Phys 582 – General Field Theory
Problem Set No.3 Solutions

Di Zhou

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Before proceeding, I want to mention that I use Eq.(n) to denote the nth equation in my solution, while eq.(n) for the nth equation in Professor Fradkin's problem set.

1 Spin waves in a quantum Heisenberg antiferromagnet

1. Let us denote e to be even, and o to be odd, in the Heisenberg representation, the time derivative of the operators are,

$$i\partial_0 \hat{A}(x) = [\hat{A}(x), \hat{H}] \tag{1}$$

This, as an important conclusion could be derived below. In the Heisenberg Representation, the operator $\hat{A}$ is,

$$\hat{A} = e^{i\hat{H}t} \hat{A}_0 e^{-i\hat{H}t} \tag{2}$$

Where $\hat{A}_0$ is the Schrodinger Representation operator. Taking partial derivative on the time,

$$i\partial_0 \hat{A} = e^{i\hat{H}t}\left(\hat{H}\hat{A}_0 - \hat{A}_0\hat{H}\right)e^{-i\hat{H}t} = \left(\hat{H}e^{i\hat{H}t}\hat{A}_0 e^{-i\hat{H}t} - e^{i\hat{H}t}\hat{A}_0 e^{-i\hat{H}t}\hat{H}\right) \tag{3}$$

This is what we want to get for Eq.(1). Note the Pauli Spin Operator commutation relations. For operators on different sites, they commute while when they are on the same site; one has the commutation relations,

$$[\hat{S}^+, \hat{S}^-] = 2\hat{S}_3; \quad [\hat{S}_3, \hat{S}^+] = \hat{S}^+, \quad [\hat{S}_3, \hat{S}^-] = -\hat{S}^- \tag{4}$$

We want to use $\hat{S}^+, \hat{S}^-, \hat{S}_3$ instead of $\hat{S}_1, \hat{S}_2, \hat{S}_3$ in the Heisenberg Hamiltonian,

$$\begin{aligned} 2\hat{S}_k(j)\hat{S}_k(j+1) &= 2\hat{S}_1(j)\hat{S}_1(j+1) + 2\hat{S}_2(j)\hat{S}_2(j+1) + 2\hat{S}_3(j)\hat{S}_3(j+1) \\ &= \hat{S}^-(j)\hat{S}^+(j+1) + \hat{S}^+(j)\hat{S}^-(j+1) + 2\hat{S}_3(j)\hat{S}_3(j+1) \end{aligned} \tag{5}$$

Thus the Heisenberg Hamiltonian is reduced into the form as,

$$\hat{H} = \frac{J}{4}\sum_j \left(\hat{S}^-(j)\hat{S}^+(j+1) + \hat{S}^+(j)\hat{S}^-(j+1) + 2\hat{S}_3(j)\hat{S}_3(j+1)\right)$$
$$+\frac{J}{4}\sum_j \left(\hat{S}^-(j)\hat{S}^+(j-1) + \hat{S}^+(j)\hat{S}^-(j-1) + 2\hat{S}_3(j)\hat{S}_3(j-1)\right) \quad (6)$$

If we want to obtain the equations of motion, first we have to investigate the value of:

$$[\hat{S}^+(j), \hat{H}] = \frac{J}{2}\left[\hat{S}_3(j)\left(\hat{S}^+(j+1) + \hat{S}^+(j-1)\right) - \hat{S}^+(j)\left(\hat{S}_3(j+1) + \hat{S}_3(j-1)\right)\right] \quad (7)$$

$$[\hat{S}^-(j), \hat{H}] = \frac{J}{2}\left[\hat{S}^-(j)\left(\hat{S}_3(j+1) + \hat{S}_3(j-1)\right) - \hat{S}_3(j)\left(\hat{S}^-(j+1) + \hat{S}^-(j-1)\right)\right] \quad (8)$$

$$[\hat{S}_3(j), \hat{H}] = \frac{J}{4}\left[\hat{S}^+(j)\left(\hat{S}^-(j+1) + \hat{S}^-(j-1)\right) - \hat{S}^-(j)\left(\hat{S}^+(j+1) + \hat{S}^+(j-1)\right)\right] \quad (9)$$

Therefore the equations of motion are,

$$\begin{aligned} i\partial_0 S^+(j) &= \frac{J}{2}\left[\hat{S}_3(j)\left(\hat{S}^+(j+1) + \hat{S}^+(j-1)\right) - \hat{S}^+(j)\left(\hat{S}_3(j+1) + \hat{S}_3(j-1)\right)\right] \\ i\partial_0 S^-(j) &= \frac{J}{2}\left[\hat{S}^-(j)\left(\hat{S}_3(j+1) + \hat{S}_3(j-1)\right) - \hat{S}_3(j)\left(\hat{S}^-(j+1) + \hat{S}^-(j-1)\right)\right] \\ i\partial_0 S_3(j) &= \frac{J}{4}\left[\hat{S}^+(j)\left(\hat{S}^-(j+1) + \hat{S}^-(j-1)\right) - \hat{S}^-(j)\left(\hat{S}^+(j+1) + \hat{S}^+(j-1)\right)\right] \end{aligned} \quad (10)$$

for both j odd and even. These equations nonlinear because on the r.h.s. the operators are bilinear.

2. For even sites, from eq.(8) of the problem set,

$$\begin{aligned} \hat{S}^+(j)|n\rangle &= \sqrt{2S}\left[1 - \frac{\hat{n}(j)}{2S}\right]^{\frac{1}{2}}\sqrt{n}|n-1\rangle = \left[2S\left(1 - \frac{n-1}{2S}\right)n\right]^{\frac{1}{2}}|n-1\rangle \\ \hat{S}^-(j)|n\rangle &= \sqrt{2S}\hat{a}^+(j)\left(1 - \frac{n}{2S}\right)^{\frac{1}{2}}|n\rangle = \left[2S\,(n+1)\left(1 - \frac{n}{2S}\right)\right]^{\frac{1}{2}}|n+1\rangle \end{aligned} \quad (11)$$

For odd sites,

$$\begin{aligned} \hat{S}^-(j)|n\rangle &= \sqrt{2S}\left[1 - \frac{\hat{n}(j)}{2S}\right]^{\frac{1}{2}}\sqrt{n}|n-1\rangle = \left[2S\left(1 - \frac{n-1}{2S}\right)n\right]^{\frac{1}{2}}|n-1\rangle \\ \hat{S}^+(j)|n\rangle &= \sqrt{2S}\hat{b}^+(j)\left(1 - \frac{n}{2S}\right)^{\frac{1}{2}}|n\rangle = \left[2S\,(n+1)\left(1 - \frac{n}{2S}\right)\right]^{\frac{1}{2}}|n+1\rangle \end{aligned} \quad (12)$$

Which is the same as that of eq.(5) in the problem set.

3. Let us write the Hamiltonian as the even part and the odd part,

$$\hat{H} = \frac{J}{2}\sum_{j_o+j_e}\left(\hat{S}^-(j)\hat{S}^+(j+1) + \hat{S}^+(j)\hat{S}^-(j+1) + 2\hat{S}_3(j)\hat{S}_3(j+1)\right) \quad (13)$$

Use the Hamiltonian with ladder operators S^+, S^-, S_z, and to apply the relations in eq.(8) and eq.(9), we can write the even and odd Hamiltonian parts as follows,

$$
\begin{aligned}
\hat{H}_o &= J\sum_{j_o}\left(S\left[1-\frac{n(j)}{2S}\right]^{\frac{1}{2}}\left[1-\frac{n(j+1)}{2S}\right]^{\frac{1}{2}} a(j+1)b(j)\right) \\
&+ J\sum_{j_o}\left(Sa^+(j+1)b^+(j)\left[1-\frac{n(j)}{2S}\right]^{\frac{1}{2}}\left[1-\frac{n(j+1)}{2S}\right]^{\frac{1}{2}}\right) \\
&- J\sum_{j_o}\left([S-n(j)][S-n(j+1)]\right) \qquad (14)
\end{aligned}
$$

H_e has the same form except for the interchange between a and b,

$$
\begin{aligned}
\hat{H}_e &= J\sum_{j_e}\left(S\left[1-\frac{n(j)}{2S}\right]^{\frac{1}{2}}\left[1-\frac{n(j+1)}{2S}\right]^{\frac{1}{2}} a(j)b(j+1)\right) \\
&+ J\sum_{j_e}\left(Sa^+(j)b^+(j+1)\left[1-\frac{n(j)}{2S}\right]^{\frac{1}{2}}\left[1-\frac{n(j+1)}{2S}\right]^{\frac{1}{2}}\right) \\
&- J\sum_{j_e}\left([S-n(j)][S-n(j+1)]\right) \qquad (15)
\end{aligned}
$$

4. Let us investigate Hamiltonian H_o for example, the first part is already quadratic for operators $a(j+1)b(j)$, and higher orders in the bracket can be dropped; the second is the same, since $a^+(j+1)b^+(j)$ is quadratic; the third should be approximated to the first order in the bracket, because S^2 is the 0th-order operator, and $Sn(j)$ is the quadratic operator. Therefore the approximated operator should be written as,

$$
\begin{aligned}
\hat{H}_o &= JS\sum_{j_o}\big(a(j+1)b(j)+a^+(j+1)b^+(j)+b^+(j)b(j)+a^+(j+1)a(j+1)-S\big) \\
\hat{H}_e &= JS\sum_{j_e}\big(a(j)b(j+1)+a^+(j)b^+(j+1)+a^+(j)a(j)+b^+(j+1)b(j+1)-S\big) \qquad (16)
\end{aligned}
$$

It is obvious that these two Hamiltonians are quadratic in operators a and b. And the total Hamiltonian is,

$$
\hat{H} = \hat{H}_e + \hat{H}_o \qquad (17)
$$

5. Use the equations of motion for Heisenberg Representaton, Eq.(10), and taking the classical limit again with $S \to \infty$ I keep things to leading non-vanishing order in S^1, and drop S^0 orders and lower, i.e., to drop terms of $\frac{n}{S}$ and $\frac{n^2}{S^2}$. For j even(corresponds to $\partial_0 a(j)$ terms) and j odd(corresponds to

$\partial_0 b(j)$ terms),

$$
\begin{aligned}
i\partial_0 a(j) &= \frac{JS}{2}\left[b^+(j-1)+b^+(j+1)+2a(j)\right] \\
-i\partial_0 a^+(j) &= \frac{JS}{2}\left[b(j-1)+b(j+1)+2a^+(j)\right] \\
i\partial_0 b(j) &= \frac{JS}{2}\left[a^+(j-1)+a^+(j+1)+2b(j)\right] \\
-i\partial_0 b^+(j) &= \frac{JS}{2}\left[a(j-1)+a(j+1)+2b^+(j)\right] \qquad (18)
\end{aligned}
$$

and,

$$
\begin{aligned}
i\partial_0 n(j) &= \frac{JS}{2}\left\{b^+(j)\left[a^+(j+1)+a^+(j-1)\right]-b(j)\left[a(j+1)+a(j-1))\right]\right\} \\
-i\partial_0 n(j) &= \frac{JS}{2}\left\{a(j)\left[b(j+1)+b(j-1)\right]-a^+(j)\left[b^+(j+1)+b^+(j-1))\right]\right\}
\end{aligned}
\qquad (19)
$$

The four equations in Eq.(17) is the new equations of motion for a, b operators, and, we can find it becomes linear in the quadratic approximation of the Hamiltonian. Eq.(18), however, needs more discussions. Since by definition $n(j) = S - S_3$ for even sites and $n(j) = S_3 - S$ for odd sites, the time derivative of $n(j)$ corresponds to the time derivative of S_3 operator. What is more, if we look back to the Heisenberg Hamiltonian, Eq.(13), we find that the Ground State seems to be, $S_3(j) = \frac{1}{2}, \quad S_3(j+1) = -\frac{1}{2}$. (Although this is not the exact Ground State of antiferromagnetic system, it is a good approximation for your physical picture. The real ground state should include zero-point energy.) This is called the antiferromagnetic case: each spin align in the opposite direction. Therefore, the physical purpose to use Holstein-Primakoff Transformation, i.e., the equations in the problem set eq.(4)-(9), is to carriy out the Ground State background and consider excitations upon it. Now it is clear that the quadratic approximation is to consider the excitations of this system, and the operator S_3 should be small to S for low energy excitations. Thus we should expect $\partial_0 n(j) \to 0$ and hence, we reach the conclusion from Eq.(18) that,

$$
\begin{aligned}
b^+(j)\left[a^+(j+1)+a^+(j-1)\right] &= b(j)\left[a(j+1)+a(j-1))\right] \\
a(j)\left[b(j+1)+b(j-1)\right] &= a^+(j)\left[b^+(j+1)+b^+(j-1))\right]
\end{aligned}
\qquad (20)
$$

6. The lattice constant is assumed to be $a = 1$ so the Fourier Transformation in the problem set eq.(10) use the lattice constant as 1. The Fourier Transformation in eq.(10), however, is the transform for lattice constant $A = 2a = 2$ because it transforms over only the even or odd lattice sites. Therefore we can prove the identity that,

$$
\frac{2}{N}\sum_q e^{iqj} = \frac{2}{N}\sum_{\substack{q\,=\,2\pi\frac{n}{N} \\ n\text{even/odd}}}^{-\frac{1}{2}N\le n\le\frac{1}{2}N} e^{iqj} = \delta_{j,0} \qquad (21)
$$

Let us find what $a(j)$, $b(j)$ is in terms of $a(q)$, $b(q)$.

$$\begin{aligned}
\sqrt{\frac{2}{N}}\sum_q e^{-iqn_e}\hat{a}(q) &= \frac{2}{N}\sum_{j_e}\sum_q e^{-iqn_e+iqj_e}\hat{a}(j_e) = \sum_{j_e}\delta_{n_e,j_e}\hat{a}(j_e) = \hat{a}(n_e) \\
\sqrt{\frac{2}{N}}\sum_q e^{+iqn_o}\hat{b}(q) &= \frac{2}{N}\sum_{j_o}\sum_q e^{+iqn_o-iqj_o}\hat{b}(j_o) = \sum_{j_o}\delta_{n_o,j_o}\hat{b}(j_o) = \hat{b}(n_o)
\end{aligned} \tag{22}$$

Similarly

$$\begin{aligned}
\sqrt{\frac{2}{N}}\sum_q e^{+iqn_e}\hat{a}^+(q) &= \frac{2}{N}\sum_{j_e}\sum_q e^{+iqn_e-iqj_e}\hat{a}^+(j_e) = \sum_{j_e}\delta_{n_e,j_e}\hat{a}^+(j_e) = \hat{a}^+(n_e) \\
\sqrt{\frac{2}{N}}\sum_q e^{-iqn_o}\hat{b}^+(q) &= \frac{2}{N}\sum_{j_o}\sum_q e^{-iqn_o+iqj_o}\hat{b}^+(j_o) = \sum_{j_o}\delta_{n_o,j_o}\hat{b}^+(j_o) = \hat{b}^+(n_o)
\end{aligned} \tag{23}$$

What is more, the momentum-space operators also obey bosonic statistics,

$$\begin{aligned}
\left[\hat{a}(q'),\hat{a}^+(q)\right] &= \frac{2}{N}\sum_{jj'} e^{+iq'j'-iqj}\left[\hat{a}(j'),\hat{a}^+(j)\right] = \frac{2}{N}\sum_{jj'} e^{+iq'j'-iqj}\delta_{jj'} = \delta_{qq'} \\
\left[\hat{b}(q'),\hat{b}^+(q)\right] &= \frac{2}{N}\sum_{jj'} e^{-iq'j'+iqj}\left[\hat{b}(j'),\hat{b}^+(j)\right] = \frac{2}{N}\sum_{jj'} e^{-iq'j'+iqj}\delta_{jj'} = \delta_{qq'} \\
\left[\hat{a}(q'),\hat{b}(q)\right] &= \frac{2}{N}\sum_{jj'} e^{iq'j'-iqj}\left[\hat{a}(j'),\hat{b}(j)\right] = 0 \\
\left[\hat{a}^+(q'),\hat{b}^+(q)\right] &= \frac{2}{N}\sum_{jj'} e^{-iq'j'+iqj}\left[\hat{a}^+(j'),\hat{b}^+(j)\right] = 0
\end{aligned} \tag{24}$$

Plug these Fourier Transformations back into our quadratic Hamiltonian, Eq.(16) and Eq.(17),

$$\begin{aligned}
\hat{H} = -JS^2N \;&+\; JS\sum_{j_eqq'}\frac{2}{N}\left[e^{-iqj+iq'j-iq}\hat{a}(q)\hat{b}(q') + e^{+iqj-iq'j+iq}\hat{a}^+(q)\hat{b}^+(q')\right] \\
&+\; JS\sum_{j_eqq'}\frac{2}{N}\left[e^{-iqj+iq'j}\hat{b}^+(q)\hat{b}(q') + e^{+iq(j+1)-iq'(j+1)}\hat{a}^+(q)\hat{a}(q')\right] \\
&+\; JS\sum_{j_oqq'}\frac{2}{N}\left[e^{-iqj+iq'j+iq'}\hat{a}(q)\hat{b}(q') + e^{iqj-iq'j-iq'}\hat{a}^+(q)\hat{b}^+(q')\right] \\
&+\; JS\sum_{j_oqq'}\frac{2}{N}\left[e^{+iqj-iq'j}\hat{a}^+(q)\hat{a}(q') + e^{-iq(j+1)+iq'(j+1)}\hat{b}^+(q)\hat{b}(q')\right]
\end{aligned} \tag{25}$$

Using the identity Eq.(21), and denote $-JS^2N = H_0$ this can be reduced into,

$$\hat{H} = H_0 \, + \, JS\sum_q \left[e^{-iq}\hat{a}(q)\hat{b}(q) + e^{+iq}\hat{a}^+(q)\hat{b}^+(q) + \hat{b}^+(q)\hat{b}(q) + \hat{a}^+(q)\hat{a}(q)\right]$$
$$+ \, JS\sum_q \left[e^{+iq}\hat{a}(q)\hat{b}(q) + e^{-iq}\hat{a}^+(q)\hat{b}^+(q) + \hat{a}^+(q)\hat{a}(q) + \hat{b}^+(q)\hat{b}(q)\right] \tag{26}$$

Where we use the notation in Eq.(21) that,

$$\sum_q = \sum_{\substack{q\,=\,2\pi\frac{n}{N} \\ n\text{even/odd}}}^{-\frac{1}{2}N\le n\le\frac{1}{2}N} \tag{27}$$

Use the identity $e^{iq} + e^{-iq} = 2\cos q$, the Hamiltonian reduces into,

$$\hat{H} = H_0 + 2JS\sum_q \left[\hat{a}(q)\hat{b}(q)\cos q + \hat{a}^+(q)\hat{b}^+(q)\cos q + \hat{b}^+(q)\hat{b}(q) + \hat{a}^+(q)\hat{a}(q)\right] \tag{28}$$

Now it's the time to diagonalize the Hamiltonian by using Bogoliubov Transformation. Following eq.(11) in the problem set, we find

$$\begin{aligned}
\hat{c}^+(q) &= \cosh(\theta(q))\hat{a}^+(q) + \sinh(\theta(q))\hat{b}(q) \\
\hat{d}^+(q) &= \cosh(\theta(q))\hat{b}^+(q) + \sinh(\theta(q))\hat{a}(q) \\
\hat{c}(q) &= \cosh(\theta(q))\hat{a}(q) + \sinh(\theta(q))\hat{b}^+(q) \\
\hat{d}(q) &= \cosh(\theta(q))\hat{b}(q) + \sinh(\theta(q))\hat{a}^+(q)
\end{aligned} \tag{29}$$

NOTE: This is a little bit problematic, since no evidence shows the coefficient $\cosh(\theta(q)), \sinh(\theta(q))$ are REAL! However, we can prove these two coefficients are real numbers by setting them to be complex numbers, and prove the imaginary parts are 0. For simplicity I will only give a self-consistent argument on the real coefficients later. Now let us use $\hat{c}(q), \hat{d}(q)$ for $\hat{a}(q), \hat{b}(q)$ operators:

$$\begin{aligned}
\hat{a}^+(q) &= \cosh(\theta(q))\hat{c}^+(q) - \sinh(\theta(q))\hat{d}(q) \\
\hat{b}^+(q) &= \cosh(\theta(q))\hat{d}^+(q) - \sinh(\theta(q))\hat{c}(q) \\
\hat{a}(q) &= \cosh(\theta(q))\hat{c}(q) - \sinh(\theta(q))\hat{d}^+(q) \\
\hat{b}(q) &= \cosh(\theta(q))\hat{d}(q) - \sinh(\theta(q))\hat{c}^+(q)
\end{aligned} \tag{30}$$

And plug this relation into Eq.(28), we get the following terms' coefficients,

$$\begin{aligned}
\hat{c}^+(q)\hat{c}(q) &: \sinh^2(\theta(q)) + \cosh^2(\theta(q)) - 2\sinh(\theta(q))\cosh(\theta(q))\cos(q) \\
\hat{d}^+(q)\hat{d}(q) &: \sinh^2(\theta(q)) + \cosh^2(\theta(q)) - 2\sinh(\theta(q))\cosh(\theta(q))\cos(q) \\
\hat{c}(q)\hat{d}(q) &: \left(\cosh^2(\theta(q)) + \sinh^2(\theta(q))\right)\cos(q) - 2\sinh(\theta(q))\cosh(\theta(q)) \\
\hat{c}^+(q)\hat{d}^+(q) &: \left(\cosh^2(\theta(q)) + \sinh^2(\theta(q))\right)\cos(q) - 2\sinh(\theta(q))\cosh(\theta(q))
\end{aligned} \tag{31}$$

and an extra term comes from $cc^+ = 1 + c^+c$ and $dd^+ = 1 + d^+d$:

$$2\sinh^2(\theta) - 2\sinh(\theta)\cosh(\theta)\cos(q) \tag{32}$$

Using the identities,

$$2\sinh(x)\cosh(x) = \sinh(2x); \quad \cosh^2(x) + \sinh^2(x) = \cosh(2x) \tag{33}$$

To diagonalize the Hamiltonian, we want the off-diagonal terms to be zero, that is, the third and fourth equations of Eq.(31) to be zero, we obtain:

$$\tanh(2\theta) = \cos(q) \Rightarrow \theta(q) = -\frac{1}{2}\ln\left(\tanh\left(\frac{1}{2}q\right)\right) \tag{34}$$

The extra term becomes,

$$|\sin(q)| - 1 \tag{35}$$

With this choice of $\theta(q)$, the diagonalized Hamiltonian is,

$$\hat{H} = -JS^2N + 2JS\sum_q \left[|\sin(q)|\left(\hat{c}^+(q)\hat{c}(q) + \hat{d}^+(q)\hat{d}(q)\right) + (|\sin(q)| - 1)\right] \tag{36}$$

Using Eq.(27), we can find

$$\sum_q = N\int_{-\pi}^{\pi}\frac{dq}{4\pi} = N\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{dq}{2\pi} \tag{37}$$

Since

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{dq}{2\pi}(|\sin(q)| - 1) = \frac{1}{\pi} - \frac{1}{2} \tag{38}$$

Therefore the Hamiltonian,

$$\hat{H} = -JS(S+\gamma)N + 2JSN\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\frac{dq}{2\pi}|\sin(q)|\left(\hat{c}^+(q)\hat{c}(q) + \hat{d}^+(q)\hat{d}(q)\right) \tag{39}$$

where

$$\gamma = 1 - \frac{2}{\pi} < 1 \tag{40}$$

In conclusion, the explicit expression for both $\theta(q)$ and $\omega(q)$ are:

$$\tanh(2\theta) = \cos(q), \quad \omega(q) = |\sin(q)| \tag{41}$$

7. The ground state of this approximation, is, when $n(c) = n(d) = 0$. They are states where $\hat{c}(q)|0\rangle = 0$ and $\hat{d}(q)|0\rangle = 0$. Note that these are not the original degrees of freedom, the a and b. They are some collective configuration of the

original spins.
8. The single particle states are $\hat{c}^{+}|0\rangle$ and $\hat{d}^{+}|0\rangle$.These states have the same energy which can be seen by simply finding the expectation value of the energy for each state. The one-quasiparticle excitation energy is, by the Hamiltonian in Eq.(39),

$$E(q) = 2JSN|\sin(q)| \tag{42}$$

This is of course zero for $q = n\pi$, but since $-\frac{\pi}{2} \leq q \leq \frac{\pi}{2}$, $n = 0$ is the only choice. At this $q = 0$ point, $|\sin(q)| \to |q|$. Hence, there's a linear dispersion relation $E(q) \to 2NJS|q|$. Since $E(q) \to 2NJSq$ for positive q and $E(q) \to -2NJSq$ for negative q, there are two branches of dispersion relations. The wave velocity is given by $v_s = d\omega/dq = 2NJS$ near the $q = 0$ point.

2 Two-Component Complex Scalar Field

1. (a) The canonical momentum Π_a conjugate to the field ϕ_a is,

$$\Pi_a = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a} = \frac{1}{2}\left(\partial^0 \phi_a\right)^*, \quad \Pi_a^* = \frac{\delta \mathcal{L}}{\delta \partial_0 \phi_a^*} = \frac{1}{2}\partial^0 \phi_a \tag{43}$$

(b) The Hamiltonian is defined as,

$$H = \Pi_a \partial_0 \phi_a + \Pi_a^* \partial_0 \phi_a^* - \mathcal{L} = 2\Pi_a \Pi_a^* + \frac{1}{2}\left|\nabla \phi_a\right|^2 + \frac{1}{2} m_0^2 |\phi_a|^2 \tag{44}$$

(c) The total momentum P^i is given by,

$$\begin{aligned} P^i = \int d^3x\, T^{0i} &= \int d^3x \left[\frac{\delta \mathcal{L}}{\delta \partial^0 \phi_a}\partial^i \phi_a + \frac{\delta \mathcal{L}}{\delta \partial^0 \phi_a^*}\partial^i \phi_a^*\right] \\ &= \int d^3x \left(\Pi_a \partial^i \phi_a + \Pi_a^* \partial^i \phi_a^*\right) \end{aligned} \tag{45}$$

2. The transformations between two components are:

$$\phi'_a(x) = U_{ab}\phi_b(x), \quad \phi'^*_a(x) = U_{ab}^{-1}\phi_b^*(x) \tag{46}$$

Since

$$\left(e^{i\vec{\theta}\cdot\vec{\sigma}_{ab}}\right)^{+} = e^{-i\theta^i \sigma_{ab}^{i+}} = e^{-i\theta^i \sigma_{ab}^i} \tag{47}$$

The variation of the fields are,

$$\begin{aligned} \delta\phi_a(x) = i\left(\vec{\theta}\cdot\vec{\sigma}_{ab}\right)\phi_b &\Rightarrow \delta\phi_a^{+}(x) = -i\phi_b^{+}(x)\left(\vec{\theta}\cdot\vec{\sigma}_{ab}\right) \\ &\Rightarrow \delta\phi_a^*(x) = -i\left(\vec{\theta}\cdot\vec{\sigma}_{ba}\right)\phi_b^*(x) \end{aligned} \tag{48}$$

Where in the last step we take transverse at the l.h.s. and r.h.s. of the equation. Note σ_{ab} changes into σ_{ba} now. Here, there are the three Pauli matrices, $i =$

1, 2, 3. In addition, there is also the 2×2 identity matrix, $i = 0$. Hence, $i = 0, 1, 2, 3$ are the four generators, and there should be four conserved charges corresponding to these four generators. The variation of the action is,

$$\delta S = \int d^d x \, \delta \mathcal{L} = \int d^d x \left(\frac{\delta \mathcal{L}}{\delta \phi_a} \delta \phi_a + \frac{\delta \mathcal{L}}{\delta \left(\partial_\mu \phi_a \right)} \delta \left(\partial_\mu \phi_a \right) + c.c. \right) \tag{49}$$

Using the equations of motion,

$$\frac{\delta \mathcal{L}}{\delta \phi_a} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \left(\partial_\mu \phi_a \right)}, \quad \frac{\delta \mathcal{L}}{\delta \phi_a^*} = \partial_\mu \frac{\delta \mathcal{L}}{\delta \left(\partial_\mu \phi_a^* \right)} \tag{50}$$

The action reduces into the boundary term,

$$\delta S = \int d^d x \, \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \left(\partial_\mu \phi_a \right)} \delta \phi_a + c.c. \right) = \frac{1}{2} \int d^d x \, \theta_i \partial_\mu \left(i \partial^\mu \phi_a^* \left(\sigma_{ab}^i \right) \phi_b + c.c. \right) \tag{51}$$

NOTE: the a b contractions above represents ALL contractions over indices $a = 1, b = 2$, $a = 2, b = 1$, $a = 1, b = 1$, $a = 2, b = 2$. To make the variation of action always zero, we want the current to be conserved, i.e.,

$$\partial^\mu j_\mu = 0, \; j_\mu^i = \frac{i}{2} \left(\partial_\mu \phi_a^* \sigma_{ab}^i \phi_b - \partial_\mu \phi_a \sigma_{ba}^i \phi_b^* \right) = \frac{i}{2} \left(\partial_\mu \phi_a^* \sigma_{ab}^i \phi_b - \partial_\mu \phi_b \sigma_{ab}^i \phi_a^* \right) \tag{52}$$

There are $i = 0, 1, 2, 3$ conserved currents for each generator as expected. There are 4 constant of motion. The constant of motion, i.e., the space integral of zero component of the 4-current, is

$$Q^i = \int d^3 x \, j_{\mu=0}^i = i \int d^3 x \, \left(\Pi_a \sigma_{ab}^i \phi_b - \Pi_b^* \sigma_{ab}^i \phi_a^* \right) \tag{53}$$

3. By imposing canonical commutation relations,

$$[\hat{\phi}_a(x), \hat{\Pi}_b(y)] = i\hbar \delta(x - y) \delta_{ab} \tag{54}$$

and,

$$[\hat{\phi}_a^+(x), \hat{\Pi}_b^+(y)] = i\hbar \delta(x - y) \delta_{ab} \tag{55}$$

The quantum mechanical Hamiltonian is written as:

$$\hat{H} = \int d^3 x \left(2 \hat{\Pi}_a^+ \hat{\Pi}_a + \frac{1}{2} \nabla \hat{\phi}_a^+ \nabla \hat{\phi}_a + \frac{m_0^2}{2} \hat{\phi}_a^+ \hat{\phi}_a \right) \tag{56}$$

The momentum operator is written as:

$$\begin{aligned} \hat{P}^i &= \int d^3 x \, \hat{T}^{0i} = \int d^3 x \, \left(\frac{\delta \mathcal{L}}{\delta \partial_0 \hat{\phi}_a} \partial^i \hat{\phi}_a + \frac{\delta \mathcal{L}}{\delta \partial_0 \hat{\phi}_a^+} \partial^i \hat{\phi}_a^+ \right) \\ &= \frac{1}{2} \int d^3 x \left(\partial_0 \hat{\phi}_a^+ \partial^i \hat{\phi}_a + \partial_0 \hat{\phi}_a \partial^i \hat{\phi}_a^+ \right) = \int d^3 x \left(\hat{\Pi}_a \partial^i \hat{\phi}_a + \hat{\Pi}_a^+ \partial^i \hat{\phi}_a^+ \right) \end{aligned} \tag{57}$$

4. In the quantum theory the operator A is conserved if its time derivative obeys:

$$i\frac{d\hat{A}_H}{dt} = [\hat{A}_H, \hat{H}] = 0 \Rightarrow i\frac{d\hat{Q}^i}{dt} = [\hat{Q}^i, \hat{H}] = 0 \tag{58}$$

where we have set $\hbar = 1$. Then the constants of motion, derived in the classical field theory, must commute with the Hamiltonian in the quantum theory. The classical constant of motion, defined in Eq.(53), just need to be changed into the operator form for the quantum theory as below:

$$\hat{Q}^i = i\int d^3x \left(\hat{\Pi}_a \sigma^i_{ab}\hat{\phi}_b - \hat{\Pi}^+_b \sigma^i_{ab}\hat{\phi}^+_a\right) \tag{59}$$

Let us make sure the commutator of $\hat{Q}$ and $\hat{H}$ is zero. To prove this we just need to prove that the charge density and Hamiltonian density commute:

$$\left[\hat{q}, \hat{h}\right] = \left[\left(\hat{\Pi}_a \sigma^i_{ab}\hat{\phi}_b - \hat{\Pi}^+_b \sigma^i_{ab}\hat{\phi}^+_a\right), \left(2\hat{\Pi}^+_a\hat{\Pi}_a + \frac{1}{2}\nabla\hat{\phi}^+_a\nabla\hat{\phi}_a + \frac{m_0^2}{2}\hat{\phi}^+_a\hat{\phi}_a\right)\right] \tag{60}$$

By using the conclusions that,

$$\begin{aligned}
\left[\hat{\phi}_b(x), \hat{\Pi}^+_a(y)\hat{\Pi}_a(y)\right] &= i\hbar\hat{\Pi}^+_a(y)\delta_{ab}\delta(x-y) \\
\left[\hat{\phi}^+_b(y), \hat{\Pi}^+_a(x)\hat{\Pi}_a(x)\right] &= i\hbar\hat{\Pi}_a(x)\delta_{ab}\delta(x-y) \\
\left[\hat{\phi}^+_a(y)\hat{\phi}_a(y), \hat{\Pi}_a(x)\right] &= i\hbar\hat{\phi}^+_a(y)\delta_{ab}\delta(x-y) \\
\left[\hat{\phi}^+_a(y)\hat{\phi}_a(y), \hat{\Pi}^+_a(x)\right] &= i\hbar\hat{\phi}_a(y)\delta_{ab}\delta(x-y)
\end{aligned} \tag{61}$$

We can find the charge and Hamiltonian operators indeed commute. Therefore the charges constitute a representation of the generators of the Lie group in the Hilbert space of the theory. One safely can write the quantum mechanical generators of transformations of operator $\hat{Q}^i$ as Eq.(59).

5. In the Heisenberg Representation, the equations of motion is

$$\frac{\partial}{\partial t}\hat{A} = -i[\hat{A}, \hat{H}] \tag{62}$$

For the field and momentum operators, and using the conclusions from Eq.(61),

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\phi}_a(x) = -i[\hat{\phi}_a(x), \hat{H}] &= -i\int d^3y \left[\hat{\phi}_a(x), \left(2\hat{\Pi}^+_b\hat{\Pi}_b + \frac{1}{2}\nabla\hat{\phi}^+_b\nabla\hat{\phi}_b + \frac{m_0^2}{2}\hat{\phi}^+_b\hat{\phi}_b\right)\right] \\
&= 2\int d^3y\delta(x-y)\delta_{ab}\hat{\Pi}^+_b(y) = 2\hat{\Pi}^+_a(x)
\end{aligned} \tag{63}$$

with this relation we can also easily find: $\partial_0\hat{\phi}^+_a(x) = 2\hat{\Pi}_a(x)$.

The other operator's time detivative, $\hat{\Pi}_a(x)$, is given as:

$$\begin{aligned}
\frac{\partial}{\partial t}\hat{\Pi}_a(x) = -i[\hat{\Pi}_a(x), \hat{H}] &= -i\int d^3y \left[\hat{\Pi}_a(x), \left(2\hat{\Pi}^+_b\hat{\Pi}_b + \frac{1}{2}\nabla\hat{\phi}^+_b\nabla\hat{\phi}_b + \frac{m_0^2}{2}\hat{\phi}^+_b\hat{\phi}_b\right)\right] \\
&= -i\int d^3y \left[\hat{\Pi}_a(x), \left(\frac{1}{2}\nabla\hat{\phi}^+_b\nabla\hat{\phi}_b + \frac{m_0^2}{2}\hat{\phi}^+_b\hat{\phi}_b\right)\right]
\end{aligned} \tag{64}$$

Here we are left to treat two terms:

$$\left[\hat{\Pi}_a(x), \nabla\hat{\phi}_b^+(y)\nabla\hat{\phi}_b(y)\right], \quad \left[\hat{\Pi}_a(x), \hat{\phi}_b^+(y)\hat{\phi}_b(y)\right] \tag{65}$$

The second term is easy to obtain, i.e., $-i\hbar\delta(x-y)\delta_{ab}\hat{\phi}_b^+(y)$, by using Eq.(61). The first term, however, need more treatment. NOTE: the space gradient on $\hat{\phi}_b(y)$ is actually ∇_y. Therefore we have

$$\nabla_y\hat{\Pi}_a(x) = 0 \tag{66}$$

Thus for the first term in Eq.(65) we have,

$$\begin{aligned}
\left[\hat{\Pi}_a(x), \nabla_y\hat{\phi}_b^+(y)\nabla_y\hat{\phi}_b(y)\right] &= +\hat{\Pi}_a(x)\nabla_y\hat{\phi}_b^+(y)\nabla_y\hat{\phi}_b(y) - \nabla_y\hat{\phi}_b^+(y)\nabla_y\hat{\phi}_b(y)\hat{\Pi}_a(x) \\
&= -\nabla_y\hat{\Pi}_a(x)\hat{\phi}_b^+(y)\nabla_y\hat{\phi}_b(y) - \hat{\Pi}_a(x)\hat{\phi}_b^+(y)\nabla_y^2\hat{\phi}_b(y) \\
&\quad +\hat{\phi}_b^+(y)\nabla_y\hat{\phi}_b(y)\nabla_y\hat{\Pi}_a(x) + \hat{\phi}_b^+(y)\nabla_y^2\hat{\phi}_b(y)\hat{\Pi}_a(x) \\
&\Rightarrow -\hat{\Pi}_a(x)\hat{\phi}_b^+(y)\nabla_y^2\hat{\phi}_b(y) + \hat{\phi}_b^+(y)\nabla_y^2\hat{\phi}_b(y)\hat{\Pi}_a(x) \\
&= -\hat{\phi}_b^+(y)\hat{\Pi}_a(x)\nabla_y^2\hat{\phi}_b(y) + \hat{\phi}_b^+(y)\nabla_y^2\hat{\phi}_b(y)\hat{\Pi}_a(x)
\end{aligned} \tag{67}$$

Where we have dropped the boundary terms above. Again use the relation Eq.(66),

$$\begin{aligned}
-\hat{\phi}_b^+(y)\hat{\Pi}_a(x)\nabla_y^2\hat{\phi}_b(y) + \hat{\phi}_b^+(y)\nabla_y^2\hat{\phi}_b(y)\hat{\Pi}_a(x) &= \hat{\phi}_b^+(y)\nabla_y^2\left[\hat{\phi}_b(y), \hat{\Pi}_a(x)\right] \\
&= i\hbar\hat{\phi}_a^+(y)\nabla_y^2\delta(x-y) \\
&= i\hbar\delta(x-y)\nabla_y^2\hat{\phi}_a^+(y)
\end{aligned} \tag{68}$$

Finally set $\hbar = 1$, we get the equation of motion for the momentum operator that,

$$\frac{\partial}{\partial t}\hat{\Pi}_a(x) = \frac{1}{2}\left(\nabla^2 - m_0^2\right)\hat{\phi}_a^+(x) \tag{69}$$

Combining both Eq.(63) and Eq.(69), we obtain the famous Klein-Gordon Equation:

$$\partial_0^2\hat{\phi}_a(x) = \left(\nabla^2 - m_0^2\right)\hat{\phi}_a(x) \tag{70}$$

where the spectrum is a little different from before,

$$\omega = \sqrt{k^2 + m^2} \tag{71}$$

6. Now since the scalar field is complex, we can divide the field into two parts:

$$\begin{aligned}
\hat{\phi}_1(x) &= \frac{1}{\sqrt{2}}\left[\hat{\phi}_{1r}(x) + i\hat{\phi}_{1i}(x)\right] \\
\hat{\phi}_2(x) &= \frac{1}{\sqrt{2}}\left[\hat{\phi}_{2r}(x) + i\hat{\phi}_{2i}(x)\right]
\end{aligned} \tag{72}$$

And the momentum operator decomposes into

$$\begin{aligned}\hat{\Pi}_1(x) &= \frac{1}{\sqrt{2}}\left[\hat{\Pi}_{1r}(x)+i\hat{\Pi}_{1i}(x)\right] = \frac{1}{2\sqrt{2}}\left[\partial_0\hat{\phi}_{1r}(x)-i\partial_0\hat{\phi}_{1i}(x)\right] \\ \hat{\Pi}_2(x) &= \frac{1}{\sqrt{2}}\left[\hat{\Pi}_{2r}(x)+i\hat{\Pi}_{2i}(x)\right] = \frac{1}{2\sqrt{2}}\left[\partial_0\hat{\phi}_{2r}(x)-i\partial_0\hat{\phi}_{2i}(x)\right] \end{aligned} \tag{73}$$

where r denotes real, and i denotes imaginary. Use the Fourier Transformation,

$$\hat{\phi}_{a(r,i)}(x) = \int \frac{d^3k}{(2\pi)^3}\hat{\phi}_{a(r,i)}(\vec{k},x_0)e^{i\vec{k}\cdot\vec{x}} \tag{74}$$

Since $\hat{\phi}_{a(r,i)}(x)$ are real and Hermitian field, $\hat{\phi}_{a(r,i)}(\vec{k},x_0)$ must satisfy:

$$\hat{\phi}^+_{a(r,i)}(\vec{k},x_0) = \hat{\phi}_{a(r,i)}(-\vec{k},x_0) \tag{75}$$

Now define the Forward Moving wave function component, and the Backward Moving wave function component as follows:

$$\hat{\phi}^+_{a(r,i)}(\vec{k},x_0) = \hat{\phi}_{a(r,i)}(-\vec{k},x_0) \tag{76}$$

Therefore let us define $\hat{a}$, $\hat{b}$ operators as follows:

$$\begin{aligned}\hat{\phi}_{ar}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left[\hat{a}_a(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}}+\hat{a}^+_a(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right] \\ \hat{\phi}_{ai}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left[\hat{b}_a(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}}+\hat{b}^+_a(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right]\end{aligned} \tag{77}$$

$$\begin{aligned}\hat{\Pi}_{ar}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\frac{\omega(\vec{k})}{2}\left[-\hat{b}_a(\vec{k})e^{-i\omega x+i\vec{k}\cdot\vec{x}}+\hat{b}^+_a(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right] \\ \hat{\Pi}_{ai}(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\frac{\omega(\vec{k})}{2}\left[-\hat{a}_a(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}}+\hat{a}^+_a(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right]\end{aligned} \tag{78}$$

However, this set of operators cannot diagonalize the Hamiltonian. Let us define $\hat{a}_{1a}(\vec{k}),\hat{a}_{2a}(\vec{k}),\hat{a}_{2a}(\vec{k}),\hat{a}^+_{2a}(\vec{k})$ as follows:

$$\begin{aligned}\hat{a}_{1a}(\vec{k}) &= \frac{1}{\sqrt{2}}\left[\hat{a}_a(\vec{k})+i\hat{b}_a(\vec{k})\right], \quad \hat{a}^+_{1a}(\vec{k}) = \frac{1}{\sqrt{2}}\left[\hat{a}^+_a(\vec{k})-i\hat{b}^+_a(\vec{k})\right] \\ \hat{a}_{2a}(\vec{k}) &= \frac{1}{\sqrt{2}}\left[\hat{a}_a(\vec{k})-i\hat{b}_a(\vec{k})\right], \quad \hat{a}^+_{2a}(\vec{k}) = \frac{1}{\sqrt{2}}\left[\hat{a}^+_a(\vec{k})+i\hat{b}^+_a(\vec{k})\right]\end{aligned} \tag{79}$$

From the above definition we can find $\hat{a}_{2a}(\vec{k}),\hat{a}^+_{2a}(\vec{k})$ as follows:

$$\begin{aligned}\hat{\phi}_a(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left[\hat{a}_{1a}(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}}+\hat{a}^+_{2a}(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right] \\ \hat{\phi}^+_a(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\left[\hat{a}_{2a}(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}}+\hat{a}^+_{1a}(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right]\end{aligned} \tag{80}$$

and

$$\hat{\Pi}^{+}_{a}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\frac{\omega(\vec{k})}{2}\left[-i\hat{a}_{1a}(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}} + i\hat{a}^{+}_{2a}(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right]$$
$$\hat{\Pi}_{a}(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(\vec{k})}\frac{\omega(\vec{k})}{2}\left[-i\hat{a}_{2a}(\vec{k})e^{-i\omega x_0+i\vec{k}\cdot\vec{x}} + i\hat{a}^{+}_{1a}(\vec{k})e^{i\omega x_0-i\vec{k}\cdot\vec{x}}\right] \tag{81}$$

Use the quantization condition of Eq.(54) and Eq.(55), we find the commutation relation for the defined new operators are:

$$\left[\hat{a}_{ia}(\vec{p}), \hat{a}^{+}_{jb}(\vec{q})\right] = (2\pi)^3 4\omega(\vec{k})\delta(\vec{p}-\vec{q})\delta_{ij}\delta_{ab}, \quad [\hat{a}_{ia}(\vec{p}), \hat{a}_{jb}(\vec{q})] = 0 \tag{82}$$

In conclusion, there are 4 species of creation and annihilation operators we need: $\hat{a}_{11}(\vec{k}), \hat{a}^{+}_{11}(\vec{k}), \hat{a}_{12}(\vec{k}), \hat{a}^{+}_{12}(\vec{k}), \hat{a}_{21}(\vec{k}), \hat{a}^{+}_{21}(\vec{k}), \hat{a}_{22}(\vec{k}), \hat{a}^{+}_{22}(\vec{k})$.
7. The SU(2) generator is just the 4-constant of motions Q^i:

$$\hat{Q}^i = i\int d^3x \left(\hat{\Pi}_a \sigma^i_{ab}\hat{\phi}_b - \hat{\Pi}^{+}_b \sigma^i_{ab}\hat{\phi}^{+}_a\right) \quad \text{Together with Eq.(80),(81).} \tag{83}$$

The above Eq.(83) has four components, two are diagonal, two are off-diagonal. The diagonal terms are:

$$+\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(p)}\frac{\omega(p)}{2}\int \frac{d^3q}{(2\pi)^3 2\omega(q)}\left[\hat{a}_{2a}(\vec{p})\hat{a}^{+}_{2b}(\vec{q}) + \hat{a}^{+}_{2a}(\vec{p})\hat{a}_{2b}(\vec{q})\right]e^{i(-p_\mu+q_\mu)x^\mu}$$
$$-\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(p)}\frac{\omega(p)}{2}\int \frac{d^3q}{(2\pi)^3 2\omega(q)}\left[\hat{a}^{+}_{1a}(\vec{p})\hat{a}_{1b}(\vec{q}) + \hat{a}_{1a}(\vec{p})\hat{a}^{+}_{1b}(\vec{q})\right]e^{i(+p_\mu-q_\mu)x^\mu} \tag{84}$$

Taking the integral over $\int d^3x$ first and use

$$\int d^3x e^{i(p_\mu-q_\mu)x^\mu} = e^{i(\omega(p)-\omega(q))t}\int d^3x e^{-i(\vec{p}-\vec{q})\cdot\vec{x}} = (2\pi)^3\delta(\vec{p}-\vec{q}) \tag{85}$$

The diagonal terms reduces into

$$\int \frac{\sigma^k_{ab}d^3p}{(2\pi)^3 8\omega(p)}\left[\hat{a}_{2a}(\vec{p})\hat{a}^{+}_{2b}(\vec{p}) + \hat{a}^{+}_{2a}(\vec{p})\hat{a}_{2b}(\vec{p}) - \hat{a}_{1a}(\vec{p})\hat{a}^{+}_{1b}(\vec{p}) - \hat{a}^{+}_{1a}(\vec{p})\hat{a}_{1b}(\vec{p})\right] \tag{86}$$

Next, the off-diagonal terms are:

$$+\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(p)}\frac{\omega(p)}{2}\int \frac{d^3q}{(2\pi)^3 2\omega(q)}\left[\hat{a}_{2a}(\vec{p})\hat{a}_{1b}(\vec{q}) + \hat{a}^{+}_{2a}(\vec{p})\hat{a}^{+}_{1b}(\vec{q})\right]e^{i(-p_\mu-q_\mu)x^\mu}$$
$$-\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(p)}\frac{\omega(p)}{2}\int \frac{d^3q}{(2\pi)^3 2\omega(q)}\left[\hat{a}_{1a}(\vec{p})\hat{a}_{2b}(\vec{q}) + \hat{a}^{+}_{1a}(\vec{p})\hat{a}^{+}_{2b}(\vec{q})\right]e^{i(+p_\mu+q_\mu)x^\mu} \tag{87}$$

Taking the integral over $\int d^3x$ again and use

$$\int d^3x e^{i(p_\mu+q_\mu)x^\mu} = e^{i(\omega(p)+\omega(q))t}\int d^3x e^{-i(\vec{p}+\vec{q})\cdot\vec{x}} = (2\pi)^3\delta(\vec{p}+\vec{q})e^{2i\omega t} \tag{88}$$

Now the off-diagonal term reduces into:

$$\int \frac{\sigma^k_{ab} d^3p}{(2\pi)^3 8\omega(p)} \left[\hat{a}_{2a}(\vec{p})\hat{a}_{1b}(-\vec{p}) + \hat{a}^+_{2a}(\vec{p})\hat{a}^+_{1b}(-\vec{p})\right] e^{-2i\omega t}$$
$$-\int \frac{\sigma^k_{ab} d^3p}{(2\pi)^3 8\omega(p)} \left[\hat{a}_{1a}(\vec{p})\hat{a}_{2b}(-\vec{p}) + \hat{a}^+_{1a}(\vec{p})\hat{a}^+_{2b}(-\vec{p})\right] e^{+2i\omega t} \quad (89)$$

This term is, a little problematic, however, if we look back to Eq.(59), the charge operator, and take the Hermitian conjugation on that operator, we can find:

$$\hat{Q}^{i+} = \hat{Q}^i \quad (90)$$

Where we have used $\sigma^{k+}_{ab} = \sigma^k_{ab}$. The diagonal part of $\hat{Q}^i$ is of course Hermitian, therefore, applying Eq.(90) on the off-diagonal part, Eq.(89), one finds the off-diagonal term equals:

$$\int \frac{\sigma^k_{ba} d^3p}{(2\pi)^3 8\omega(p)} \left[\hat{a}_{2a}(\vec{p})\hat{a}_{1b}(-\vec{p}) + \hat{a}^+_{2a}(\vec{p})\hat{a}^+_{1b}(-\vec{p})\right] e^{+2i\omega t}$$
$$-\int \frac{\sigma^k_{ba} d^3p}{(2\pi)^3 8\omega(p)} \left[\hat{a}_{1a}(\vec{p})\hat{a}_{2b}(-\vec{p}) + \hat{a}^+_{1a}(\vec{p})\hat{a}^+_{2b}(-\vec{p})\right] e^{-2i\omega t} \quad (91)$$

Let us take the summation on Eq.(91)+Eq.(89), and note:

$$\int d^3p \left[\hat{a}_{2a}(\vec{p})\sigma^k_{ab}\hat{a}_{1b}(-\vec{p}) - \hat{a}_{1a}(\vec{p})\sigma^k_{ba}\hat{a}_{2b}(-\vec{p})\right] e^{+2i\omega t}$$
$$= \int d^3p \left[\hat{a}_{2a}(\vec{p})\sigma^k_{ab}\hat{a}_{1b}(-\vec{p}) - \hat{a}_{1b}(\vec{p})\sigma^k_{ab}\hat{a}_{2a}(-\vec{p})\right] e^{+2i\omega t} \quad (92)$$

Using

$$\int d^3p f(\vec{p}) = \int d^3(-q) f(-\vec{q}) = \int d^3q f(-\vec{q}) \quad (93)$$

Eq.(92) must be zero. In addition, other 3 terms in the off-diagonal term also vanish with the same arguments.
Finally we reach the conclusion that the charge operator only has diagonal terms,

$$\hat{Q}^k = \frac{\sigma^k_{ab}}{2} \int \frac{d^3p}{(2\pi)^3 4\omega(p)} \left[\left\{\hat{a}_{2a}(\vec{p}), \hat{a}^+_{2b}(\vec{p})\right\} - \left\{\hat{a}_{1a}(\vec{p}), \hat{a}^+_{1b}(\vec{p})\right\}\right] \quad (94)$$

As noted before, there are two diagonal operators in the $\{a, b\}$ basis where the Hamiltonian is also diagonal. Hence, the good quantum numbers are labelled by the eigenvalues of the Hamiltonian, σ^0_{ab} and σ^3_{ab}. Notice that for σ^0_{ab}, $\hat{a}^+$ particles and $\hat{b}^+$ have eigenvalues with opposite sign in $\hat{Q}^0$ while for $\hat{H}$ and σ^3_{ab} in $\hat{Q}^3$ they have the same sign. In this sense, one has particles with the same energy and spin, but opposite charge given by σ^0_{ab}.

8. The Hamiltonian can be written in terms of creation annihilation operators, the diagonal terms are:

$$
\begin{aligned}
&\frac{1}{2}\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \int \frac{d^3q}{(2\pi)^3 2\omega(\vec{q})} \times e^{i(q_\mu - p_\mu)x^\mu} \times \\
&\left\{\omega(\vec{p})\omega(\vec{q})\hat{a}_{1a}(\vec{p})\hat{a}^+_{1a}(\vec{q}) + m_0^2 \hat{a}_{2a}(\vec{p})\hat{a}^+_{2a}(\vec{q}) + \vec{p}\cdot\vec{q}\hat{a}_{2a}(\vec{p})\hat{a}^+_{2a}(\vec{q})\right\} \\
&+\frac{1}{2}\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \int \frac{d^3q}{(2\pi)^3 2\omega(\vec{q})} \times e^{i(p_\mu - q_\mu)x^\mu} \times \\
&\left\{\omega(\vec{p})\omega(\vec{q})\hat{a}^+_{2a}(\vec{p})\hat{a}_{2a}(\vec{q}) + m_0^2 \hat{a}^+_{1a}(\vec{p})\hat{a}_{1a}(\vec{q}) + \vec{p}\cdot\vec{q}\hat{a}^+_{1a}(\vec{p})\hat{a}^+_{1a}(\vec{q})\right\}
\end{aligned} \tag{95}
$$

Again, taking integral on $\int d^3x$ and integral on $\int d^3q$, and use $\omega^2(\vec{k}) = (k^2 + m_0^2)$ we get the reduced diagonal term as:

$$
\int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\omega(\vec{p})}{4} \left(\hat{a}_{1a}\hat{a}^+_{1a} + \hat{a}_{2a}\hat{a}^+_{2a} + \hat{a}^+_{1a}\hat{a}_{1a} + \hat{a}^+_{2a}\hat{a}_{2a}\right) \tag{96}
$$

The off-diagonal terms, however, vanishes. For example, let us consider $\hat{a}_{1a}\hat{a}_{2a}$ term's coefficient:

$$
\begin{aligned}
&\frac{1}{2}\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \int \frac{d^3q}{(2\pi)^3 2\omega(\vec{q})} \times e^{i(-q_\mu - p_\mu)x^\mu} \times \\
&\left\{-\omega(\vec{p})\omega(\vec{q})\hat{a}_{1a}(\vec{p})\hat{a}_{2a}(\vec{q}) + m_0^2 \hat{a}_{2a}(\vec{p})\hat{a}_{1a}(\vec{q}) - \vec{p}\cdot\vec{q}\hat{a}_{2a}(\vec{p})\hat{a}_{1a}(\vec{q})\right\}
\end{aligned} \tag{97}
$$

Again take integral over $\int d^3x$, however, due to the exponential term in Eq.(97), which gives $\delta(\vec{p}+\vec{q})$, $\vec{q} = -\vec{p}$, the coefficient gives $-\omega^2 + (p^2 + m_0^2) = 0$.
Thus the Hamiltonian could be written as:

$$
\hat{H} = \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\omega(\vec{p})}{4} \left(\hat{a}_{1a}\hat{a}^+_{1a} + \hat{a}_{2a}\hat{a}^+_{2a} + \hat{a}^+_{1a}\hat{a}_{1a} + \hat{a}^+_{2a}\hat{a}_{2a}\right) \tag{98}
$$

Using the normal-order of the Hamiltonian,

$$
: \hat{H} := \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\omega(\vec{p})}{2} \left(\hat{a}^+_{1a}\hat{a}_{1a} + \hat{a}^+_{2a}\hat{a}_{2a}\right), \quad E_0 = \int \omega(p) d^3p \tag{99}
$$

The momentum operator, is also diagonal. To prove this, let us have a look at the off-diagonal terms. For example, the off-diagonal term of $\hat{a}_{1a}(\vec{q})\hat{a}_{2a}(\vec{p})$

$$
\begin{aligned}
&\int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\omega(\vec{p})}{2} \{-\vec{p}\hat{a}_{2a}(\vec{p})\hat{a}_{1a}(-\vec{p}) - \vec{p}\hat{a}_{1a}(\vec{p})\hat{a}_{2a}(-\vec{p}))\} \\
&= \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\omega(\vec{p})}{2} \{-\vec{p}\hat{a}_{2a}(\vec{p})\hat{a}_{1a}(-\vec{p}) + \vec{p}\hat{a}_{1a}(-\vec{p})\hat{a}_{2a}(\vec{p}))\}
\end{aligned} \tag{100}
$$

This means the off-diagonal term must be zero.
The diagonal term is:

$$
\begin{aligned}
&\int d^3x \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\omega(\vec{p})}{2} \int \frac{d^3q}{(2\pi)^3 2\omega(\vec{q})} \times \vec{q} e^{i(q_\mu - p_\mu)x^\mu} \times \\
&\left\{\hat{a}_{2a}(\vec{p})\hat{a}^+_{2a}(\vec{q}) + \hat{a}^+_{1a}(\vec{p})\hat{a}_{1a}(\vec{q})) + \hat{a}_{1a}(\vec{p})\hat{a}^+_{1a}(\vec{q}) + \hat{a}^+_{2a}(\vec{p})\hat{a}_{2a}(\vec{q}))\right\}
\end{aligned} \tag{101}
$$

Thus the momentum operatpr $\hat{P}$ could be written as:

$$\hat{P} = \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\vec{p}}{4} \left(\hat{a}^+_{1a}\hat{a}_{1a} + \hat{a}_{1a}\hat{a}^+_{1a} + \hat{a}^+_{2a}\hat{a}_{2a} + \hat{a}_{2a}\hat{a}^+_{2a} \right) \tag{102}$$

The normal-ordered operator, is,

$$: \hat{P} := \int \frac{d^3p}{(2\pi)^3 2\omega(\vec{p})} \frac{\vec{p}}{2} \left(\hat{a}^+_{1a}\hat{a}_{1a} + \hat{a}^+_{2a}\hat{a}_{2a} \right) \tag{103}$$

The ground state of this system are states where:

$$\hat{a}_{i=(1,2),j=(1,2)} |\text{gnd}\rangle = 0 \tag{104}$$

The generator acts on the ground state is also zero,

$$\hat{Q}^i |0\rangle = 0 \tag{105}$$

9. As mentioned before, the dispersion relation is given by $\omega = \sqrt{k^2 + m_0^2}$ Each particle type has $E(k) = \omega(k)$. This is four fold degenerate. This can be seen directly from the Hamiltonian. There are four terms, each of which have the same energy. The $\hat{a}_{2j}$ particles have charge $+1$ while the $\hat{a}_{1j}$ particles have charge -1. In addition to this, there are two components for each of these. I label them by $j = 1$ and $j = 2$. Hence, states can be labelled by

$$\begin{array}{ll} \hat{a}^+_{21}(k)|0\rangle = |+, 1, k\rangle & \hat{a}^+_{22}(k)|0\rangle = |+, 2, k\rangle \\ \hat{a}^+_{11}(k)|0\rangle = |-, 1, k\rangle & \hat{a}^+_{12}(k)|0\rangle = |-, 2, k\rangle \end{array} \tag{106}$$