Phys 582 – General Field Theory
Problem Set No.4 Solutions

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1 Path Integral for a particle in a double well potential

1. The real time path-integral is:

$$\langle q_f, t_f | q_i, t_i \rangle = \int DpDq e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt[p\dot{q} - H(p,q)]} = \int Dq e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt \mathcal{L}(q,\dot{q})} \tag{1}$$

Taking the imaginary time $t \to -i\tau$, we have:

$$S = \frac{i}{\hbar}\int_{t_i}^{t_f} dt \mathcal{L}(q,\dot{q}) \to S = \frac{i}{\hbar}\int d(-i\tau)\left(-\frac{1}{2}m\dot{q}_\tau^2 - V(q)\right) \tag{2}$$

The Lagrangian in Euclidean space time is defined as:

$$\mathcal{L}_E = \frac{1}{2}m\dot{q}_\tau^2 + \lambda(q^2 - q_0^2)^2 \tag{3}$$

Now the path-integral is

$$\langle q_f, T/2 | q_i, -T/2 \rangle = \int Dq e^{-\frac{1}{\hbar}\int_{-T/2}^{T/2} d\tau \left(\frac{1}{2}m\dot{q}_\tau^2 + \lambda(q^2 - q_0^2)^2\right)} \tag{4}$$

The relation between Euclidean Lagrangian and Minkowski Lagrangian is:

$$\begin{aligned} \mathcal{L}_M &= \frac{1}{2}m\left(\frac{\partial q}{\partial t}\right)^2 - \lambda(q^2 - q_0^2)^2 \\ \mathcal{L}_E &= \frac{1}{2}m\left(\frac{\partial q}{\partial \tau}\right)^2 + \lambda(q^2 - q_0^2)^2 \end{aligned} \tag{5}$$

Where the potential term has the opposite sign.
2. The equations of motion corresponds to the imaginary time is,

$$\frac{\partial \mathcal{L}}{\partial q} = \frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_\tau}\right) \tag{6}$$

Plug in Eq.(3), the explicit form of Euclidean Lagrangian, we obtain the equations of motion as,

$$4\lambda(q^2 - q_0^2)q = m\ddot{q}_\tau \Rightarrow \vec{F} = -m\vec{a} \tag{7}$$

Looking back to the case of real-time equations of motion,

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial q} = \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) &\Rightarrow -4\lambda(q^2 - q_0^2)q = m\ddot{q} \\ &\Rightarrow \vec{F} = m\vec{a} \end{aligned} \tag{8}$$

Comparing the results between imaginary and real time solutions, the minus sign appears in Eq.(7) looks like a funny. Use the energy in imaginary time,

$$E = \frac{1}{2}m\left(\frac{\partial q}{\partial \tau}\right)^2 - \lambda(q^2 - q_0^2)^2 \Rightarrow \frac{\partial q}{\partial \tau} = \pm\left\{\frac{2}{m}\left[\lambda\left(q^2 - q_0^2\right)^2 + E\right]\right\}^{\frac{1}{2}} \tag{9}$$

Which gives,

$$\pm\sqrt{\frac{m}{2}}\int_{q_i}^{q_f} \frac{dq}{\sqrt{\lambda\left(q^2 - q_0^2\right)^2 + E}} = \tau - \tau_0 \tag{10}$$

The solution is not unique since all the solutions that take the system from $-q_0 \to +q_0$ has $T \to \infty$. There is still an arbitrary parameter, the origin of imaginary time τ_0 in this solution (and hence there is a family of such solutions). This has to be the case for all solutions since the lagrangian does not depend explicitly on time and hence the origin of the time coordinate is arbitrary. This is the multiplicity of solutions that matters.

The $\pm$ sign in the above equation results the right-moving and left-moving particle which is time-reversal to each other. However, The time-reversed solution is always there because the system is invariant under time reversal.

The boundary conditions are of course incorporated into the limits of integration. Note that working in the limit $T \to \infty$, the right hand side blows up. In addition, the particle at $T \to \infty$ is at $-q_0$. The only way for the left hand side to blow up as well is for $E = 0$. One can also think of this as the situation where the particle begins at rest in one of the wells at $T \to -\infty$ and at some point tunnels through the barrier and is at rest in the opposite well at $T \to \infty$. The second case is reminiscent of calculations in classical mechanics of planetary bodies. Solve the integral as $E = 0$ and the positive sign (the negative is just the time reversal, so the particle's behavior is similar):

$$\tau - \tau_0 = \int_{-q_0}^{q} \sqrt{\frac{m}{2\lambda}}\left(\frac{dq}{q_0^2 - q^2}\right) = \sqrt{\frac{m}{8\lambda q_0^2}}\ln\left(\frac{q_0 + q}{q_0 - q}\right) \tag{11}$$

Rewriting q in terms of $\tau - \tau_0$, we obtain

$$q_c(\tau) = q_0 \tanh\left(\sqrt{\frac{2\lambda q_0^2}{m}}(\tau - \tau_0)\right) \tag{12}$$

Here I write q_c instead of q because I want to emphasize this is the classical trajectory. Therefore at $q = -q_0$, $\tau = -T/2$; at $q = q_0$, $\tau = T/2$, and $T \to \infty$. Although there are two solutions, every trajectory is unique. The physical interpretation of this trajectory is from Newton's second law, and the amplitude of this is the probability for a particle to move from $-q_0$ to q_0, which is nearly zero.

3. The imaginary action is, by definition,

$$S = \int_{-T/2}^{T/2} d\tau \left[\frac{1}{2} m \left(\frac{\partial q}{\partial \tau} \right)^2 + \lambda (q^2 - q_0^2)^2 \right] \tag{13}$$

Using

$$E = \frac{1}{2} m \left(\frac{\partial q}{\partial \tau} \right)^2 - \lambda (q^2 - q_0^2)^2 = 0, \quad d\tau = \sqrt{\frac{m}{2\lambda}} \frac{dq}{(q_0^2 - q^2)} \tag{14}$$

We can obtain the action as:

$$S = \sqrt{2m\lambda} \int_{-q_0}^{q_0} (q_0^2 - q^2) dq = \frac{4}{3} q_0^3 \sqrt{2m\lambda} \tag{15}$$

4. Expand $q = q_c + q_1$, where q_c is the classical trajectory and q_1 is the quantum fluctuation.

$$S = S_{q_c} + \int_{-\frac{T}{2}}^{\frac{T}{2}} d\tau \left\{ \left[\frac{1}{2} m \left(\frac{\partial q_1}{\partial \tau} \right)^2 + \frac{1}{2} \frac{\partial^2 V_{q_c}}{\partial q^2} q_1^2 \right] + \frac{d}{d\tau} (m q_1 \dot{q}_c) + \left(\frac{\partial V_{q_c}}{\partial q} - m \ddot{q}_c \right) q_1 \right\} \tag{16}$$

The third term in the bracket, is the boundary term and we can set it into zero; the fourth term, is the Newton's second law, also to be zero. Therefore we obtain the relation that,

$$S = S(q_c) + \int_{-T/2}^{T/2} d\tau \left[\frac{1}{2} m \left(\frac{\partial q_1}{\partial \tau} \right)^2 + \frac{1}{2} \frac{\partial^2 V(q_c)}{\partial q^2} q_1^2 \right] \tag{17}$$

therefore for the path integral,

$$\langle q_0, T/2 | -q_0, -T/2 \rangle = \langle q_0, T/2 | e^{-S(q_c)} (1 - S(q_1) + ...) | -q_0, -T/2 \rangle \tag{18}$$

Since

$$\langle q_0, T/2 | -q_0, -T/2 \rangle = 0 \tag{19}$$

The leading order term is the first order expansion of Eq.(17):

$$-e^{-\frac{4}{3} q_0^3 \sqrt{2m\lambda}} \langle q_0, \frac{T}{2} | S(q_1) | -q_0, -\frac{T}{2} \rangle, S(q_1) = \frac{1}{2} \left(\frac{\partial q_1}{\partial \tau} \right)^2 + \frac{1}{2} \frac{\partial^2 V(q_c)}{\partial q^2} q_1^2 \tag{20}$$

Therefore the operator is

$$\hat{A} = -\frac{1}{2}\left(\frac{\partial}{\partial\tau}\right)^2 + \frac{1}{2}\frac{\partial^2 V(q_c)}{\partial q^2} = -\frac{1}{2}\left(\frac{\partial}{\partial\tau}\right)^2 + 2\lambda(3q_c^2 - q_0^2)$$
$$= -\frac{1}{2}\left(\frac{\partial}{\partial\tau}\right)^2 + 2\lambda q_0^2\left(2 - \frac{3}{\cosh^2\left[\sqrt{\frac{2\lambda q_0^2}{m}}(\tau-\tau_0)\right]}\right) \tag{21}$$

2 Path Integral for a charged particle moving on a plane in the presence of a perpendicular magnetic field

1. We're given the Hamiltonian for a particle in a magnetic field and so this can be expressed in the Heisenberg picture where operators carry time dependence and the states do not,

$$\langle\vec{r}_0, t_f|\vec{r}_0, t_i\rangle = \langle\vec{r}_0|e^{-\int_{t_i}^{t_f} i\hat{H}dt}|\vec{r}_0\rangle \tag{22}$$

Inserting a complete set of momentum states and position states, one can arrive at the standard expression for the path integral as in the usual case, same as that of 3-dimensional case, that,

$$\langle\vec{r}_0, t_f|\vec{r}_0, t_i\rangle = \int DpDqe^{-i\int dt\left[\vec{p}\cdot\dot{\vec{q}} - H(\vec{p},\vec{q})\right]} \tag{23}$$

Use the eq.(3) in the problem set,

$$\vec{p}\cdot\dot{\vec{q}} - H(\vec{p},\vec{q}) = -\frac{1}{2m}\left(\vec{p} - m\dot{\vec{q}} + \frac{e}{c}\vec{A}\right)^2 + \frac{1}{2}m\dot{\vec{q}}^2 - \frac{e}{c}\dot{\vec{q}}\cdot\vec{A} \tag{24}$$

Taking the integral over $\vec{p}$, which give a constant out of the path integral, we obtain the final path integral that,

$$\langle\vec{r}_0, t_f|\vec{r}_0, t_i\rangle = \mathcal{N}\int D\vec{q}\exp\left\{\int_{t_i}^{t_f}\frac{i}{\hbar}\left(\frac{1}{2}m\dot{\vec{q}}^2 - \frac{e}{c}\dot{\vec{q}}\cdot\vec{A}\right)dt\right\} \tag{25}$$

2. For the ultra-quantum limit $m \to 0$,

$$\frac{1}{2}m\dot{\vec{q}}^2 - \frac{e}{c}\dot{\vec{q}}\cdot\vec{A} \to -\frac{e}{c}\dot{\vec{q}}\cdot\vec{A} = \frac{eB}{2c}(\dot{x}y - \dot{y}x) \tag{26}$$

Therefore the action becomes,

$$\frac{1}{\hbar}S = \frac{eB}{2\hbar c}\int_{t_i}^{t_f} dt\,(\dot{x}y - \dot{y}x) = \frac{eB}{2\hbar c}\oint(ydx - xdy) = \frac{eB}{2\hbar c}\oint(y\hat{e}_x - x\hat{e}_y)\cdot d\vec{l}$$
$$= \frac{eB}{2\hbar c}\int_S \nabla\times(y\hat{e}_x - x\hat{e}_y)\cdot d\vec{S} = -\frac{e}{\hbar c}\Phi \tag{27}$$

where Φ is the flux inclosed by the path. The line integral around a closed path came from the boundary conditions on allowed paths: they had to begin and end at the same point. Of course, there's no condition on how long the path is, what the initial point is, and what the shape of the path is.
3. The ambiguity comes from the path's enclosed area. We only fixed the particle's begin and end point at the same point and have not specified a normal vector, which will lead to the ambiguity of inside and outside boundary. Geometrically on the plane, one could say that the area enclosed is the inside surface, Ω_1 or the outside surface Ω_2. The difference is in the definition of the normal vector to the boundary. In both cases, One should find that

$$\frac{1}{\hbar}S = -\frac{e}{\hbar c}\Phi_{\Omega_1}, \quad \frac{1}{\hbar}S = -\frac{e}{\hbar c}\Phi_{\Omega_2} \tag{28}$$

Since the path enclose the area is in the positive direction while the path corresponds to the outside area is in the opposite direction, the actions above should be written as:

$$\frac{1}{\hbar}S = -\frac{e}{\hbar c}\Phi_{\Omega_1}, \quad \frac{1}{\hbar}S = -\frac{e}{\hbar c}\left(-BL^2 + \Phi_{\Omega_1}\right) \tag{29}$$

Finally because this ambiguity do not matter in the physical transition amplitude, the difference must be $2n\pi$:

$$-\frac{e}{\hbar c}\Phi_{\Omega_1} + 2n\pi = -\frac{e}{\hbar c}\left(-BL^2 + \Phi_{\Omega_1}\right) \tag{30}$$

Denote $\Phi = BL^2$ and define flux quantum as,

$$\Phi_0 = \frac{hc}{e} \Rightarrow \Phi = n\Phi_0 \tag{31}$$

This gives a quantization condition for the total flux in units of flux quantum.

3 Path Integrals for a Scalar Field Theory

1. The vacuum persistent amplitude is given by:

$$Z[J, J^*] = {}_J\langle 0|0\rangle_J = \int D\phi D\phi^* e^{i\int d^4x(\mathcal{L} - J\phi^* - J^*\phi)} \tag{32}$$

In imaginary space-time, by changing $t \to -i\tau$,

$$Z[J, J^*]_E = \left({}_J\langle 0|0\rangle_J\right)_E = \int D\phi D\phi^* e^{-\int d\tau d^3x(\mathcal{L}_E + J\phi^* + J^*\phi)} \tag{33}$$

where

$$\mathcal{L}_E = (\partial_0\phi)^* (\partial_0\phi) + (\nabla\phi)^* (\nabla\phi) + m^2\phi^*\phi \tag{34}$$

2. Using the method discussed in class, we want to expand the field into the classical solution term and fluctuation term:

$$\phi = \phi_0 + \psi \tag{35}$$

where ϕ_0 is the classical solution. In Minkowski space-time, the Lagrangian is expanded as:

$$\mathcal{L} = \mathcal{L}(\phi_0) + \mathcal{L}(\psi) - \left(\partial^2 \phi_0^* + m^2 \phi_0^* + J^*\right)\psi - \left(\partial^2 \phi_0 + m^2 \phi_0 + J\right)\psi^* \tag{36}$$

where

$$\mathcal{L}(\phi_0) = \partial_\mu \phi_0^* \partial^\mu \phi_0 - m^2 \phi_0^* \phi_0 - J\phi_0^* - J^* \phi_0, \quad \mathcal{L}(\psi) = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi \tag{37}$$

Let us choose the classical solution ϕ_0 to make the linear term in ψ vanish, which requires

$$\partial^2 \phi_0 + m^2 \phi_0 + J = 0, \quad \partial^2 \phi_0^* + m^2 \phi_0^* + J^* = 0 \tag{38}$$

Define the Green Function as

$$\left(\partial^2 + m^2\right) G(x - x') = \delta(x - x') \tag{39}$$

The solution of the classical field is,

$$\phi_0(x) = -\int d^4x' G(x - x') J(x') \tag{40}$$

Plugging this back into the original Lagrangian and the path integral, one gets

$$Z[J, J^*] = Z[0] e^{i \int d^4x \int d^4x' J^*(x') G(x - x') J(x)} \tag{41}$$

where

$$Z[0] = \int D\psi D\psi^* e^{-i \int d^4x \psi^* \left(\partial^2 + m^2\right) \psi} \tag{42}$$

Next in the Euclidean space-time, expand the Lagrangian as:

$$\mathcal{L}_E = \mathcal{L}_E(\phi_0) + \mathcal{L}_E(\psi) + \left(-\partial_\mu^2 \phi_0^* + m^2 \phi_0^* + J^*\right)\psi + \left(-\partial_\mu^2 \phi_0 + m^2 \phi_0 + J\right)\psi^* \tag{43}$$

where

$$\mathcal{L}_E(\phi_0) = \partial_\mu \phi_0^* \partial_\mu \phi_0 + m^2 \phi_0^* \phi_0 + J\phi_0^* + J^* \phi_0, \quad \mathcal{L}_E(\psi) = \partial_\mu \psi^* \partial_\mu \psi + m^2 \psi^* \psi \tag{44}$$

The equations for the classical field should satisfy:

$$-\partial_\mu^2 \phi_0^* + m^2 \phi_0^* + J^* = 0, \quad -\partial_\mu^2 \phi_0 + m^2 \phi_0 + J = 0 \tag{45}$$

Define the Euclidean Green Function as,

$$\left(-\partial_\mu^2 + m^2\right) G_E(x - x') = \delta(x - x') \tag{46}$$

The classical solution is given by:

$$\phi_0(x) = -\int d^4x' G_E(x-x')J(x') \tag{47}$$

The path-integral becomes,

$$Z_E[J,J^*] = Z_E[0]e^{\int d^4x \int d^4x' J^*(x') G_E(x-x') J(x)} \tag{48}$$

where

$$Z_E[0] = \int D\psi D\psi^* e^{-\int d^4x \psi^* \left(-\partial_\mu^2 + m^2\right)\psi} \tag{49}$$

3. Correlation functions are compute by taking the function derivative with respect to the sources. In this problem I would like to use the Minkowski Space-Time:

$$\begin{aligned} G_2^*(x-x') &= \frac{1}{Z[0]}\left(\frac{\delta^2 Z[J]}{\delta(-iJ(x))\delta(-iJ^*(x'))}\right)_{J,J^*=0} = -iG_M(x-x') \\ G_2(x-x') &= \frac{1}{Z[0]}\left(\frac{\delta^2 Z[J]}{\delta(-iJ^*(x))\delta(-iJ(x'))}\right)_{J,J^*=0} = -iG_M(x-x') \\ G_2'(x-x') &= \langle 0|T\phi(x)\phi(x')|0\rangle = G_2'^*(x-x') = \langle 0|T\phi^*(x)\phi^*(x')|0\rangle = 0 \end{aligned} \tag{50}$$

For the last two term, one could compute the functional derivatives in the same way as above to arrive at this result, or one could realize that the combinations $\phi^*\phi^*$ and $\phi\phi$ do not respect the $U(1)$ symmetry of the system and should vanish.
4. Recall Eq.(45), the differential equation for Euclidean Space, this gives the Green Function as,

$$G_E(x-x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{i(x-x')\cdot k}}{k^2+m^2} \tag{51}$$

Use Schwinger Transformation that,

$$\frac{1}{k^2+m^2} = \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\alpha\left(k^2+m^2\right)} d\alpha \tag{52}$$

The important fact is now the k^2+m^2 is in the exponential and one can complete the square, shift the momentum integration and change into a Gaussian integral. The remaining integral yields a Bessel function (details are explicitly in the lecture notes). The result is: (to plug in $D=4$ in the lecture notes)

$$G_E(x-x') = \frac{1}{(2\pi)^2}\left(\frac{m}{|x-x'|}\right) K_1\left(m|x-x'|\right) \tag{53}$$

On the other hand, the Minkowski space-time Green Function is give by $t \to -i\tau$, from the lecture notes we obtain

$$G_M(x-x') = \frac{i}{(2\pi)^2}\left(\frac{m}{\sqrt{-s^2}}\right) K_1\left(m\sqrt{-s^2}\right) \tag{54}$$

where

$$s^2 = \Delta_\mu x \Delta^\mu x \tag{55}$$

The asymptotic behavior of the Bessel functions can be obtained from the lecture notes. The Euclidean Green Function is, the long range behavior is,

$$G_E(x - x') \approx \frac{\sqrt{\pi/2} m^2 e^{-m|x-x'|}}{(2\pi)^2 (m|x - x'|)^{\frac{3}{2}}} \tag{56}$$

while the short range behavior is

$$G_E(x - x') \approx \frac{1}{4\pi^2 |x - x'|^2} \tag{57}$$

Hence, there is power law decay at short distances and exponential decay at longer distances.
For Minkowski space, there are two regimes. Space like $s^2 < 0$ and time like $s^2 > 0$. For space like separations, these correspond to the regime where $-s^2 > 0$, so the asymptotic behavior at long and short distances is similar as the Euclidean space version. For timelike separations, $-s^2 < 0$ and so the Green Function's long range behavior is

$$G_M(x - x') \approx \frac{\sqrt{\pi/2} m^2 e^{im\sqrt{s^2}}}{(2\pi)^2 (m\sqrt{s^2})^{\frac{3}{2}}} \tag{58}$$

it is exponential vibration. While for short range, it is

$$G_E(x - x') \approx \frac{1}{4\pi^2 s^2} \tag{59}$$

still has power law decay.
5. The four point functions can be computed by taking the desired functional derivatives with respect to the sources. This yields familiar expressions that one could obtain via Wick's theorem. Taking all possible combinations between ϕ^* and ϕ, the four point field have two point combinations as:

$$\begin{aligned} \phi^*(x_1)\phi^*(x_2)\phi(x_3)\phi(x_4) &\to [\phi^*(x_1)\phi(x_3)]\,[\phi^*(x_2)\phi(x_4)] \\ &+ [\phi^*(x_1)\phi(x_4)]\,[\phi^*(x_2)\phi(x_3)] \end{aligned} \tag{60}$$

Therefore, the four point Green Function is,

$$\begin{aligned} &\langle 0|T\phi^*(x_1)\phi^*(x_2)\phi(x_3)\phi(x_4)|0\rangle = G(x_1, x_2, x_3, x_4) \\ &= G(x_3 - x_1)G(x_4 - x_2) + G(x_4 - x_1)G(x_3 - x_2) \end{aligned} \tag{61}$$

and

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \langle 0|T\phi^*(x_1)\phi^*(x_2)\phi^*(x_3)\phi^*(x_4)|0\rangle = 0 \tag{62}$$

the other two terms are just similar with the first 4-point Green Function:

$$\langle 0|T\phi^*(x_1)\phi(x_2)\phi^*(x_3)\phi(x_4)|0\rangle = G(x_1, x_3, x_2, x_4)$$
$$\langle 0|T\phi^*(x_1)\phi(x_2)\phi(x_3)\phi^*(x_4)|0\rangle = G(x_1, x_4, x_3, x_2) \tag{63}$$

The rule, from performing the functional derivatives explicitly is that only contractions (two point functions) appear for pairs $\langle\phi^*\phi\rangle$. Other contractions vanish. This is expected since the other combinations do not preserve the symmetry of the system. The non-vanishing three 4-point Green Functions' relation is just to permute the position of $x_1, x_{2,3}, x_4$ because the position change of * symble just affect the contration combination of the field operators.