ is

$$
\begin{aligned}
Z[J] &= \int \mathcal{D}\phi \; e^{-S_0[\phi] + \int d^D x J(x)\phi(x) - S_{\text{int}}[\phi]} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!}\right)^n \\
&\quad \times \int d^D x_1 \ldots \int d^D x_n \int \mathcal{D}\phi \; \phi^4(x_1)\ldots\phi^4(x_n) \; e^{-S_0[\phi] + \int d^D x J(x)\phi(x)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!}\right)^n \int d^D x_1 \ldots \int d^D x_n \; \frac{\delta^4}{\delta J(x_1)^4} \cdots \frac{\delta^4}{\delta J(x_n)^4} \; Z_0[J] \\
&\equiv e^{-\frac{\lambda}{4!} \int d^D x \; \frac{\delta^4}{\delta J(x)^4}} \; Z_0[J]
\end{aligned}
\tag{13}
$$

where the operator of the last line is defined by its power series expansion. We see that this amounts to the formal replacement

$$
S_{\text{int}}[\phi] \leftrightarrow S_{\text{int}}\left(\frac{\delta}{\delta J}\right) \tag{14}
$$

This expression allows us to write the generating function of the full theory $Z[J]$ in terms of $Z_0[J]$, the generating function of the free field theory,

$$
Z[J] = e^{-S_{\text{int}}\left(\frac{\delta}{\delta J}\right)} \; Z_0[J] \tag{15}
$$

Notice that this expression holds for *any* theory, not just a ϕ^4 interaction. This result is the starting point of perturbation theory.

Before we embark on explicit calculations in perturbation theory it is worthwhile to see what assumptions we have made along the way. We assumed, (a) that the fields obey Bose commutation relations, and (b) that the vacuum (or ground state) is non-degenerate.

The restriction to Bose statistics was made at the level of the path integral of the scalar field. This approach is, however, of general validity, and it also applies to theories with Fermi fields, path integrals over Grassmann fields. As we saw before, in all cases the generating functional yields vacuum (or ground state) expectation values of time ordered products of fields.

The restriction to a non-degenerate vacuum state has more subtle physical consequences. We have mentioned that, in a number of cases, the vacuum may be degenerate if a *global symmetry* is spontaneously broken. Here, the thermodynamic limit plays an essential role. For example, if the vacuum is doubly degenerate, we can do perturbation theory on one of the two vacuum states.

If they are related by a global symmetry, the number of orders in perturbation theory which are necessary to have a mixing with its degenerate partner is approximately equal to the total number of degrees of freedom. Thus, in the thermodynamic limit, they will not mix unless the vacuum is unstable. Such an instability usually shows up in the form of infrared divergent contributions in the perturbation expansion. This is not a sickness of the expansion but the consequence of having an inadequate starting point for the ground state! In general, the "safe procedure" to deal with degeneracies that are due to symmetry is to add an additional term (like a source) to the Lagrangian that breaks the symmetry and to do all the calculations in the presence of such symmetry breaking term. This term should be removed only *after* the thermodynamic limit is taken.

The case of *gauge symmetries* has other and important subtleties. In the case of Maxwell's Electrodynamics we saw that the ground state is locally gauge invariant and that, as a result, it is unique. It turns out that this is a generic feature of theories that are locally gauge invariant for all symmetry groups and in all dimensions. There is a very powerful theorem (due to S. Elitzur, and which we will discuss later) which states that not only the ground state of theories with gauge invariance is unique and invariant, but that this restriction extends to the entire spectrum of the system. Thus only gauge invariant operators have non-zero expectation values and only such operators can generate the physical states. To put it differently, a local symmetry cannot be broken spontaneously even in the thermodynamic limit. The physical reason behind this statement is that if the symmetry is local it takes only a *finite* order in perturbation theory to mix all symmetry related states.

Thus whatever may happen at the boundaries of the system has no consequence on what happens in the interior and the thermodynamic limit does not play a role any larger. This is why we can fix the gauge and remove the enormous *redundancy* of the description of the states. Nevertheless we have to be very careful about two issues. Firstly, the gauge fixing procedure must select one and only one state from each gauge class. Secondly, the perturbation theory is based on the propagator of the gauge fields $i\langle 0|TA_\mu(x)A_\nu(x')|0\rangle$ which is *not* gauge invariant and, unless a gauge is fixed, it is zero. If a gauge is fixed, this propagator has contributions that depend on the choice of gauge. But the *poles* of this propagator do *not* depend on the choice of gauge since they describe physical excitations, *e.g.,* photons. Furthermore although the propagator is gauge dependent it will only appear in combination with *matter currents* which are *conserved*. Thus, the gauge-dependent terms of the propagator do not contribute to physical processes.

Except for these caveats, we can now proceed to do perturbation theory for all field theories of interest.

11.2 Perturbative Expansion for the Two-Point Function

Let us discuss the perturbative computation of the two-point function in ϕ^4 field theory in D-dimensional Euclidean space-time. Recall that under a Wick rotation, the analytic continuation to imaginary time $ix_0 \to x_D$, the two-point function in D-dimensional Minkowski space-time, $\langle 0|T\phi(x_1)\phi(x_2)|0\rangle$ maps onto the two-field correlation function of D-dimensional Euclidean space-time,

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle \;\longleftrightarrow\; \langle \phi(x_1)\phi(x_2)\rangle \tag{16}$$

Let us formally write the two point function $G^{(2)}(x_1 - x_2)$ as a power series in the coupling constant λ,

$$G^{(2)}(x_1 - x_2) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, G_n^{(2)}(x_1 - x_2) \tag{17}$$

However, using the generating functional $Z[J]$ we can write

$$G^{(2)}(x_1 - x_2) = \frac{1}{Z[J]}\, \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\, Z[J]\Big|_{J=0} \tag{18}$$

where

$$Z[J] = e^{-S_{\rm int}\left(\frac{\delta}{\delta J}\right)}\, Z_0[J] \tag{19}$$

Hence, the two-point function can be expressed as a ratio of two series expansions in powers of the coupling constant. The numerator is given by the expansion of $\frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\, Z[J]\Big|_{J=0}$,

$$\frac{\delta^2}{\delta J(x_1)\delta J(x_2)} Z[J]\Big|_{J=0} =$$
$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!}\right)^n \int d^D y_1 \ldots \int d^D y_n\, \frac{\delta^2}{\delta J(x_1)\delta J(x_2)} \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_n)^4}\, Z_0[J]\Big|_{J=0} \tag{20}$$

and the denominator by the expansion of $Z[0]$, which leads to a similar expression but without a contribution due to the external legs, corresponding to the functional derivatives with respect to the source at the external points x_1 and x_2. The equivalent expression in Minkowski space-time is obtained by the replacement,

$$-\lambda \longleftrightarrow i\lambda \tag{21}$$

at every order in the expansion.

We will now look at the form of the first few terms of the expansion of the two-point function in perturbation theory.

11.2.1 Zeroth Order in λ.

To zeroth order in λ (*i.e.*, $O(\lambda^0)$), the numerator reduces to

$$\begin{aligned}\frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} Z[J]\Big|_{J=0} &= \frac{\delta}{\delta J(x_1)} \frac{\delta}{\delta J(x_2)} Z_0[J]\Big|_{J=0} + O(\lambda) \\ &= G_0(x_1 - x_2) + O(\lambda)\end{aligned} \tag{22}$$

while the denominator is simply equal to one

$$Z[0] = Z_0[0] + O(\lambda) = 1 + 0(\lambda) \tag{23}$$

Hence,

$$G^{(2)}(x_1 - x_2) = G_0(x_1 - x_2) + O(\lambda) \tag{24}$$

11.2.2 First Order in λ.

To first order in λ, the denominator $Z[0]$ is given by

$$Z[0] = 1 + \frac{-1}{1!}\left(\frac{\lambda}{4!}\right)\int dy \frac{\delta^4}{\delta J(y)^4} Z_0[J]|_{J=0} + O(\lambda^2) \tag{25}$$

The expression inside the integrand can be calculated from the Taylor expansion of $Z_0[J]$ in powers of $J(x)$. To find a non-zero contribution we need to bring down from the exponent enough factors of J so that they can be cancelled by the functional derivatives. Since the argument of the exponential factor in $Z_0[J]$ is *bilinear* in $J(x)$,

$$Z_0[J] = \left[\text{Det}\left(-\partial^2 + m^2\right)\right]^{-1/2} e^{\frac{1}{2}\int d^Dx \int d^Dy J(x) G_0(x-y) J(y)}, \tag{26}$$

only an even number of derivatives in $J(x)$ can be cancelled to a given order. In particular, to first order in λ, we have to cancel four derivatives. This means that we need to expand the exponential in $Z_0[J]$ to second order in its argument to obtain the only non-vanishing contribution to first order in λ to $Z[J]$ at $J = 0$,

$$\begin{aligned} Z[0] = 1 + &\\ \frac{(-1)}{1!}\left(\frac{\lambda}{4!}\right)&\int d^Dx \frac{\delta^4}{\delta J(x)^4} \frac{1}{2!}\left(\frac{1}{2}\int d^Dy_1 \int d^Dy_2 J(y_1) G_0(y_1 - y_2) J(y_2)\right)^2\Big|_{J=0} \\ &+O(\lambda^2) \end{aligned} \tag{27}$$

The derivatives yield a set of δ-functions

$$\frac{\delta^4}{\delta J(x)^4}[J(y_1)\cdots J(y_4)]\Big|_{J=0} = \sum_P \prod_{j=1}^4 \delta(y_{pj} - xy) \tag{28}$$

where P runs over the 4! permutations of the four arguments y_1, y_2, y_3 and y_4.

We can now write the first order correction to $Z[0]$, in the form

$$\begin{aligned} Z[0] = 1 + &\\ \left(\frac{-1}{1!}\right)\left(\frac{\lambda}{4!}\right)\frac{1}{2!}\left(\frac{1}{2}\right)^2 &\int d^Dx \int d^Dy_1 \ldots d^Dy_4 G_0(y_1 - y_2) G_0(y_3 - y_4) \sum_P \prod_{j=1}^4 \delta(y_{pj} - x) \\ &+O(\lambda^2) \\ &= 1 + \frac{(-1)}{1!}\left(\frac{\lambda}{4!}\right) S \int d^Dx\, G_0(x,x)\, G_0(x,x) + O(\lambda^2) \end{aligned} \tag{29}$$

where

$$S = \frac{4!}{2!\, 2^2} = 3 \tag{30}$$

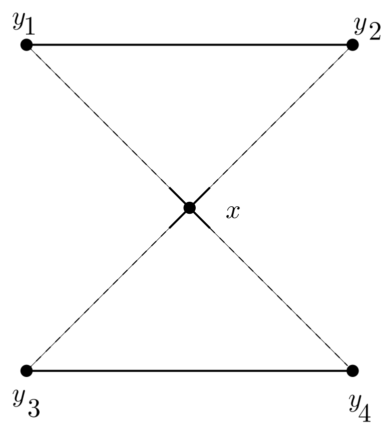


Figure 1: A vertex and its four contractions.

It is useful to introduce a picture or *diagram* to represent this contribution. Let us mark four points $y_1, \ldots, y_4$ and an additional point at x (which we will call a *vertex*) with four legs coming out of it. Let us join y_1 and y_2 by a line and y_3 with y_4 by another line. To each line we assign a factor of $G_0(y_1 - y_2)$ and $G_0(y_3 - y_4)$ respectively, as in the figure:

$G_0(x - y) =$ 

Next, because of the δ-functions, we have to identify each of the points $y_1, \ldots, y_4$ with each one of the legs attached to y in all possible ways (as shown in figure 1)

The result has to be integrated over all values of the the coordinates and of x. The result is

$$Z[0] = 1 - \left(\frac{\lambda}{8}\right) \int dx \, (G_0(x,x))^2 + O(\lambda^2) \tag{31}$$

Physically, the first order contribution represents corrections to the ground state energy due to *vacuum fluctuations*. This expression can be represented more simply by the *Feynman diagram* shown in Fig.2.

Here, and below, we denote by a full line the bare propagator $G_0(x - y)$.

Let us now compute the first order corrections to $\displaystyle \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \, Z[J]\Big|_{J=0}$.

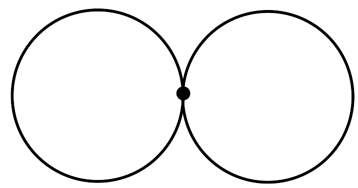


Figure 2: Vacuum fluctuations: first order correction to the denominator, $Z[0]$.

We obtain

$$
\begin{aligned}
&\frac{(-1)}{1!}\;\frac{\lambda}{4!}\int d^D y \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\;\frac{\delta^4}{\delta J(y)^4}\;Z[J]\Big|_{J=0} \\
=&\frac{(-1)}{1!}\;\frac{\lambda}{4!}\int d^D y \frac{\delta^2}{\delta J(x_1)\delta J(x_2)}\;\frac{\delta^4}{\delta J(y)^4}\;\frac{1}{3!}\;\left(\frac{1}{2}\int d^D z_1 \int d^D z_2 J(z_1) G_0(z_1-z_2) J(z_2)\right)^3\Big|_{J=0}
\end{aligned}
\tag{32}
$$

The non-vanishing contributions are obtained by matching the derivatives in Eq.(32) with an equal number of powers of J. We see that we have six factors of the source J at points $z_1, \ldots, z_6$ and six derivatives, one at x_1 and at x_2, and four at y. To match derivatives with powers amounts to find all possible pair-wise contractions of these two sets of points. Once the delta functions have acted we are left with just one integral over the position y of the internal vertex. Hence, the result amounts to finding all possible contractions among the external legs at x_1 and x_2, with each other and/or with the internal vertex at y. Notice that for each contraction we get a power of the bare (unperturbed) propagator G_0.

The only non-vanishing terms resulting from this process are represented by the Feynman diagrams of Fig.3.

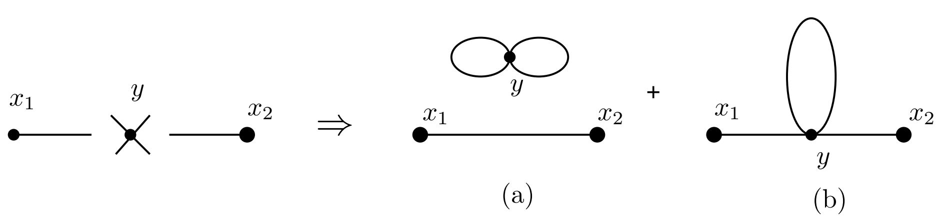


Figure 3: First order contribution to the two-point function.

The first contribution, Fig.3(a), is the product of the bare propagator $G_0(x_1 - x_2)$ between the external points and the first order correction of the vacuum diagrams:

$$
(a) = G_0(x_1 - x_2) \times \left[-\left(\frac{\lambda}{8}\right)\int d^D y\,\Big(G_0(y,y)\Big)^2\right] \tag{33}
$$

The second term, the “tadpole” diagram of Fig.3(b), is given by the expression

$$(b) = -\left(\frac{\lambda}{4!}\right)\ S\ \int d^D y\ G_0(x_1, y) G_0(y, y) G_)(y, x_2) \tag{34}$$

where the multiplicity factor S is

$$S = 4 \times 3 \tag{35}$$

It counts the number of ways (12) of contracting the external points to the internal vertex: there are four different ways of contracting one external point to one of the four lines attached to the internal vertex at y, and three different ways of contracting the remaining external point to the one of the three remaining lines of the internal vertex. There is only one way to contract the two leftover internal lines attached to the vertex at y.

By collecting terms we get the result shown in Fig. 4.

To first order in λ, the expansion of the two-point function can be written in the form shown in Fig. 5.

At least at this order in perturbation theory, a number of diagrams which contribute to the expansion of the numerator get exactly cancelled by the expansion of the denominator, the vacuum diagrams. The diagrams that get cancelled are unlinked in the sense that one can split a diagram in two by drawing a line the does not cut any of the propagator lines. These diagrams contain a factor consisting of terms of the vacuum diagrams. We will see shortly that this is a feature of this expansion to all orders in λ.

Thus, to first order in λ, the two-point function is given by (see figure)

$$G^{(2)}(x_1, x_2) = G_0^{(2)}(x_1, x_2) - \frac{\lambda}{2} \int d^D y\ G_0^{(2)}(x_1, y) G_0^{(2)}(y, y) G_0^{(2)}(y, x_2) + O(\lambda^2) \tag{36}$$

as shown in Fig.6.

11.3 Cancellation of the vacuum diagrams

The cancellation of the vacuum diagrams is a general feature of perturbation theory. Let us reexamine this issue in more general terms. We will give the arguments for the case of the two-point function, but they are trivial to generalize to any N-point function. This feature also holds in all theories, relativistic or not,

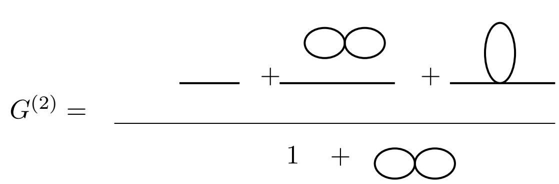


Figure 4: Feynman diagrams for the two-point function to first order in λ.

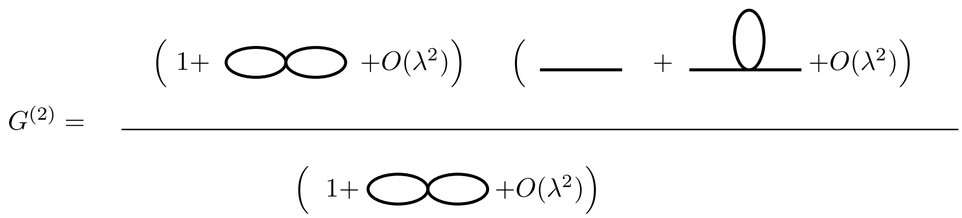


Figure 5: Factorization of Feynman diagrams for the two-point function to first order in λ.

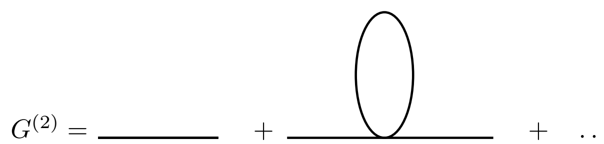


Figure 6: The two-point function to first order in λ.

bosonic or fermionic, provided the fields satisfy local canonical commutation (or anti-commutation) relations.

The expansion of the two point function has the form

$$\langle \phi(x_1)\phi(x_2)\rangle = \frac{1}{Z[0]} \sum_{n=0}^{\infty} \int d^D y_1 \ldots d^D y_n \, \frac{(-1)^n}{n!} \langle \phi(x_1)\phi(x_2) \prod_{j=1}^{n} \mathcal{L}_{\text{int}} \left(\phi(y_j)\right)\rangle_0 \tag{37}$$

The denominator factor $Z[0]$ has a similar expansion

$$Z[0] = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int d^D y_1 \ldots d^D y_n \, \langle \prod_{j=1}^{N} \mathcal{L}_{\text{int}} \left(\phi(y_j)\right)\rangle_0 \tag{38}$$

where $\langle A(\phi)\rangle_0$ denotes an expectation value of the operator $A(\phi)$ in the free field theory.

Let us consider first the numerator. Each expectation value involves a sum of products of pairwise contractions. If we assign a Feynman diagram to each contribution, it is clear that we can classify these terms into two classes: (a) linked and (b) unlinked diagrams. A diagram is said to be *unlinked* if it contains a *sub-diagram* in which a set of internal vertices are linked with each other but not to an external vertex. The *linked* diagrams satisfy the opposite property. Since the vacuum diagrams by definition do not contain any external vertices, they are unlinked.

All the expectation values that appear in the numerator can be written as a sum of terms, each of the form of a linked diagram times a vacuum graph, *i.e.,*

$$\begin{aligned}&\langle\phi(x_1)\phi(x_2)\mathcal{L}_{\text{int}}\left(\phi(y_1)\right)\dots\mathcal{L}_{\text{int}}\left(\phi(y_n)\right)\rangle_0\\&=\sum_{k=0}^{n}\binom{n}{k}\langle\phi(x_1)\phi(x_2)\prod_{j=1}^{k}\mathcal{L}_{\text{int}}\left(\phi(y_j)\right)\rangle_0^{\ell}\,\langle\prod_{j=k+1}^{n}\mathcal{L}_{\text{int}}\left(\phi(y_j)\right)\rangle_0\end{aligned}\tag{39}$$

where the super-index ℓ denotes a *linked* factor, *i.e.,* a factor that does not contain any vacuum sub-diagram. Thus, the numerator has the form

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^n}{n!}\binom{n}{k}\langle\phi(x_1)\phi(x_2)\prod_{j=1}^{k}\mathcal{L}_{\text{int}}(y_k)\rangle_0^{\ell}\langle\prod_{j=k+1}^{N}\mathcal{L}_{\text{int}}\left(\phi(y_j)\right)\rangle_0\tag{40}$$

which factorizes into

$$\begin{aligned}&\Big(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int d^Dy_1\dots d^Dy_k\ \langle\phi(x_1)\phi(x_2)\prod_{j=1}^{k}\mathcal{L}_{\text{int}}\left(\phi(y_j)\right)\rangle_0^{\ell}\Big)\\&\qquad\qquad\times\Big(\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^Dy_1\dots d^Dy_n\langle\prod_{j=1}^{n}\mathcal{L}_{\text{int}}\left(\phi(y_j)\right)\rangle_0\end{aligned}\tag{41}$$

We can clearly recognize that the second factor is *exactly* equal to the denominator $Z[0]$.

Hence we find that we can write the two-point function as a sum of linked Feynman diagrams:

$$\langle\phi(x_1)\phi(x_2)\rangle=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int d^Dy_1\dots d^Dy_n\ \langle\phi(x_1)\phi(x_2)\prod_{j=1}^{n}\mathcal{L}_{\text{int}}\left(\phi(y_n)\right)\rangle_0^{\ell}\tag{42}$$

This result is known as the *linked-cluster theorem.* This theorem, which proves that the vacuum diagrams cancel exactly out to all orders in perturbation theory, is valid for all the N-point functions (not just for two-point function) and for any local theory. It also holds in Minkowski space-time upon the replacement $(-1)^n \leftrightarrow i^n$. It holds for all theories with a local canonical structure, relativistic or not.

11.4 Summary of Feynman Rules for ϕ^4 theory

11.4.1 Position Space

The general rules to construct the diagrams for the N-point function $\langle 0|T\phi(x_1)\dots\phi(x_N)|0\rangle$ in ϕ^4 theory in Minkowski space and $\langle\phi(x_1)\dots\phi(x_N)\rangle$ in Euclidean space, in *position space* are

1. A general graph for the N-point function has N external points (ore vertices) and n interaction vertices, where n is the order in perturbation theory. Each vertex is a point with a coordinate label and 4 lines (for a ϕ^4 theory) coming out of it.

2. Draw all topologically distinct graphs by connecting the external points and the internal vertices in all possible ways. Discard all graphs which contain sub-diagrams not linked to at least one external point.

3. The following weight is assigned to each graph:

   (a) For every vertex a factor of $-i\frac{\lambda}{4!}$ in Minkowski space and $-\frac{\lambda}{4!}$ in Euclidean space .

   (b) For every line connecting a pair of points z_1 and z_2, a factor of $\langle 0|T\phi(z_1)\phi(z_2)|0\rangle_0 = -iG_0^{(2)}(z_1, z_2)$ in Minkowski space, or $\langle\phi(z_1)\phi(z_2)\rangle_0 = G_0(z_1 - z_2)$ in Euclidean space.

   (c) An overall factor of $\frac{1}{n!}$.

   (d) A multiplicity factor which counts the number of ways in which the lives can be joined without altering the topology of the graph.

   (e) Integrate over all internal coordinates.

For example, the 4-point function

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle \tag{43}$$

has the contribution at order λ^2 shown in the diagram of figure 7. These two

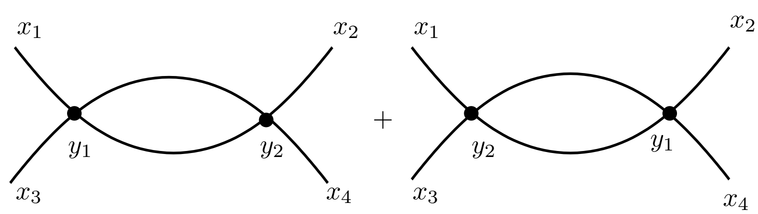


Figure 7: Two contributions to the four-point function to order λ^2.

diagrams have exactly the same weight (if $G_0^{(2)}(1,2) = G_0^{(2)}(2,1)$), and their total contribution to the 4-point function is

$$\frac{1}{2!}\left(\frac{-\lambda}{4!}\right)^2 S \int d^D y_1 \int d^D y_2 \, G_0^{(2)}(x_1, y_1)\, G_0^{(2)}(x_2, y_1) \left[G_0^{(2)}(y_1, y_2)\right]^2 G_0^{(2)}(x_3, y_2)\, G_0^{(2)}(x_4, y_2) \tag{44}$$

but are *topologically distinct* and, thus, count as separate contributions. The multiplicity factor S is $S = (4 \times 3)^2 \times 2$.

11.4.2 Momentum Space

The N-point functions can also be computed in momentum space. The rules for constructing Feynman diagrams in momentum space are:

1. A graph has N external legs, labelled by a set of external momenta $k_1, \ldots, k_N$, flowing into the diagram and n internal vertices (for order n in perturbation theory). Each vertex has 4 lines (for ϕ^4 theory) each carrying a momentum $q_1, \ldots q_4$ (out of the vertex). All lines must be connected in pairs. All vacuum terms have to be discarded.

2. Draw all the topologically different graphs.

3. Weight of each diagram:

   (a) for each vertex, a factor of $(\frac{-\lambda}{4!})(2\pi)^D \delta^4(\sum_{i=1}^{D} q_i)$.

   (b) each line carries a momentum p_μ and contributes to the weight with a factor $G_0^{(2)}(p) = \frac{1}{p^2+m^2}$, in Euclidean space. In Minkowski space it becomes $-iG_0^{(2)} = \frac{i}{p^2-m^2+i\epsilon}$.

   (c) all the numerical factors are the same as in position space.

   (d) we must integrate over all the internal momenta.

For example, the first order contribution to the two-point function is the tadpole diagram shown in Fig.8. It has the algebraic weight

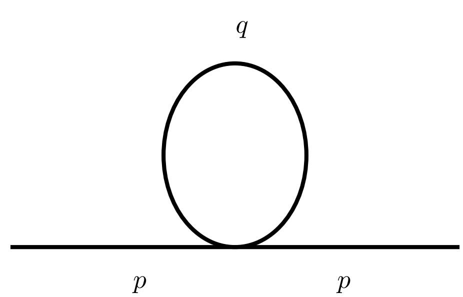


Figure 8: First order contribution to the two-point function in momentum space.

$$\left(\frac{-\lambda}{4!}\right) \frac{1}{1!}(4 \times 3) \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+m^2} \left(\frac{1}{p^2+m^2}\right)^2 \tag{45}$$

11.5 The two-point function and the self-energy

To first order in λ, and in momentum space, the two point function in Euclidean space is

$$G^{(2)}(p) = \frac{1}{p^2+m^2} + \frac{-\lambda}{4!}(4 \times 3)\left(\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2+m^2}\right)\left(\frac{1}{p^2+m^2}\right)^2 + O(\lambda^2) \tag{46}$$

Let us define by μ^2 the *effective* or *renormalized mass* (squared) such that

$$\frac{1}{p^2+\mu^2} = \frac{1}{p^2+m^2}\left\{1 - \frac{\lambda}{2}\left(\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2+m^2}\right)\left(\frac{1}{p^2+m^2}\right) + O(\lambda^2)\right\} \tag{47}$$

Again, to first order in λ, we can write the equivalent expression

$$G^{(2)}(p) = \frac{1}{p^2+m^2+\frac{\lambda}{2}\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2+m^2}} + O(\lambda^2) \tag{48}$$

This expression leads us to define μ^2 to be

$$\mu^2 = m^2 + \frac{\lambda}{2}\int \frac{d^D q}{(2\pi)^D}\frac{1}{q^2+m^2} + \dots \tag{49}$$

This equation is equivalent to a sum of a large number of diagrams with higher order in λ. How do we know that it is consistent? Let us first note that we have summed diagrams of the form shown in figure 9. These diagrams have the very

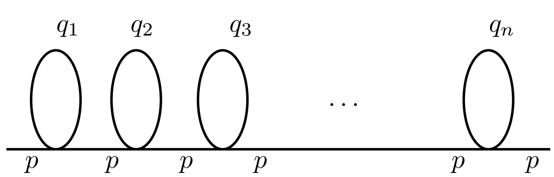


Figure 9: The set of all one-particle reducible diagrams of the two-point function to leading order in λ.

special feature that it is possible to split the diagram into two sub-diagrams by cutting only a *single* internal line. Momentum conservation requires that the momentum of that line be *equal* to the momentum on the incoming external leg. Thus, once again we have two types of diagrams: (a) one-particle *reducible* diagrams (which satisfy the property defined above) and (b) the one-particle *irreducible* graphs which do not. Hence, the total contribution to the two-point function is the solution of the equation shown in Fig 10.

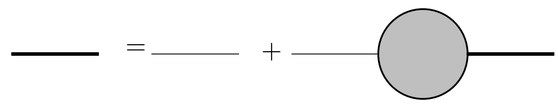


Figure 10: The Dyson equation for the two-point function.

Here the thick lines are the *full* propagator, the thin line is the *bare* propagator, and the *shaded blob* represents the irreducible diagrams, *i.e.,* diagrams with amputated external legs. We represent the blob by the *self-energy* operator $\Sigma(p)$, shown in Fig.11

Thus, the total sum satisfies the Dyson equation

$$G^{(2)}(p) = G_0^{(2)}(p) + G_0^{(2)}(p)\,\Sigma(p)\,G^{(2)}(p) \tag{50}$$

The inverse of $G^{(2)}(p)$, $\Gamma^{(2)}(p)$, satisfies

$$\Gamma^{(2)}(p) = G_0^{(2)}(p)^{-1} - \Sigma(p) = p^2 + m^2 - \Sigma(p) \tag{51}$$

To first order in $\lambda, \Sigma(p)$ is just the tadpole term

$$\Sigma(p) = -\frac{\lambda}{2}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+m^2} + O(\lambda^2) \tag{52}$$

which happens to be independent of the external momentum p_μ. Of course, the higher order terms in general will be functions of p_μ.

In terms of the renormalized mass μ^2, to order one-loop (*i.e.,* $O(\lambda)$) we get

$$\mu^2 = m^2 - \Sigma(p) = m^2 + \frac{\lambda}{2}\int \frac{d^Dq}{(2\pi)^D}\frac{1}{q^2+m^2} + O(\lambda^2) \tag{53}$$

Thus, we conclude that vacuum fluctuations renormalize the mass. However, a quick look at Eq.(53) reveals that this is very large renormalization. Indeed, fluctuations *of all momenta*, ranging from long wave-lengths (and low energies) with $q \sim 0$, to short wave-lengths (or high energies) contribute to the mass renormalization. In fact, the high-energy fluctuations, with $q^2 \gg m^2$, yield the *largest* contributions to Eq.(53), since the mass effectively cuts-off the contributions in the infrared, IR $q \to 0$. Moreover, for all dimensions $D \geq 2$ the high-energy (or ultraviolet, UV, $q \to \infty$) contribution is *divergent.* If we were to cutoff the integral at a high-momentum scale Λ, in general space-time dimension

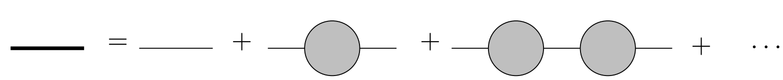


Figure 11: Feynman diagrams summed by the Dyson equation.

D the diagram diverges as Λ^{D-2}. In particular, the tadpole contribution to the mass renormalization is *logarithmically divergent* for $D = 2$ $(1+1)$ dimensions, and *quadratically divergent* for $D = 4$ dimensions.

Thus, although it is consistent (and quite physical) to regard the leading effect of fluctuations as a mass renormalization, they amount to a divergent change. The reason for this divergence is that all wavelengths contribute, from the IR to the UV. This happens since space-time is continuous: we assumed that there is no intrinsic short-distance scale below which local field theory would not be valid.

There is a way to think about this problem. The problem of how to understand the physics of these singular contributions, indeed of how the *continuum limit* (a theory without cutoff) of quantum field theory is the central purpose of the Renormalization Group (RG). We will study this approach in detail next term. Here we will discuss some qualitative features. From the point of view of the RG the problem is that the continuum theory (*i.e.,* defining a theory without a UV cutoff) cannot be done naively. We will see next term that for such a procedure to work it is necessary to be able to define the theory in a regime in which there is no scale, *i.e.,* in a scale-invariant regime. This requirement means that one should look at a regime in which the renormalized mass becomes arbitrarily small, $\mu^2 \to 0$. As we will see below, this requires to fine tune the *bare coupling constant* and the *bare mass* to some determined *critical values*. It turns out that, near such a *critical point* a continuum field theory (without a UV cutoff) can be defined. The RG point of view relates the problem of the *definition* of a Quantum Field Theory to that of finding a *continuous phase transition*, a central problem in Statistical Physics.

However, there are alternative descriptions, such as String Theory, that postulate that local field theory is not the correct description at short distances, typically near the *Planck scale*, $\ell_{\text{Planck}} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}\text{cm}$ (!), where G is Newton's gravitation constant. From this view point, these singular contributions at high energies signal a breakdown of the theory at those scales.

Before we try to compute $\Sigma(p)$, it is worth to mention the *Hartree Approximation.* It consists in summing up all tadpole diagrams (and only the tadpole diagrams) to all orders in λ. A typical graph is shown in figure 12.

The sum of all the tadpole diagrams can be done by means of a very simple trick. Let us modify the expression for the self-energy to make it self consistent, *i.e.,*

$$\Sigma_0(p) = -\frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2 - \Sigma_0(q)} \tag{54}$$

This formula is equivalent to a Dyson equation in which the internal propagator is replaced by the full propagator, as in figure 13. This approximation becomes exact for a theory of an N-component real scalar field $\phi_a(x)$ $(a = 1, \ldots N)$, with $O(N)$ symmetry, and interaction

$$\mathcal{L}_{\text{int}}[\phi] = \frac{\lambda}{4!} \left(\left(\vec{\phi} \right)^2 \right)^2 = \frac{\lambda}{4!} \left(\sum_{a=1}^{N} \phi_a(x) \phi_a(x) \right)^2 \tag{55}$$

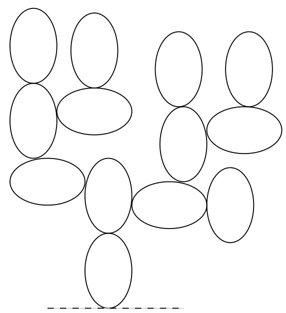


Figure 12: A typical tree diagram.

in the large N limit, $N \to \infty$. Otherwise, the solution of this integral equation just yields the leading correction.

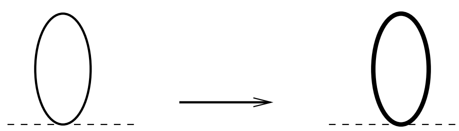


Figure 13: The self-energy in the one-loop (Hartree) approximation.

Eq.(54) is an integral non-linear equation for $\Sigma_0(p)$. Equations of this type are common in many-body physics. For example, the gap equation of the BCS theory of superconductivity has a similar form.

Let us now evaluate the integral in the equation for $\Sigma_0(p)$, Eq.(54). Clearly $\Sigma_0(p)$ is a correction due to *virtual* fluctuations with momenta q_μ ranging from zero to infinity. These fluctuations do *not* obey the mass shell condition $p^2 = m^2$. Notice that, at this level of approximation, $\Sigma(p)$ is *independent of the momentum*. This is only correct to order one-loop.

Before computing the integral, let us rewrite Eq.(54) in terms of the effective or renormalized mass μ^2,

$$\mu^2 = m^2 - \Sigma(p) = m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + \mu^2} \tag{56}$$

Let us denote by m_c^2 the value of the bare mass such that $\mu^2 = 0$:

$$0 = m_c^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \, \frac{1}{q^2} \tag{57}$$

Clearly, m_c^2 is IR divergent for $D \leq 2$ and UV divergent for $D \geq 2$. Let us now express the renormalized mass μ^2 in terms of m_c^2, and define $\delta m^2 = m^2 - m_c^2$.

We find,

$$
\begin{aligned}
\mu^2 &= \delta m^2 + m_c^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+\mu^2} \\
&= \delta m^2 + \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \left(\frac{1}{q^2+\mu^2} - \frac{1}{q^2} \right) \\
&= \delta m^2 - \mu^2 \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2(q^2+\mu^2)}
\end{aligned}
\tag{58}
$$

In $D = 4$ dimensions the integral contributes with a logarithmic divergence (*c.f.* Eq.(92) of Appendix A):

$$
\mu^2 = \delta m^2 - \frac{\lambda}{2} \mu^2 \left(\frac{1}{8\pi^2} \ln \left(\frac{\Lambda}{\mu} \right) - \frac{\gamma}{16\pi^2} \right) \tag{59}
$$

Thus, although the stronger, quadratic divergence, was absorbed by a renormalization of the mass, a weaker logarithmic singularity remains. It turns out, as we will discuss next semester, that this remaining singular contribution can absorbed only by a renormalization of the coupling constant λ. In any case, what is clear is that, already at the lowest order in perturbation theory, the leading corrections can (and do) yield a larger contribution to the behavior of physical quantities than the bare, unperturbed, values, and that this corrections are not small.

We will not give a thorough discussion of these singular contributions right now. A complete discussion of this problem involves the development of the idea of renormalization and of the renormalization group, which we will do next term. However, it is worth to discuss some of the physical issues behind these problems. In a relativistic field theory there is no natural cutoff since a cutoff would break Lorentz invariance. However, if a field theory like the present one is regarded as an *effective theory* (*i.e.,* not "fundamental") which is only correct at distances larger than some scale ξ, we can legitimately cutoff the integrals at a momentum $\Lambda \geq 1/\xi$. But, in this case, we have to argue that, at length scales shorter than ξ, there is a consistent theory which is free of this divergence. Except from some radical new approach (such as String Theory), all theories known to date contain divergencies. Does it mean that they are meaningless? For a long time (*i.e.,* from the 30's to the 60's) it was assumed that the divergencies signaled that QFT was incomplete. However, in the late 60's and early 70's a new approach was found that made sense of such theories. This approach, known as the Renormalization Group, tells us that the apparent shortcomings of perturbation theory are due to perturbation theory, not to the theory itself. In fact, the Renormalization Group gives a framework to define the theory at so-called non-trivial fixed points of a certain set of transformations, in which the results are physical. From this point of view, the problems are not the theories but our clumsy computational tools.

We will end with a discussion of what these large perturbative corrections mean in the context of the theory of phase transitions since, after all, it is also described by a theory with the same form. We saw before that in the Landau theory of phase transitions the mass (squared) is related to the difference $m^2 \equiv T - T_0$, the distance to the mean field critical temperature, T_0. We can think of $m_c^2 \equiv T_c - T_0$, defining a corrected critical temperature, T_c,

$$T_c = T_0 - \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2} \tag{60}$$

which shows that fluctuations suppress T_c *downwards* from T_0 due to (precisely) the large contributions from short distances. Thus, the UV singular effects can be absorbed in a new (and lower) T_c. The *subtracted* mass, $\delta m^2 \equiv T - T_c$, is now the new control parameter, as shown in the last line of Eq.(58).

Returning to the mass renormalization of Eq.(58), we can use the integral of Eq.(91) (of Appendix A), to find the result

$$\mu^2 = \delta m^2 - \frac{\lambda}{2} \frac{(\mu^2)^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\frac{D}{2} - 1} \tag{61}$$

Since $\delta m^2 = T - T_c$, we can write this result in the suggestive form

$$\delta m^2 = T - T_c = \mu^2 + \left[\frac{\lambda}{2} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\frac{D}{2} - 1} \right] (\mu^2)^{\frac{D}{2}-1} \tag{62}$$

We will now look at the IR behavior, where the renormalized mass $\mu^2 \to 0$. Let us recall that the relation between the susceptibility and the effective (or renormalized) mass: $\mu^2 = \chi^{-1}$. For $D < 4$ the second term of Eq.(62) vanishes *more slowly* than the first term (linear in μ^2), whereas for $D > 4$ the first term always wins. Hence, for $D < 4$, the one-loop perturbative correction becomes more important than the bare (linear in μ^2) term, leading us to expect that for $D < 4$ the second term in Eq.(62) should give the important contribution, whereas for $D > 4$ this term becomes negligible as $T \to T_c$. Hence, at one-loop order, we would predict that

$$\chi(T) \propto \begin{cases} (T - T_c)^{-\frac{2}{D-2}} & : \quad D < 4 \\ (T - T_c)^{-1} & : \quad D > 4 \\ (T - T_c)^{-1} \times \text{small logarithmic corrections} & : \quad D = 4 \end{cases} \tag{63}$$

We see that one effect of these fluctuations can be to change the dependence of a physical quantity, such as the susceptibility, on the control parameter, $T - T_c$, that sets how close the theory is the *massless* or *critical* regime. A key purpose of the renormalization group program is the prediction of *critical exponents* such as the one we fund in Eq.(63). We will see that this result is actually the beginning of a set of controlled approximations to the exact values. Just as important will be the fact that the renormalization group will give a deeper interpretation of the meaning of renormalization beyond a process of "hiding infinities".

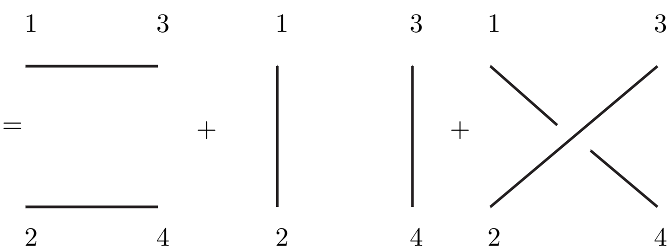
12 The four point function and the effective coupling constant

We will now discuss briefly the perturbative contributions to the four point function,

$$G^{(4)}(x_1, x_2, x_3, x_4) = \langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle \tag{64}$$

which is also known as the two-particle Green function. We will discuss its connection with the effective (or renormalized) coupling constant.

To zeroth order in perturbation theory, $O(\lambda^0)$, the four point function factorizes into a product of all (three) possible two point functions obtained by pair-wise contractions of the four field operators.

$G^{(4)}(1,2,3,4) =$  $+O(\lambda)$

In other words,

$$\begin{aligned} &G^{(4)}(x_1, x_2, x_3, x_4) = \\ &\qquad G_0(x_1, x_3)G_0(x_2, x_4) + G_0(x_1, x_2)G_0(x_3, x_4) + G_0(x_1, x_4)G_0(x_2, x_3) + O(\lambda) \end{aligned} \tag{65}$$

As it is apparent, to zeroth order in λ, the four point function reduces to just products of bare two-point functions and hence nothing new is learned from it. We will show next semester that to all orders on perturbation theory the four-point function has the following structure:

$$\begin{aligned} &G^{(4)}(x_1, x_2, x_3, x_4) = \\ &\qquad G^{(2)}(x_1, x_3)G^{(2)}(x_2, x_4) + G^{(2)}(x_1, x_2)G^{(2)}(x_3, x_4) + G^{(2)}(x_1, x_4)G^{(2)}(x_2, x_3) \\ &\qquad + \int d^D y_1 \dots d^D y_4 \, G^{(2)}(x_1, y_1)G^{(2)}(x_2, y_2)G^{(2)}(x_3, y_3)G^{(2)}(x_4, y_4)\, \Gamma^{(4)}(y_1, y_2, y_3, y_4) \end{aligned} \tag{66}$$

where the factors of $G^{(2)}(x, x')$ represent the *exact two-point function*, and the new four-point function, $\Gamma^{(4)}(y_1, y_2, y_3, y_4)$, is known as the four-point *vertex function*. The vertex function is defined as the set of one-particle irreducible (1PI) Feynman diagrams, *i.e.,* diagrams that cannot be split in two by cutting a single propagator line, with the external lines "amputated" (they are already accounted for in the propagator factors).

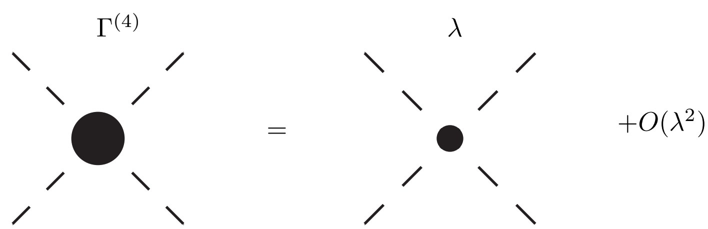
In momentum space, due to momentum conservation at the vertex, $\Gamma^{(4)}$ has the form

$$\Gamma^{(4)}(p_1, \dots, p_4) = (2\pi)^D \ \delta^D \left(\sum_{i=1}^{4} p_i \right) \ \overline{\Gamma^{(4)}}(p_1, \dots, p_4) \tag{67}$$

The *lowest order* contribution to $\Gamma^{(4)}(y_1, y_2, y_3, y_4)$ appear at order λ

$$\Gamma^{(4)}(y_1, y_2, y_3, y_4) = \lambda + O(\lambda^2) \tag{68}$$

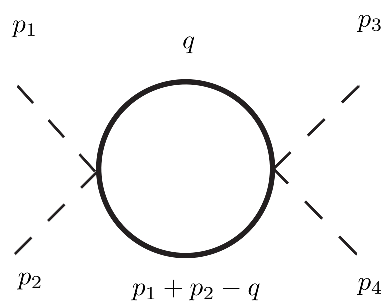
depicted by the *tree level* diagrams:



which, in momentum space is

$$\overline{\Gamma^{(4)}}(p_1, \dots, p_4) = \lambda + O(\lambda^2) \tag{69}$$

To one-loop order, $O(\lambda^2)$, the four-point vertex function is a sum of (three) Feynman diagrams of the form



The total contribution to the vertex function $\overline{\Gamma^{(4)}}$, to order one-loop, is

$$\overline{\Gamma^{(4)}}(p_1, \dots, p_4) =$$
$$\lambda - \frac{\lambda^2}{2} \left\{ \int \frac{d^D q}{(2\pi)^D} \ \frac{1}{(q^2 + m^2)\,((p_1 + p_2 - q)^2 + m^2)} + \ \text{two permutations} \right\} + O(\lambda^3) \tag{70}$$

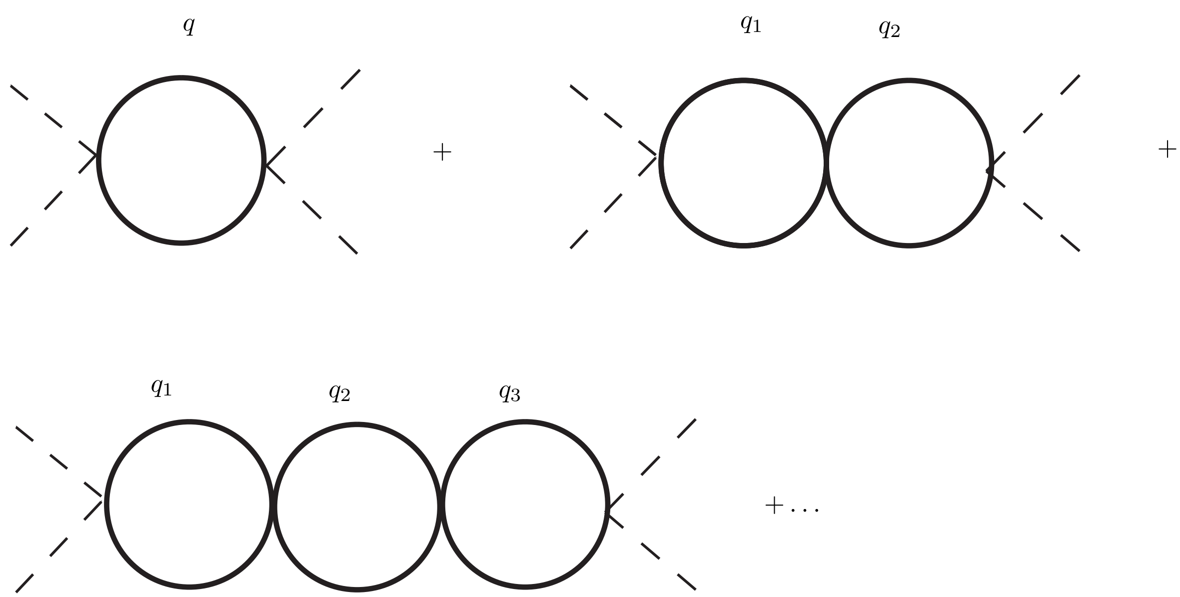
This expression has a logarithmic UV divergence in $D = 4$, and more severe divergencies for $D > 4$. To address this problem let us proceed by analogy with the mass renormalization and define the physical or *renormalized* coupling constant g by the value of $\overline{\Gamma^{(4)}}(p_1, \dots, p_4)$ at zero external momenta, $p_1 = \dots = p_4 = 0$. (It is up to us to define it at any momentum scale we wish.)

$$g \equiv \lim_{p_i \to 0} \overline{\Gamma^{(4)}}(p_1, \dots, p_4) = \overline{\Gamma^{(4)}}(0, \dots, 0) \tag{71}$$

This definition is convenient and simple but it is problematic if the renormalized mass μ^2 vanishes (*i.e.,* in the massless or critical theory). To order one-loop, the renormalized coupling constant g is

$$g = \lambda - 3\frac{\lambda^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{\left(q^2 + m^2\right)^2} + O(\lambda^3) \tag{72}$$

Using the same line of argument we used to the dine the self energy, $\gamma^{(2)}$, we will now sum all the one loop diagrams, as shown in the figure:



This "bubble" sum is a geometric series, and it is equivalent to the replacement of Eq.(72) by

$$g = \lambda - 3\frac{g^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{\left(q^2 + m^2\right)^2} + O(\lambda^3) \tag{73}$$

or, alternatively, to write the bare coupling constant λ in terms of the renormalized coupling g as

$$\lambda = g + \frac{3}{2}\, g^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{\left(q^2 + \mu^2\right)^2} + O(g^3) \tag{74}$$

where we have replaced the bare mass m^2 with the renormalized mass μ^2. This amounts to add the insertions of tadpole diagrams in the internal propagators. This si a consistent at this order in perturbation theory.

Written in terms of the renormalized coupling constant g and of the renormalized mass μ^2, the vertex function becomes

$$\overline{\Gamma^{(4)}}(p_1,\ldots,p_4) = g - \frac{g^2}{2}\int \frac{d^Dq}{(2\pi)^D}\left[\frac{1}{(q^2+\mu^2)\left((p_1+p_2-q)^2+\mu^2\right)} - \frac{1}{(q^2+\mu^2)^2}\right] + \text{two permutations} + O(g^3) \tag{75}$$

which is UV finite for $D<6$. Thus, the renormalization of the coupling constant leads to a subtraction of the singular expression for the vertex function.

After the renormalization of the coupling constant, the singular behavior of the integral now appears only in the relation between the bare coupling constant λ and the renormalized coupling constant g, given in Eq.(74). Clearly there are a number of ways to interpret the meaning of this relation. One interpretation is to say that at a fixed value of the bare coupling constant λ, Eq.(74) relates the regulator Λ (which is a momentum scale) and the renormalized coupling constant. In other terms, the effective of renormalized coupling constant g has become a function of a momentum (or energy) scale!. Conversely, we can fix the renormalized coupling and ask how do we have to change the bare coupling constant λ as we send the regulator to infinity.

It will be useful to work with dimensionless quantities. Since the bare and the renormalized coupling constants have units of Λ^{4-D}, we define the *dimensionless* bare coupling constant u by

$$\lambda = \Lambda^{4-D} u \tag{76}$$

Then

$$\begin{aligned} u &= \Lambda^{D-4}\left(g + \frac{3}{2}\,g^2\int\frac{d^Dq}{(2\pi)^D}\frac{1}{(q^2)^2} + O(g^3)\right) \\ &= \Lambda^{D-4}\left(g + \frac{1}{(4\pi)^{D/2}}\frac{6}{(D-2)(D-4)}\Lambda^{D-4}g^2 + O(g^3)\right) \end{aligned} \tag{77}$$

We will vary the momentum scale Λ and the dimensionless bare coupling constant u at fixed g. The differential change of the dimensionless coupling constant is the renormalization group *beta-function*,

$$\beta(u) = -\Lambda\frac{\partial u}{\partial \Lambda}\Big|_g \tag{78}$$

Hence (for $D \to 4$)

$$\beta(u) = (4-D)u - \frac{3}{16\pi^2}u^2 + O(u^3) \tag{79}$$

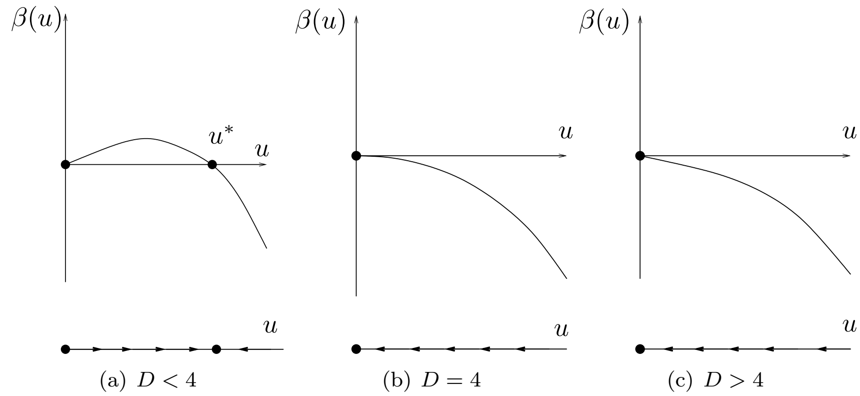


Figure 14: The beta-function (top) and the IR RG flows (bottom) for $D < 4$, $D = 4$ and $D > 4$. The UV RG flows are the reverse.

The behavior of the Renormalization group beta-function (or Gell-Mann-Low) for $D < 4$, $D = 4$ and $D > 4$ is shown in Fig.14. The infrared (IR) RG *flows*, *i.e.,* the flow of the dimensionless coupling constant u as the momentum scale λ is *decreased*, is also shown in Fig.14 (bottom). The UV flows, *i.e.,* the flow of u as the momentum scale is *increased*, is obtained by reversing the direction of the RG flows.

Clearly for $D \geq 4$ the dimensionless coupling constant flows to 0 in the infrared (at low energies and long distances). That means that in the IR regime, $\Lambda \to 0$, the theory becomes *weakly coupled*, and perturbation theory becomes reliable in that regime. However, at short distances (or at high energies), $\Lambda \to \infty$, the opposite happens: the dimensionless coupling becomes large and perturbation theory breaks down. Four dimensions is special in that the approach and departure from the decoupled limit, $u = 0$, is very slow, and leads to logarithmic corrections to the free field values.

However, for $D < 4$ something new happens: there is a *non-trivial fixed point* at u^* where $\beta(u^*) = 0$. *At the fixed point*, the coupling constant does not flow as the momentum scale changes. Hence, *at a fixed point* it is possible to send the momentum scale $\Lambda \to \infty$ and effectively have a theory without a cutoff. Notice that even infinitesimally away from the fixed point the UV flows are unstable. Also, and for the same reason, at the fixed point it is also possible to go into the deep infrared regime and have a theory with a finite coupling constant u^*. It turns out that this behavior is central for the theory of phase transitions. We will come back to the problem of the renormalization group next term where we will develop it and discuss its application to different theories.

A Integrals

We introduce a momentum cutoff Λ and to suppress the contributions at large momenta, $q \gg \Lambda$ of integrals of the form

$$I_D\left(\frac{\mu^2}{\Lambda^2}\right) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2+\mu^2}\, e^{-\dfrac{q^2}{\Lambda^2}} \tag{80}$$

where we used a Gaussian cutoff function (or *regulator*). We will only be interested in the regime $\mu^2 \ll \Lambda^2$. Using a Feynman-Schwinger parametrization we can write

$$\begin{aligned} I_D\left(\frac{\mu^2}{\Lambda^2}\right) &= \int_0^\infty d\alpha \int \frac{d^D q}{(2\pi)^D}\, e^{-\dfrac{q^2}{\Lambda^2} - \alpha(q^2+\mu^2)} \\ &= \int_0^\infty d\alpha\, e^{-\alpha\mu^2} \int \frac{d^D q}{(2\pi)^D}\, e^{-\left(\dfrac{1}{\Lambda^2}+\alpha\right) q^2} \\ &= e^{\frac{\mu^2}{\Lambda^2}} \frac{\left(\mu^2\right)^{\frac{D}{2}-1}}{(4\pi)^{D/2}} \int_{\mu^2/\Lambda^2}^\infty dt\, t^{-\frac{D}{2}} e^{-t} \end{aligned} \tag{81}$$

Hence

$$I_D\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{\left(\mu^2\right)^{\frac{D}{2}-1}}{(4\pi)^{D/2}}\, \Gamma\left(1-\frac{D}{2}, \frac{\mu^2}{\Lambda^2}\right)\, e^{\frac{\mu^2}{\Lambda^2}} \tag{82}$$

where $\Gamma(\nu, z)$ is the incomplete gamma function, with $z = \frac{\mu^2}{\Lambda^2}$ and $\nu = 1 - \frac{D}{2}$,

$$\Gamma(\nu, z) = \int_z^\infty dt\, t^{\nu-1} e^{-t} \tag{83}$$

and $\Gamma(\nu, 0) = \Gamma(\nu)$ is the Gamma function

$$\Gamma(\nu) = \int_0^\infty dt\, t^{\nu-1} e^{-t} \tag{84}$$

If the regulator Λ is removed (*i.e.,* if we take the limit $\Lambda \to \infty$), $I_D(\mu^2)$ formally becomes:

$$I_D(\mu^2) = \frac{\left(\mu^2\right)^{\frac{D}{2}-1}}{(4\pi)^{D/2}}\, \Gamma\left(1-\frac{D}{2}\right) \tag{85}$$

For general D, $\Gamma(1-D/2)$ is a meromorphic function of the complex variable D, and has simple poles for $\nu = 0$ or any *negative* integer, $\Gamma(1-D/2)$ has poles for $D = 2, 4, 6, \ldots$

In $D = 4$ dimensions, $\nu = -1$ where $\Gamma(\nu)$ has a pole, the incomplete Gamma function at $\nu = -1$ is (as $z \to 0$)

$$\Gamma(-1, z) = \left(\frac{1}{z} - \ln\frac{1}{z}\right)\, e^{-z} + \gamma \tag{86}$$

where γ is the Euler-Mascheroni constant

$$\gamma = -\int_0^\infty dt\; e^{-t}\; \ln t = 0.5772\ldots \tag{87}$$

Hence, for $\mu^2 \ll \Lambda^2$, I_4 is

$$I_4\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{\Lambda^2}{16\pi^2} - \frac{\mu^2}{8\pi^2}\;\ln\left(\frac{\Lambda}{\mu}\right) + \frac{\gamma}{16\pi^2}\;\mu^2 \tag{88}$$

Here we see that the leading singularity is quadratic in the regulator Λ, with a sub-leading logarithmic piece.

In two dimensions I_2 has instead a logarithmic singularity for $\mu^2 \ll \Lambda^2$

$$\begin{aligned} I_2\left(\frac{\mu^2}{\Lambda^2}\right) &= \frac{1}{4\pi}\;\Gamma\left(0, \frac{\mu^2}{\Lambda^2}\right) \\ &= \frac{1}{2\pi}\;\ln\left(\frac{\Lambda}{\mu}\right) - \frac{\gamma}{4\pi} \end{aligned} \tag{89}$$

A second integral of interest is

$$\begin{aligned} J_D\left(\frac{\mu^2}{\Lambda^2}\right) &= \int \frac{d^Dq}{(2\pi)^D}\;\frac{1}{q^2\,(q^2+\mu^2)} \\ &= \frac{1}{\mu^2}\left(I_D(0) - I_D\left(\frac{\mu^2}{\Lambda^2}\right)\right) \end{aligned} \tag{90}$$

$J_D(\mu^2/\Lambda^2)$ is UV finite if $D<4$, where it is given by

$$J_D\left(\mu^2\right) = \frac{\left(\mu^2\right)^{\frac{D}{2}-2}}{(4\pi)^{D/2}}\;\frac{\Gamma\left(\frac{D}{2}-2\right)}{\frac{D}{2}-1} \tag{91}$$

In four dimensions, J_4, has a logarithmic divergence

$$J_4\left(\frac{\mu^2}{\Lambda^2}\right) = \frac{1}{8\pi^2}\;\ln\left(\frac{\Lambda}{\mu}\right) - \frac{\gamma}{16\pi^2} \tag{92}$$

A third useful integral is

$$I'_D(\frac{\mu^2}{\Lambda^2}) = \int \frac{d^Dq}{(2\pi)^D}\frac{1}{\left(q^2+\mu^2\right)^2}\;e^{-\frac{q^2}{\Lambda^2}} = -\frac{\partial I_D}{\partial \mu^2} \tag{93}$$

In the massless limit it becomes

$$I'_D(0) = \frac{1}{(4\pi)^{D/2}}\frac{4}{(D-2)(D-4)}\Lambda^{D-4} \tag{94}$$

In four dimensions it becomes

$$I'_4(\frac{\mu^2}{\Lambda^2}) = \frac{1}{8\pi^2}\;\ln\left(\frac{\Lambda}{\mu}\right) - \left(\frac{\gamma+1}{16\pi^2}\right) \tag{95}$$