Phys 582 – General Field Theory
Problem Set No.5 Solutions

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1 Propagators and Correlation Functions for the One-Dimensional Quantum Heisenberg Antiferromagnet

1. First of all we need to have a look at the time-evolution of the operators of c_q and d_q in Problem Set No.3: Using the Heisenberg Representation of Equations of Motion,

$$\begin{aligned} i\partial_t c_q = E_q c_q \quad i\partial_t c_q^+ = -E_q c_q^+ \Rightarrow c_q(t) = e^{-iE_q t} c_q \quad c_q^+(t) = e^{iE_q t} c_q^+ \\ i\partial_t d_q = E_q d_q \quad i\partial_t d_q^+ = -E_q d_q^+ \Rightarrow d_q(t) = e^{-iE_q t} d_q \quad d_q^+(t) = e^{iE_q t} d_q^+ \end{aligned} \tag{1}$$

where

$$E_q = \frac{JSN}{\pi} |\sin q| \tag{2}$$

Looking back to a_q, b_q operators, which gives,

$$\begin{aligned} a_q(t) &= e^{-iE_q t} u_q c_q - e^{iE_q t} v_q d_q^+ \\ a_q^+(t) &= e^{iE_q t} u_q c_q^+ - e^{-iE_q t} v_q d_q \\ b_q(t) &= e^{-iE_q t} u_q d_q - e^{iE_q t} v_q c_q^+ \\ b_q^+(t) &= e^{iE_q t} u_q d_q^+ - e^{-iE_q t} v_q c_q \end{aligned} \tag{3}$$

where we define

$$u_q = \cosh\theta_q \qquad v_q = \sinh\theta_q \tag{4}$$

the momentum space operators satisfies a Fourier Transformation back to the real space operators, since the Fourier Transformation only acts on the momentum while does nothing on time variable, the real space operator still satisfies the relation in Eq.(3).
Now let us consider the correlation functions, there are four possibilities,

(1) n even, n' even, (2) n odd, n' odd,
(3) n odd, n' even, (4) n even, n' odd.

First let us consider the case for $D_{\pm}(q,\omega)$:
(1) n even, n' even, the correlation function is,

$$\begin{aligned} & D_{\pm}(a,a) \\ = & -2Si\langle 0|T\, a(n,t)a^{+}(n',t')|0\rangle = -i\frac{4S}{N}\langle 0|T\sum_{q} e^{-iqn}a_q(t)\sum_{p} e^{ipn'}a_p^{+}(t')|0\rangle \\ = & -i\frac{4S}{N}\sum_{pq} e^{-iqn+ipn'}\langle 0|T\left(e^{-iE_qt}u_qc_q - e^{iE_qt}v_qd_q^{+}\right)\left(e^{iE_pt'}u_pc_p^{+} - e^{-iE_pt'}v_pd_p\right)|0\rangle \end{aligned} \quad (5)$$

Due to the time-ordered operator T, there are two combinations, on the other hand, since only operators like $c_qc_q^{+}$ and $d_pd_p^{+}$ acting on the ground state have a non-vanishing value, we can simplify the above equation as: (Here I begin to use $\Delta n = n - n'$ and $\Delta t = t - t'$)

$$D_{\pm}(a,a) = -i\frac{4S}{N}\sum_{p} e^{-ip\Delta n}\left[\theta(\Delta t)e^{-iE_p\Delta t}u_p^2 + \theta(-\Delta t)e^{iE_p\Delta t}v_p^2\right] \quad (6)$$

Using the definition of the Fourier Transformation,

$$D_{\pm}^{q,\omega}(a,a) = \sum_{\Delta n_e}\int d(\Delta t)\, e^{iq\Delta n - i\omega\Delta t}D_{\pm}^{\Delta n,\Delta t}(a,a) \quad (7)$$

redefine $\Delta n \to n$ and $\Delta t \to t$, we obtain,

$$D_{\pm}^{q,\omega}(a,a) = -\frac{4Si}{N}\sum_{n,p}\int dt\, e^{i(q-p)n}\left[\theta(t)e^{-i(E_p+\omega)t}u_p^2 + \theta(-t)e^{i(E_p-\omega)t}v_p^2\right] \quad (8)$$

In the double lattice model we can use the identity,

$$\frac{2}{N}\sum_{n_e,n_o} e^{iqn} = \delta(q,0) \quad (9)$$

Eq.(8) reduces into:

$$\begin{aligned} & D_{\pm}^{q,\omega}(a,a) \\ = & -2Si\int dt\left[\theta(t)e^{-i(E_q+\omega)t}u_q^2 + \theta(-t)e^{i(E_q-\omega)t}v_q^2\right] \\ = & -2Si\left\{\int_0^{\infty} dt\left[e^{-i(E_q+\omega)t-\epsilon t}u_q^2\right] + \int_{-\infty}^{0} dt\left[e^{i(E_q-\omega)t+\epsilon t}v_q^2\right]\right\} \\ = & \frac{2S}{|\sin q|}\left(\frac{1-|\sin q|}{\omega - E_q + i\epsilon} - \frac{1+|\sin q|}{\omega + E_q - i\epsilon}\right) \end{aligned} \quad (10)$$

(2) n odd, n' odd, the correlation function is, almost the same as the conclusion above, but the little difference occurs at an exchange of two coefficients. Look back to Eq.(6), the difference results from,

$$D_{\pm}^{q,\omega}(b,b) = \frac{2S}{|\sin q|}\left(\frac{1+|\sin q|}{\omega - E_q + i\epsilon} - \frac{1-|\sin q|}{\omega + E_q - i\epsilon}\right) \tag{11}$$

(4) n even, n' odd, similar steps give the correlation function as,

$$D_{\pm}(a,b) = \frac{4Si}{N}\sum_p e^{-ip\Delta n}\sinh\theta_p\cosh\theta_p\left[e^{-iE_p\Delta t}\theta(\Delta t) + e^{iE_p\Delta t}\theta(-\Delta t)\right] \tag{12}$$

Fourier Transform to the momentum space,

$$D_{\pm}^{q,\omega}(a,b) = 2S\frac{\cos q}{|\sin q|}\left(\frac{1}{\omega + E_q - i\epsilon} - \frac{1}{\omega - E_q + i\epsilon}\right) \tag{13}$$

(3) n odd, n' even,

$$D_{\pm}^{q,\omega}(b,a) = 2S\frac{\cos q}{|\sin q|}\left(\frac{1}{\omega + E_q - i\epsilon} - \frac{1}{\omega - E_q + i\epsilon}\right) \tag{14}$$

NOTE: the final correlation function, $D_{\pm}^{q,\omega}$ is the summation of the four terms $D_{\pm}^{q,\omega}(a,a)$, $D_{\pm}^{q,\omega}(b,b)$, $D_{\pm}^{q,\omega}(a,b)$, $D_{\pm}^{q,\omega}(b,a)$ therefore we obtain

$$D_{\pm}^{q,\omega} = 4S\frac{(1-\cos q)}{|\sin q|}\left(\frac{1}{\omega - E_q + i\epsilon} - \frac{1}{\omega + E_q - i\epsilon}\right) \tag{15}$$

Second, consider the case for $D_3(q,\omega)$:
(1) for both of n and n' are odd or even, this gives,

$$D_{33}(a,a) = D_{33}(b,b) = -i\langle 0|T(S - N(n,t))(S - N(n',t'))|0\rangle \tag{16}$$

Since a_q and b_q are the same form except for the interchange of c_q and d_q, we want to use a_q in this case. To obtain the value for Eq.(16), we have to evaluate:

$$\langle 0|a^+(n,t)a(n,t)|0\rangle = \frac{2}{N}\langle 0|\sum_p e^{ipn}a_p^+(t)\sum_q e^{-iqn}a_q(t)|0\rangle = 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{dq}{2\pi}v_q^2 \tag{17}$$

and :

$$\begin{aligned}
&\langle 0|Ta^+(n,t)a(n,t)a^+(n',t')a(n',t')|0\rangle \\
=\ &\frac{4}{N^2}\langle 0|\hat{T}\sum_p e^{ipn}a_p^+(t)\sum_q e^{-iqn}a_q(t)\sum_{p'} e^{ip'n'}a_{p'}^+(t')\sum_{q'} e^{-iq'n'}a_{q'}(t')|0\rangle \\
=\ &\frac{4}{N^2}\sum_{pq} v_p^2\left(v_q^2 + \hat{T}e^{i(p-q)(n-n')-i(E_q+E_p)(t-t')}u_q^2\right)
\end{aligned} \tag{18}$$

Let us denote

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dq}{2\pi} v_q^2 = C \tag{19}$$

Thus there are 4 terms of constants in the correlation function $D_{33}(nt, n't')$, since it gives the $\delta(q=0, \omega=0)$ function in the momentum-space correlation, we only take out the time-space dependent term out, and keep the δ-function later:

$$\frac{4}{N^2} \sum_{pq} v_p^2 u_q^2 e^{i(p-q)(n-n') - i(E_q+E_p)(t-t')} \left[\theta(t-t') + \theta(t'-t)\right] \tag{20}$$

Fourier Transform it back to the momentum space,

$$\begin{aligned} D_{33}^{k,\omega}(a,a) &= -iN\pi\delta(k,0)\delta(\omega,0)\left(S-2C\right)^2 + 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dq}{2\pi} v_{q-\frac{1}{2}k}^2 u_{q+\frac{1}{2}k}^2 \\ &\times \left(\frac{1}{\omega + E_{q-\frac{1}{2}k} + E_{q+\frac{1}{2}k} + i\epsilon} - \frac{1}{\omega + E_{q-\frac{1}{2}k} + E_{q+\frac{1}{2}k} - i\epsilon} \right) \\ D_{33}^{k,\omega}(b,b) &= D_{33}^{k,\omega}(a,a) \end{aligned} \tag{21}$$

(2) For n even and n' odd, the first three constants keep the same as that in (1) except for a minus sign; on the other hand, the last term becomes,

$$\begin{aligned} & -\langle 0|Ta^+(n,t)a(n,t)b^+(n',t')b(n',t')|0\rangle \\ =& -\frac{4}{N^2}\langle 0|T\sum_p e^{ipn}a_p^+(t)\sum_q e^{-iqn}a_q(t)\sum_{p'} e^{-ip'n'}b_{p'}^+(t')\sum_{q'} e^{iq'n'}b_{q'}(t')|0\rangle \\ =& -\frac{4}{N^2}\sum_{pq}\left(v_q^2 v_p^2 + \hat{T} v_q u_q v_p u_p e^{-i(E_q+E_p)\Delta t + i(p-q)\Delta n}\right) \end{aligned} \tag{22}$$

Again, redefine $\Delta t \to t$ and $\Delta n \to n$. What we are interested in is the time-ordered operator,

$$-\frac{4}{N^2}\sum_{pq}\left(v_q u_q v_p u_p e^{-i(E_q+E_p)t + i(p-q)n}\right)\left[\theta(t) + \theta(-t)\right] \tag{23}$$

Fourier Transforma back to momentum space, we obtain

$$\begin{aligned} D_{33}^{k,\omega}(a,b) &= -2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dq}{2\pi}\left(v_{q+\frac{1}{2}k}u_{q+\frac{1}{2}k}v_{q-\frac{1}{2}k}u_{q-\frac{1}{2}k}\right) \\ &\times \left(\frac{1}{E_{q+\frac{1}{2}k} + E_{q-\frac{1}{2}k} + \omega + i\epsilon} - \frac{1}{E_{q+\frac{1}{2}k} + E_{q-\frac{1}{2}k} + \omega - i\epsilon} \right) \\ &+ iN\pi\delta(k,0)\delta(\omega,0)\left(S-2C\right)^2 \\ D_{33}^{k,\omega}(a,b) &= D_{33}^{k,\omega}(b,a) \end{aligned} \tag{24}$$

Finally the total correlation function of $D_{33}^{k,\omega}$ is the summation of all four terms above,

$$\begin{aligned} D_{33}^{k,\omega} &= 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dq}{2\pi} \sinh\theta_{q-\frac{1}{2}k} \cosh\theta_{q+\frac{1}{2}k} \sinh\left(\theta_{q-\frac{1}{2}k} - \theta_{q+\frac{1}{2}k}\right) \\ &\times \left(\frac{1}{E_{q+\frac{1}{2}k} + E_{q-\frac{1}{2}k} + \omega + i\epsilon} - \frac{1}{E_{q+\frac{1}{2}k} + E_{q-\frac{1}{2}k} + \omega - i\epsilon}\right) \end{aligned} \tag{25}$$

2. Using the Kubo Formula, which is directly related to the Retarded Green Function, we obtain that

$$\chi_{\pm}(k,\omega) = \Re D_{\pm}^{k,\omega}, \quad \chi_{33}(k,\omega) = \Re D_{33}^{k,\omega} \tag{26}$$

where the Retarded Green Function and Feynman Green Function's relation is

$$\Re\, D^R = \Re\, D^F, \quad \Im\, D^R = \mathrm{sgn}(\omega) \Im\, \mathrm{D^F} \tag{27}$$

Both are the same for $D_3(q,\omega)$ and $D_{\pm}(q,\omega)$.
3. We have obtained this from the first part of the problem,

$$\begin{aligned} D_{33}^{k,\omega} &= 2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{dq}{2\pi} \sinh\theta_{q-\frac{1}{2}k} \cosh\theta_{q+\frac{1}{2}k} \sinh\left(\theta_{q-\frac{1}{2}k} - \theta_{q+\frac{1}{2}k}\right) \\ &\times \left(\frac{1}{E_{q+\frac{1}{2}k} + E_{q-\frac{1}{2}k} + \omega + i\epsilon} - \frac{1}{E_{q+\frac{1}{2}k} + E_{q-\frac{1}{2}k} + \omega - i\epsilon}\right) \end{aligned} \tag{28}$$

$$D_{\pm}^{k,\omega} = 4S\frac{(1-\cos q)}{|\sin q|}\left(\frac{1}{\omega - E_k + i\epsilon} - \frac{1}{\omega + E_k - i\epsilon}\right) \tag{29}$$

4. In the limit $\omega \to 0$, $\chi_{\pm}(q,\omega)$, also known as $D_{\pm}^{q,\omega}$, is shown above,

$$\begin{aligned} D_{\pm}^{k,0} &= 4S\frac{(1-\cos q)}{|\sin q|}\left(\frac{1}{-E_k + i\epsilon} - \frac{1}{E_k - i\epsilon}\right) \\ &= -\frac{8\pi}{JN}\frac{(1-\cos q)}{|\sin q|^2} = -\frac{4\pi}{JN}\frac{1}{\cos^2\frac{q}{2}} \end{aligned} \tag{30}$$

This equation has a pole when $|\cos\frac{q}{2}|$ vanishes, corresponding to the point of $q \to \pi$.

2 Spectral Function for the Dirac Propagator

1. If the Hamiltonian has translational invariance, then the momentum operator commutes with the Hamiltonian and the energy eigenstates are the same as the momentum eigenstates. Inserting a complete set of states,

$$1 = \sum_n |n\rangle\langle n| \tag{31}$$

where $|n\rangle$ is the eigenstate of momentum, in the continuous limit

$$\sum_n |n\rangle\langle n| = |0\rangle\langle 0| + \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\sum_{s=1}^{2}\left(|b_{ps}\rangle\langle b_{ps}| + |d_{ps}\rangle\langle d_{ps}|\right) \tag{32}$$

also we require $|n\rangle$ is space-like. For the propagator we are interested, instering the identity defined above,

$$\begin{aligned} iS^F_{\alpha\alpha'}(x,x') &= \langle 0|\hat{T}\psi_\alpha(x)\bar{\psi}_{\alpha'}(x')|0\rangle \\ &= \theta(x_0 - x'_0)\langle 0|\psi_\alpha(x)\bar{\psi}_{\alpha'}(x')|0\rangle - \theta(x'_0 - x_0)\langle 0|\bar{\psi}_{\alpha'}(x')\psi_\alpha(x)|0\rangle \\ &= \sum_n \theta(x_0 - x'_0)\langle 0|\psi_\alpha(x)|n\rangle\langle n|\bar{\psi}_{\alpha'}(x')|0\rangle \\ &- \sum_n \theta(x'_0 - x_0)\langle 0|\bar{\psi}_{\alpha'}(x')|n\rangle\langle n|\psi_\alpha(x)|0\rangle \end{aligned} \tag{33}$$

where

$$\begin{aligned} \psi_\alpha(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{m}{\omega_p}\sum_{s=1}^{2}\left(b_{p,s}u_\alpha(p,s)e^{-ip\cdot x} + d^+_{p,s}v_\alpha(p,s)e^{ip\cdot x}\right) \\ \bar{\psi}_\alpha(x') &= \int \frac{d^3p}{(2\pi)^3}\frac{m}{\omega_p}\sum_{s=1}^{2}\left(d_{p,s}\bar{v}_{\alpha'}(p,s)e^{-ip\cdot x} + b^+_{p,s}\bar{u}_{\alpha'}(p,s)e^{ip\cdot x}\right) \end{aligned} \tag{34}$$

with the commutation relation,

$$\begin{aligned} \left\{b_{ps}, b^+_{p's'}\right\} &= (2\pi)^3 2\omega_p\delta_{ss'}\delta(p-p') \\ \left\{d_{ps}, d^+_{p's'}\right\} &= (2\pi)^3 2\omega_p\delta_{ss'}\delta(p-p') \end{aligned} \tag{35}$$

Therefore we have,

$$\begin{aligned} \langle 0|\psi_\alpha(x)|b_{ps}\rangle &= 2mu_\alpha(p,s)e^{-ip\cdot x} \\ \langle 0|\bar{\psi}_{\alpha'}(x')|d_{ps}\rangle &= 2m\bar{v}_{\alpha'}(p,s)e^{-ip\cdot x'} \end{aligned} \tag{36}$$

For the interacting case we can also give the renormalization coefficient,

$$\begin{aligned} \langle 0|\psi_\alpha(x)|n\rangle &= \sqrt{Z_n}2mu_\alpha(p,s)e^{-ip\cdot x} = 2mu_\alpha(p,s)\langle 0|\phi(0)|n\rangle e^{-ip\cdot x} \\ \langle 0|\bar{\psi}_{\alpha'}(x')|n\rangle &= \sqrt{Z_n}2m\bar{v}_{\alpha'}(p,s)e^{-ip\cdot x'} = 2m\bar{v}_{\alpha'}(p,s)\langle 0|\phi(0)|n\rangle e^{-ip\cdot x'} \end{aligned} \tag{37}$$

In the free case limit $Z_n = |\langle 0|\phi(0)|n\rangle|^2 = 1$. To represent the overlap of free-field's eigentate and the interacting field's eigenstate, we need to use the parameter $Z_n = |\langle 0|\phi(0)|n\rangle|^2$. The correlation function becomes,

$$\begin{aligned} iS_{\alpha\alpha'}(x,x') &= \sum_n \theta(x_0 - x'_0)\langle 0|\psi_\alpha(x)|n\rangle\langle n|\bar{\psi}_{\alpha'}(x')|0\rangle \\ &- \sum_n \theta(x'_0 - x_0)\langle 0|\bar{\psi}_{\alpha'}(x')|n\rangle\langle n|\psi_\alpha(x)|0\rangle \end{aligned}$$

$$
\begin{aligned}
&= 2m \sum_n |\langle 0|\phi(0)|n\rangle|^2 \left[\theta(x_0 - x_0')e^{-ip_n\cdot(x-x')} \sum_s u_\alpha(p_n s)\bar{u}_{\alpha'}(p_n s)\right] \\
&- 2m \sum_n |\langle 0|\phi(0)|n\rangle|^2 \left[\theta(x_0' - x_0)e^{-ip_n\cdot(x'-x)} \sum_s v_\alpha(p_n s)\bar{v}_{\alpha'}(p_n s)\right]
\end{aligned}
\tag{38}
$$

define the density function,

$$
\begin{aligned}
\tilde{\rho}^u_{\alpha\alpha'}(p) &= 2m(2\pi)^3 \sum_n \delta^4(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 \left(\sum_s u_\alpha(p_n s)\bar{u}_{\alpha'}(p_n s)\right) \\
\tilde{\rho}^v_{\alpha\alpha'}(p) &= 2m(2\pi)^3 \sum_n \delta^4(p-p_n)|\langle 0|\phi(0)|n\rangle|^2 \left(\sum_s v_\alpha(p_n s)\bar{v}_{\alpha'}(p_n s)\right) \\
\rho(p) &= (2\pi)^3 \sum_n \delta^4(p-p_n)|\langle 0|\phi(0)|n\rangle|^2
\end{aligned}
\tag{39}
$$

and,

$$
\begin{aligned}
2m \sum_s u_\alpha(p_n, s)\bar{u}_{\alpha'}(p_n, s) &= (\not{p} + m)_{\alpha\alpha'} \\
2m \sum_s v_\alpha(p_n, s)\bar{v}_{\alpha'}(p_n, s) &= (\not{p} - m)_{\alpha\alpha'}
\end{aligned}
\tag{40}
$$

before proceeding to evaluate the correlation function, we want to have a look at the $\Delta^+_{\mathrm{KG}}(x-x')$ and $\Delta^-_{\mathrm{KG}}(x-x')$: Since from the K-G field,

$$
\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(a_p e^{-ip\cdot x} + b_p^+ e^{ip\cdot x}\right) \\
\phi^+(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\left(b_p e^{-ip\cdot x} + a_p^+ e^{ip\cdot x}\right)
\end{aligned}
\tag{41}
$$

and

$$
\begin{aligned}
\left[a_p, a_q^+\right] &= (2\pi)^3 2\omega_p \delta^3(p-q) \\
\left[b_p, b_q^+\right] &= (2\pi)^3 2\omega_p \delta^3(p-q)
\end{aligned}
\tag{42}
$$

therefore,

$$
\begin{aligned}
i\Delta^+_{\mathrm{KG}}(x-x') &= \langle 0|\phi(x)\phi^+(x')|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\langle 0|\phi(x)|p\rangle\langle p|\phi^+(x')|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip\cdot(x-x')} = \int \frac{d^4p}{(2\pi)^3}\frac{\delta(p_0-\omega_p)}{2\omega_p}e^{-ip\cdot(x-x')} \\
&= \int \frac{d^4p}{(2\pi)^3}\delta(p_0^2-\omega_p^2)\theta(p_0)e^{-ip\cdot(x-x')} \\
&= \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(p_0)e^{-ip\cdot(x-x')}
\end{aligned}
\tag{43}
$$

and

$$
\begin{aligned}
-i\Delta^{-}_{\rm KG}(x-x') &= \langle 0|\phi^+(x')\phi(x)|0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}\langle 0|\phi^+(x')|p\rangle\langle p|\phi(x)|0\rangle \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_p}e^{-ip\cdot(x'-x)} = \int \frac{d^4p}{(2\pi)^3}\frac{\delta(p_0+\omega_p)}{2\omega_p}e^{-ip\cdot(x'-x)} \\
&= \int \frac{d^4p}{(2\pi)^3}\delta(p_0^2-\omega_p^2)\theta(-p_0)e^{-ip\cdot(x'-x)} \\
&= \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\theta(-p_0)e^{-ip\cdot(x'-x)} \qquad (44)
\end{aligned}
$$

thus we can obtain the explicit form of $\Delta^F_{\rm KG}(x-x')$:

$$
\begin{aligned}
& i\Delta^F_{\rm KG}(x-x',m^2) = \langle 0|\hat{T}\phi(x)\phi^+(x')|0\rangle \\
&= \theta(x_0-x_0')\langle 0|\phi(x)\phi^+(x')|0\rangle + \theta(x_0'-x_0)\langle 0|\phi^+(x')\phi(x)|0\rangle \\
&= \theta(x_0-x_0')i\Delta^+_{\rm KG}(x-x') - \theta(x_0'-x_0)i\Delta^-_{\rm KG}(x-x') \\
&= \int \frac{d^4p}{(2\pi)^3}\delta(p^2-m^2)\left[\theta(p_0)e^{-ip\cdot(x-x')} + \theta(-p_0)e^{-ip\cdot(x'-x)}\right] \qquad (45)
\end{aligned}
$$

On the other hand, the scalar field Green Function gives,

$$
i\Delta^F_{\rm KG}(x-x',m^2) = \int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-x')}}{p^2-m^2+i\epsilon} \qquad (46)
$$

therefore,

$$
\begin{aligned}
& iS^F_{\alpha\alpha'}(x-x') \\
&= \int \frac{d^4p}{(2\pi)^3}\left[\tilde{\rho}^u_{\alpha\alpha'}(p)\theta(x_0-x_0')e^{-ip(x-x')} - \theta(x_0'-x_0)\tilde{\rho}^v_{\alpha\alpha'}(p)e^{-ip(x'-x)}\right] \\
&= \int \frac{d^4p}{(2\pi)^3}(2\pi)^3\sum_n \delta^4(p-p_n)|\langle 0|\phi(0)|n\rangle|^2\left[\theta(x_0-x_0')(i\partial\!\!\!/+m)_{\alpha\alpha'}e^{-ip(x-x')}\right] \\
&- \int \frac{d^4p}{(2\pi)^3}(2\pi)^3\sum_n \delta^4(p-p_n)|\langle 0|\phi(0)|n\rangle|^2\left[\theta(x_0'-x_0)(i\partial\!\!\!/-m)_{\alpha\alpha'}e^{-ip(x-x')}\right] \\
&= \int \frac{d^4p}{(2\pi)^3}(2\pi)^3\sum_n \delta^4(p-p_n)|\langle 0|\phi(0)|n\rangle|^2(i\partial\!\!\!/+m)_{\alpha\alpha'} \\
&\times \left[\theta(x_0-x_0')e^{-ip(x-x')} + \theta(x_0'-x_0)e^{-ip(x'-x)}\right] \\
&= (i\partial\!\!\!/+m)_{\alpha\alpha'}\int \frac{d^4p}{(2\pi)^3}\rho(p)\left[\theta(x_0-x_0')e^{-ip(x-x')} + \theta(x_0'-x_0)e^{-ip(x'-x)}\right] \\
&= (i\partial\!\!\!/+m)_{\alpha\alpha'}\int \frac{d^4p}{(2\pi)^3}\rho(p^2)\theta(p_0)\left[\theta(x_0-x_0')e^{-ip(x-x')} + \theta(x_0'-x_0)e^{-ip(x'-x)}\right] \\
&= (i\partial\!\!\!/+m)_{\alpha\alpha'}\int_0^\infty ds\,\rho(s)\int \frac{d^4p}{(2\pi)^3}\delta(p^2-s)\theta(p_0)
\end{aligned}
$$

$$\times \left[\theta(x_0 - x_0')e^{-ip(x-x')} + \theta(x_0' - x_0)e^{-ip(x'-x)}\right]$$
$$= (i\partial\!\!\!/ + m)_{\alpha\alpha'} \int_0^\infty ds\, \rho(s) \left[\theta(x_0 - x_0') i\Delta^+_{\rm KG}(x - x') + \theta(x_0' - x_0) i\Delta^+_{\rm KG}(x' - x)\right]$$
$$= (i\partial\!\!\!/ + m)_{\alpha\alpha'} \int_0^\infty ds\, \rho(s) \left[\theta(x_0 - x_0') i\Delta^+_{\rm KG}(x - x') - \theta(x_0' - x_0) i\Delta^-_{\rm KG}(x - x')\right]$$
$$= (i\partial\!\!\!/ + m)_{\alpha\alpha'} \int_0^\infty ds\, \rho(s) i\Delta^F_{\rm KG}(x - x') \tag{47}$$

Finally in conclusion,

$$\begin{aligned} iS^F_{\alpha\alpha'}(x, x') &= \int_0^{+\infty} ds\, \rho_{\alpha\alpha'}(s) i\Delta^F_{\rm KG}(x - x', s) \\ \Rightarrow iS^F_{\alpha\alpha'}(p) &= \int_0^{+\infty} ds\, \frac{\rho_{\alpha\alpha'}(s)}{p^2 - m^2 + is} \end{aligned} \tag{48}$$

where we have defined,

$$\rho_{\alpha\alpha'}(s) = (2\pi)^3 \sum_n \delta(s - p_n^2) |\langle 0|\phi(0)|n\rangle|^2 \sum_{S=1}^{2} u_\alpha(p_n, S)\bar{u}_{\alpha'}(p_n, S) \tag{49}$$

2. If the vaccuum is invariant under parity transformation we find,

$$\langle 0|\hat{T}\psi_\alpha(x)\bar{\psi}_{\alpha'}(x')|0\rangle = \langle 0|P^{-1}\hat{T}\psi_\alpha(x)\bar{\psi}_{\alpha'}(x')|0\rangle \tag{50}$$

and

$$\begin{aligned} P^{-1}S_{\alpha\alpha'}(P)P &= S_{\alpha\alpha'}(P) \\ \Rightarrow P^{-1}\rho_{\alpha\alpha'}(P)P &= \rho_{\alpha\alpha'}(P) \end{aligned} \tag{51}$$

since $\rho_{\alpha\alpha'}$ is Lorentz-invariant, $\rho_{\alpha\alpha'}$ can expand in terms of γ matrix, and

$$\rho_{\alpha\alpha'}(p) = \rho_1(p^2)p\!\!\!/_{\alpha\alpha'} + \rho_2(p^2)\delta_{\alpha\alpha'} + \rho_3(p^2)(p\!\!\!/\gamma_5)_{\alpha\alpha'} + \rho_4(p^2)(\gamma_5)_{\alpha\alpha'} \tag{52}$$

the parity transformation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{53}$$

under the parity transformation the invariant term is

$$\rho_{\alpha\alpha'}(p) = \rho_1(p^2)p\!\!\!/_{\alpha\alpha'} + \rho_2(p^2)\delta_{\alpha\alpha'} \tag{54}$$

therefore in the free case

$$\rho_{\alpha\alpha'} = (2\pi)^3\delta(p^2 - m^2) \sum_{s=1}^{2} u_\alpha(p, s)\bar{u}_{\alpha'}(p, s) = (2\pi)^3\delta(p^2 - m^2)(p\!\!\!/ + m)_{\alpha\alpha'} \tag{55}$$

For the free Dirac Theory,

$$\begin{aligned} \rho_1(p^2) &= (2\pi)^3\delta(p^2 - m^2) \\ \rho_2(p^2) &= (2\pi)^3 m\delta(p^2 - m^2) \end{aligned} \tag{56}$$

3 Wick's Theorem

1. We have a 3-component scalar field. And know that under an $O(3)$ transformation, $\phi'_a(x) = O_{ab}\phi_b(x)$ where the matrix O is orthogonal, $O^T = O^{-1}$. One just needs to see if the product of field operators is invariant under $O(3)$ transformations:
(a) $\phi_a(x')\phi_a(x) \to \phi_b(x')O_{ba}O_{ab}\phi_b(x) = \phi_a(x')\phi_a(x')$ This correlation function has the symmetry of the system and is non-zero.
(b) $\phi_a\phi_b\phi_b \to O_{ac}\phi_c\phi'_b\phi_b$, because it is the third order of the field operator ϕ_a, it behaves smilar with ϕ_a, which does not satisfy the symmetry of the system, hence the v.e.v should be zero.
(c) The correlation function is invariant. When indices are summed over, then the matrix O_{ab} from the transformation vanish. A non-vanishing correlation function must have paired indices.
2. (b) is already proved to be vanishing, we just need to expand (c). The non-vanishing contribution must be the operators with indices in pairs.

$$\begin{aligned}\langle 0|T\phi_a(x)\phi_b(x')\phi_b(x'')\phi_a(x''')|0\rangle &= \langle 0|T\phi_a(x)\phi_a(x''')|0\rangle\langle 0|T\phi_b(x')\phi_b(x'')|0\rangle \\ &+ \delta_{ab}\langle 0|T\phi_a(x)\phi_b(x')|0\rangle\langle 0|T\phi_b(x'')\phi_a(x''')|0\rangle \\ &+ \delta_{ab}\langle 0|T\phi_a(x)\phi_b(x'')|0\rangle\langle 0|T\phi_b(x')\phi_a(x''')|0\rangle \end{aligned} \tag{57}$$

4 Reduction Formulas

The photon can be described by transverse EM field, which can be written as,

$$A^\mu_{\text{in}} = \int \frac{d^3k}{(2\pi)^3 2k_0} \sum_{\lambda=1,2} \epsilon^\lambda_\mu(k) \left(a^{(\lambda)}_{\text{in}}(k)e^{ik\cdot x - ik_0 t} + a^{+(\lambda)}_{\text{in}}(k)e^{-ik\cdot x + ik_0 t}\right) \tag{58}$$

Transform back, we obtain

$$a^{+(\lambda)}_{\text{in}}(k) = i\int d^3x\, A^\mu_{\text{in}}(x)\partial_0\left(\epsilon^{(\lambda)}_\mu(k)e^{ik\cdot x - ik_0 x_0}\right) \tag{59}$$

Thus,

$$S_{fi} = \langle p^{\text{out}}_+, p^{\text{out}}_- | p^{\text{in}}_i, \lambda\rangle = \langle p^{\text{out}}_+, p^{\text{out}}_- | a^{+(\lambda)}_{\text{in}}(p_i) - a^{+\lambda}_{\text{out}}(p_i)|0\rangle \tag{60}$$

where

$$\begin{aligned}& a^{+(\lambda)}_{\text{in}}(k_i) - a^{+\lambda}_{\text{out}}(k_i) \\ &= i\int d^3x\, \left(A^\mu_{\text{in}}(x) - A^\mu_{\text{out}}(x)\right)\partial_0\left(\epsilon^\lambda_\mu(k)e^{ik\cdot x - ik_0 t}\right) \\ &= i\int d^3x\, \left[\lim_{x_0\to-\infty} A^\mu_{\text{in}}(x)\partial_0\left(\epsilon^{(\lambda)}_\mu(k)e^{ik\cdot x - ik_0 t}\right)\right]\end{aligned}$$

$$
\begin{aligned}
&- i \int d^3x \left[\lim_{x_0 \to +\infty} A^\mu_{\rm out}(x) \partial_0 \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) \right] \\
&= -\frac{i}{\sqrt{Z_A}} \left(\lim_{x_0 \to +\infty} - \lim_{x_0 \to -\infty} \right) \int d^3x A^\mu(x) \partial_0 \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) \\
&= -\frac{i}{\sqrt{Z_A}} \int d^4x \, \partial_0 \left[A^\mu(x) \partial_0 \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) \right] \\
&= -\frac{i}{\sqrt{Z_A}} \int d^4x \, \left[\left(-\partial_0^2 A^\mu(x) \right) \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) + A^\mu(x) \partial_0^2 \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) \right] \\
&= -\frac{i}{\sqrt{Z_A}} \int d^4x \, \left[\left(-\partial_0^2 A^\mu(x) \right) \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) + A^\mu(x) \nabla^2 \left(\epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \right) \right] \\
&= \frac{i}{\sqrt{Z_A}} \int d^4x \, \epsilon^{(\lambda)}_\mu(k) e^{ik\cdot x - ik_0 t} \partial^2 A^\mu(x) \qquad (61)
\end{aligned}
$$

where we have used

$$
\partial^2 e^{ik\cdot x - ik_0 t} = 0 \Leftarrow k_0^2 = \vec{k}^2 \qquad (62)
$$

Therefore the S matrix becomes

$$
\begin{aligned}
S_{fi} &= \langle p_+^{\rm out}, p_-^{\rm out} | p_i^{\rm in}, \lambda \rangle \\
&= \frac{i}{\sqrt{Z_A}} \int d^4x_i \, \epsilon^{(\lambda)}_\mu(p_i) e^{-ip_i x_i} \partial^2 \langle p_+^{\rm out}, p_-^{\rm out} | A^\mu(x_i) | 0 \rangle \qquad (63)
\end{aligned}
$$

In the same way we can write out the contribution of pion's,

$$
\begin{aligned}
& \langle p_+^{\rm out}, p_-^{\rm out} | A^\mu(x_i) | 0 \rangle \\
&= \frac{-1}{Z} \int d^4y_+ d^4y_- e^{ip_+y_+ + ip_-y_-} \left(\partial_+^2 + m_+^2 \right) \left(\partial_-^2 + m_-^2 \right) \\
&\times \langle 0 | \hat{T} \phi_+(y_+) \phi_-(y_-) A^\mu(x_i) | 0 \rangle \qquad (64)
\end{aligned}
$$

Collecting all the factors, we obtain the final result as:

$$
\begin{aligned}
& \langle p_+^{\rm out}, p_-^{\rm out} | p_i^{\rm in}, \lambda \rangle = \langle p_+, p_- | \hat{S} | p_i, \lambda \rangle \\
&= -\frac{i}{Z\sqrt{Z_A}} \int d^4x_i d^4y_+ d^4y_- e^{-ip_i x_i + ip_+y_+ + ip_-y_-} \epsilon^{(\lambda)}_\mu(p_i) \\
&\times \left(\partial_+^2 + m_+^2 \right) \left(\partial_-^2 + m_-^2 \right) \left(\partial_i^2 \right) \langle 0 | \hat{T} \phi_+(y_+) \phi_-(y_-) A^\mu(x_i) | 0 \rangle \qquad (65)
\end{aligned}
$$

where

$$
\begin{aligned}
\lim_{x_0 \to \mp\infty} \langle b | A^\mu(x_0) | a \rangle &= \sqrt{Z_A} \langle b | A^\mu_{\rm in/out} | a \rangle \\
\lim_{y_0 \to \mp\infty} \langle d | \phi_\pm(y_0) | c \rangle &= \sqrt{Z} \langle d | \phi_{\pm \rm in/out} | c \rangle \qquad (66)
\end{aligned}
$$