Phys 582 – General Field Theory
Problem Set No.2 Solutions

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1 Symmetries and Conservation Laws

1. Under a global transformation, the scalar field changes while the vector field not. The variation of these fields are:

$$\begin{aligned}\delta\phi(x) &= \phi'(x) - \phi(x) = i\theta\phi \\ \delta\phi^*(x) &= \phi'^*(x) - \phi^*(x) = -i\theta\phi^* \\ \delta A_\mu(x) &= A'_\mu(x) - A_\mu(x) = 0\end{aligned} \tag{1}$$

The total variation of the Action is,

$$\delta S = \int d^dx \left(\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi}\delta\partial_\mu\phi + \frac{\delta\mathcal{L}}{\delta\phi^*}\delta\phi^* + \frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^*}\delta\partial_\mu\phi^* + \frac{\delta\mathcal{L}}{\delta A_\mu}\delta A_\mu + \frac{\delta\mathcal{L}}{\delta\partial_\nu A_\mu}\delta\partial_\nu A_\mu \right) \tag{2}$$

In this problem, the variation of vector field is zero, we can obtain the locally conserved current as:

$$\delta S = \int d^dx \left(\left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)} \right] \delta\phi + \partial_\mu \left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\delta\phi \right] + \text{c.c.} \right) \tag{3}$$

The classical equations of motion should vanish in the first bracket above,

$$\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)} = 0, \quad \frac{\delta\mathcal{L}}{\delta\phi^*} - \partial_\mu \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^*\right)} = 0 \tag{4}$$

The above variation of Action is reduced to:

$$\delta S = \int d^dx \left(\partial_\mu \left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\delta\phi + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^*\right)}\delta\phi^* \right] \right) = \theta \int d^dx \left(\partial_\mu j^\mu\right) \tag{5}$$

The partial derivative on θ is zero because of the global transformation. In the above equation we have defined,

$$\theta j^\mu = \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\delta\phi + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^*\right)}\delta\phi^* \tag{6}$$

Now have a look at the exact form of that of the Action:

$$S = \int d^d x \left((\partial^\mu \phi^*) (\partial_\mu \phi) + ieA_\mu (\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi) + (e^2 A^2 - m_0^2)|\phi|^2 - \frac{\lambda}{4!}|\phi|^4 - \frac{1}{4}F^2 \right) \quad (7)$$

The conserved current becomes, $\partial_\mu j^\mu = 0$,

$$j_\mu = [(\partial_\mu \phi^*) - ieA_\mu \phi^*] i\phi - [(\partial_\mu \phi) + ieA_\mu \phi] i\phi^* = i \left[\phi (D_\mu \phi)^* - \phi^* (D_\mu \phi) \right] \quad (8)$$

2. As $\partial_\mu j^\mu = 0$, we get $\int d^d x \partial_\mu j^\mu = 0$. Use the Gauss theorem,

$$\int d^d x \partial_\mu j^\mu = \int_S dS_\mu j^\mu = 0 \quad (9)$$

The surface tends to infinity if the real volume's coordinates $x, y, z \to \infty$. If the current $\vec{J}$ vanishes at $x, y, z \to \infty$, from Eq.(9) we have,

$$\int_{S(t)} dS_0 j^0 = - \int_{S(t)} d\vec{S} \cdot \vec{j} = 0$$

$$\Rightarrow \int_{S(t)} dS_0 j^0 = \left(\int_{V(t+\Delta t)} - \int_{V(t)} \right) dS_0 j^0 = 0 \quad (10)$$

Therefore doesn't depend on time, more explicitly,

$$\int_{V(t+\Delta t)} dS_0 j^0 = \int_{V(t)} dS_0 j^0 \quad (11)$$

Since the surface for a fixed 0th dimension – time, is the volume in the real space, $dS_0 = d^3 x$. Finally we get

$$\int_{V(t)} d^3 x j^0 = Q \quad (12)$$

is independent of time, which is, a constant of motion. From Eq.(8) above, we can obtain the explicit form for this constant of motion that,

$$Q = \int_{V(t)} d^3 x i \left[\phi (D_0 \phi)^* - \phi^* (D_0 \phi) \right] \quad (13)$$

3. We want the Lagrangian to be invariant under this local gauge transformation, that is to make at least $(D_\mu \phi)^* (D_\mu \phi)$ to be invariant under local gauge transformation.

$$D'_\mu \phi' = (\partial_\mu + ieA_\mu + ie\partial_\mu \Lambda) \left(e^{i\theta} \phi \right) = e^{i\theta} D_\mu \phi + (ie\partial_\mu \Lambda + i\partial_\mu \theta) e^{i\theta} \phi \quad (14)$$

The l.h.s to be equal to $e^{i\theta} D_\mu \phi$, the second term must be zero. This gives

$$ie\partial_\mu \Lambda + i\partial_\mu \theta = 0 \Rightarrow \Lambda = -\frac{1}{e}\theta \quad (15)$$

On the other hand, the vector field $F_{\mu\nu}$ keeps invariant, so the requirement above is the only what we need.

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu A_\nu - \partial_\nu A_\mu + (\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\Lambda = F_{\mu\nu} \tag{16}$$

4. Now let us investigate the variation of the Action in the gauge transformation:

$$\begin{aligned} \delta S = & \int d^d x \left(\left[\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \right] \delta \phi + \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} \delta \phi \right] + \text{c.c.} \right) \\ & + \int d^d x \left(\left[\frac{\delta \mathcal{L}}{\delta A_\mu} - \partial_\nu \frac{\delta \mathcal{L}}{\delta (\partial_\nu A_\mu)} \right] \delta A_\mu + \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta (\partial_\nu A_\mu)} \delta A_\mu \right] \right) \end{aligned} \tag{17}$$

Again the first and third term above in the bracket is 0 due to the classical equations of motion. The second and fourth term corresponds to the gauge current. Using $\delta A_\mu = A'_\mu - A_\mu = \partial_\mu \Lambda$ and $\delta \phi = i\theta\phi, \delta\phi^* = -i\theta\phi^*$, we have the second and fourth term in the bracket above as follows:

$$\delta S = \int d^d x \left(\partial_\mu \left[\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi)} i\theta\phi \right] - \partial_\mu \left[\frac{\delta \mathcal{L}}{\delta (\partial_\mu \phi^*)} i\theta\phi^* \right] - \partial_\nu \left[\frac{\delta \mathcal{L}}{\delta (\partial_\nu A_\mu)} \partial_\mu \left(\frac{\theta}{e} \right) \right] \right) \tag{18}$$

Note that, now $\partial_\mu \theta \neq 0$ since it is the gauge transformation. Using $\partial_\mu j^\mu = 0$, and

$$\frac{\delta \mathcal{L}}{\delta (\partial_\nu A_\mu)} = F^{\mu\nu} \tag{19}$$

the above equation is reduced into:

$$\delta S = \int d^d x \left(j^\mu \partial_\mu \theta - \frac{1}{e} \partial_\nu (F^{\mu\nu}) \partial_\mu \theta - \frac{1}{e} F^{\mu\nu} \partial_\nu \partial_\mu \theta \right) \tag{20}$$

Note that, the above third term is antisymmetric with $\mu\nu$ indices in $F^{\mu\nu}$ while symmetric in $\partial_\mu \partial_\nu$. This term must be vanish.

$$\delta S = \int d^d x \left(j^\mu - \frac{1}{e} \partial_\nu F^{\mu\nu} \right) \partial_\mu \theta \tag{21}$$

This term vanishes since the term in parenthesis is zero by the equations of motion which follows from the 3rd term in Eq.(17). The real issue is that the gauge current J^μ, which is defined by the variation of the action with respect to the vector field, is forced by gauge invariance to be $J^\mu = ej^\mu$. Making this equation vanish, we have to set up the relation,

$$\partial_\nu F^{\mu\nu} = ej^\mu \tag{22}$$

This equation above, is known as the Maxwell's Equations. Now let us define the gauge current J^μ, satisfying $\partial_\mu J^\mu = 0$:

$$J^\mu = \partial_\nu F^{\mu\nu} \Rightarrow J^\mu = ej^\mu \tag{23}$$

The difference from j^μ is the charge e. The associated constant of motion is, similar with that of problem 2,

$$Q = \int d^3x \nabla \cdot \vec{E} = \int d^{d-1}x \left(ej^0\right) = e \int_V d^3xi \left[\phi \left(D_0\phi\right)^* - \phi^* \left(D_0\phi\right)\right] \quad (24)$$

5. Now we are going to consider under a space-time transformation, how the system reponses,

$$\delta S = \delta \left(\int d^dx\mathcal{L} \right) = \int \delta d^dx\mathcal{L} + \int d^dx\delta\mathcal{L} \quad (25)$$

Due to a translational transformation of the space-time,

$$x'_\mu = x_\mu + \delta x_\mu \Rightarrow \frac{\partial x'_\mu}{\partial x_\nu} = \frac{\partial}{\partial x_\nu}\left(x_\mu + \delta x_\mu\right) = g^\nu_\mu + \partial^\nu \delta x_\mu \quad (26)$$

The volume element d^4x changes by a Jacobian Determinent:

$$d^4x' = d^4x\, J, \quad J = \left| \det\left(\frac{\partial x'_\mu}{\partial x_j}\right)\right| = \left|\det\left(g^\nu_\mu + \partial^\nu \delta x_\mu\right)\right| \quad (27)$$

The above determinent is, mostly determined by the diagonal terms, g^ν_μ. Although $\partial^\nu \delta x_\mu$ CAN have off-diagonal terms and contribute to the determinent, it is of second-order or even higher. The first-order contribution from $\partial^\nu \delta x_\mu$ is, those diagonal terms of $\partial^\nu \delta x_\mu$. Let us denote the diagonal term of $\partial^\nu \delta x_\mu$ to be $\left(\partial^\nu \delta x_\mu\right)_{nn}$. Therefore we have the approximation that,

$$J = \prod_{n=1}^{4} \left(1 + \left(\partial^\nu \delta x_\mu\right)_{nn}\right) + O\left(\left(\partial^\nu \delta x_\mu\right)^2\right) \approx 1 + \partial^\mu \delta x_\mu \quad (28)$$

This gives,

$$\delta d^4x = \partial^\mu \delta x_\mu d^4x \quad (29)$$

This is what we want to get in the first term of Eq.(29). Next, consider $\delta\mathcal{L}(x, \phi, \partial\phi, A, \partial A)$,

$$\delta\mathcal{L} = \partial_\mu L \delta x^\mu + \left(\frac{\delta\mathcal{L}}{\delta\phi}\delta\phi + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\delta\left(\partial_\mu\phi\right) + c.c.\right) + \frac{\delta\mathcal{L}}{\delta A_\mu}\delta A_\mu + \frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu\right)}\delta\left(\partial_\nu A_\mu\right) \quad (30)$$

The total change of scalar field and vector field are:

$$\begin{aligned} \phi'(x') = \phi'(x+\delta x) = \phi'(x) + \partial_\mu\phi(x)\delta x^\mu &= \phi(x) + \delta\phi(x) + \partial_\mu\phi(x)\delta x^\mu \\ \Rightarrow \delta_T\phi &= \delta\phi(x) + \partial_\mu\phi(x)\delta x^\mu \\ A'_\mu(x') = A'_\mu(x) + \partial_\nu A_\mu(x)\delta x^\nu &= A_\mu(x) + \delta A_\mu(x) + \partial_\nu A_\mu(x)\delta x^\nu \\ \Rightarrow \delta_T A_\mu &= \delta A_\mu(x) + \partial_\nu A_\mu(x)\delta x^\nu \quad (31) \end{aligned}$$

The variation of Lagrangian can be reduced into:

$$\delta\mathcal{L} = \partial_\mu \mathcal{L}\delta x^\mu + \left[\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\delta\phi\right) + c.c. + \left[\frac{\delta\mathcal{L}}{\delta A_\mu} - \partial_\nu\left(\frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu\right)}\right)\right]\delta A_\mu + \partial_\nu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu\right)}\delta A_\mu\right] \quad (32)$$

Using the equations of motion, we further reduce it into:

$$\delta\mathcal{L} = \partial_\mu \mathcal{L}\delta x^\mu + \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\left(\delta_T\phi - \partial_\alpha\phi\delta x^\alpha\right)\right] + c.c. + \partial_\nu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu\right)}\left(\delta_T A_\mu - \partial_\alpha A_\mu\delta x^\alpha\right)\right] \quad (33)$$

Since the translation transform does not affect the real change of the fields, including both scalar and vector, the total variation of them, $\delta_T\phi = 0$, $\delta_T A_\mu = 0$.

$$\delta\mathcal{L} = \partial_\mu \mathcal{L}\delta x^\mu - \partial_\mu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\partial_\alpha\phi\delta x^\alpha\right] + c.c. - \partial_\nu\left[\frac{\delta\mathcal{L}}{\delta\left(\partial_\nu A_\mu\right)}\partial_\alpha A_\mu\delta x^\alpha\right] \quad (34)$$

Since $\delta S = \int d^4x\left(\mathcal{L}\partial^\mu\delta x_\mu + \delta\mathcal{L}\right)$, the first term in the variation of Action can combine with the first term in Eq.(34), resulting:

$$\left(\mathcal{L}\partial^\mu\delta x_\mu + \partial_\mu\mathcal{L}\delta x^\mu\right) = \partial_\mu\left(\mathcal{L}\delta x^\mu\right) = \partial_\mu\left(\mathcal{L}g^\mu_\alpha\delta x^\alpha\right) \quad (35)$$

Therefore we can further reduce the variation of the Action,

$$\delta S = \int d^4x\partial_\mu\left(\mathcal{L}g^\mu_\alpha - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\partial_\alpha\phi - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^*\right)}\partial_\alpha\phi^* - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu A_\nu\right)}\partial_\alpha A_\nu\right)\delta x^\alpha \quad (36)$$

Define the stress tensor as,

$$T^{\mu\nu} = -\mathcal{L}g^{\mu\nu} + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\partial^\nu\phi + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi^*\right)}\partial^\nu\phi^* + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu A_\alpha\right)}\partial^\nu A_\alpha \quad (37)$$

Which satisfies

$$\partial_\mu T^{\mu\nu} = 0 \quad (38)$$

Following the requirement of this problem, to divide the stress tensor into two parts, one dependent both on ϕ, A_μ and the other independent of ϕ, one gets

$$T^{\mu\nu}(A) = \frac{1}{4}g^{\mu\nu}F^2 + F^{\alpha\mu}\partial^\nu A_\alpha$$
$$T^{\mu\nu}(A,\phi) = \left(D^\mu\phi\right)^*\partial^\nu\phi + D^\mu\phi\left(\partial^\nu\phi\right)^* - g^{\mu\nu}\left(\mathcal{L} + \frac{1}{4}F^2\right) \quad (39)$$

These however, seems not so symmetric as we want. Let us use the property that, for an antisymmetric tensor $\partial_\alpha B^{\alpha\mu\nu}$ which is antisymmetric for the first two indices, then $\partial_\mu \partial_\alpha B^{\alpha\mu\nu} = 0$ is always correct. Therefore $\tilde{T}^{\mu\nu} = T^{\mu\nu} + \partial_\alpha B^{\alpha\mu\nu}$ still satisfies $\partial_\mu \tilde{T}^{\mu\nu} = 0$. Looking at the two stress tensors $T^{\mu\nu}(A), T^{\mu\nu}(A,\phi)$, if we can find some $\partial_\alpha B^{\alpha\mu\nu}$, to make them transforms into:

$$\tilde{T}^{\mu\nu}(A) = \frac{1}{4} g^{\mu\nu} F^2 - F^{\mu\alpha}\left(\partial^\nu A^\beta - \partial^\beta A^\nu\right) g_{\alpha\beta} = T^{\mu\nu}(A) + F^{\mu\alpha}\partial^\beta A^\nu g_{\alpha\beta} \tag{40}$$

and,

$$\tilde{T}^{\mu\nu}(A,\phi) = \left(D^\mu\phi\right)^* D^\nu\phi + D^\mu\phi\left(D^\nu\phi\right)^* - g^{\mu\nu}\left(\mathcal{L} - \frac{1}{4}F^2\right) \tag{41}$$

Using Eq.(8) and Eq.(23),

$$\tilde{T}^{\mu\nu}(A,\phi) = T^{\mu\nu}(A,\phi) + ieA^\nu\left(\phi\left(D^\mu\phi\right)^* - \phi^* D^\mu\phi\right) = T^{\mu\nu}(A,\phi) + J^\mu A^\nu \tag{42}$$

By Maxwell's equation, Eq.(22),

$$\tilde{T}^{\mu\nu}(A,\phi) = T^{\mu\nu}(A,\phi) + \partial_\alpha F^{\mu\alpha} A^\nu \tag{43}$$

Combining Eq.(43) and Eq.(40) together, and find

$$\partial_\alpha F^{\mu\alpha} A^\nu + F^{\mu\alpha}\partial_\alpha A^\nu = \partial_\alpha\left(F^{\mu\alpha}A^\nu\right) = \partial_\alpha\left(B^{\mu\alpha\nu}\right) \tag{44}$$

We can find that the newly defined $\tilde{T}^{\mu\nu}$ satisfies $\partial_\mu \tilde{T}^{\mu\nu} = 0$. Finally write out the new stress tensor,

$$\tilde{T}^{\mu\nu} = \left(D^\mu\phi\right)^* D^\nu\phi + D^\mu\phi\left(D^\nu\phi\right)^* - F^{\mu\alpha}F^{\nu\alpha}g_{\alpha\alpha} - g^{\mu\nu}\mathcal{L} \tag{45}$$

which is not only symmetric but also gauge invariant.
6. The tensor $T^{\mu\nu}$ is known as the energy-momentum tensor. The corresponding constant of motion is defined to be the Hamiltonian and Momentum.

$$\begin{aligned} H = \int d^3x \tilde{T}^{00} &= \int d^3x \left(\left(D^0\phi\right)^* D^0\phi + D^0\phi\left(D^0\phi\right)^* - F^{0\alpha}F^{0\alpha}g_{\alpha\alpha} - \mathcal{L}\right) \\ &= \int d^3x \left(\Pi^*\Pi + \left(D_i\phi\right)^*\left(D_i\phi\right) + V(\phi) + \frac{1}{2}\left(E^2 + B^2\right)\right) \end{aligned} \tag{46}$$

In the Hamiltonian above, the first three terms are the scalar field energy, and the fourth is the EM field energy. The Momemtum is,

$$\begin{aligned} P^\mu = \int d^3x \tilde{T}^{0\mu} &= \int d^3x \left(\left(D^0\phi\right)^* D^\mu\phi + D^0\phi\left(D^\mu\phi\right)^* - F^{0\alpha}F^{\mu\alpha}g_{\alpha\alpha}\right) \\ &= \int d^3x \left(\left(D^0\phi\right)^* D^\mu\phi + D^0\phi\left(D^\mu\phi\right)^* + \left(\vec{E}\times\vec{B}\right)^\mu\right) \end{aligned} \tag{47}$$

The first two are the scalar field momentum while the third is the EM field's momentum.
7. The coordinates transforms as below:

$$x'_\mu = x_\mu + \omega_{\mu\nu}x^\nu \Rightarrow \delta x_\mu = \omega_{\mu\nu}x^\nu \tag{48}$$

For a scalar field, note the field itself is independent of the change of the space-time. Therefore the following relation still satisfies:

$$\delta_T\phi = 0 \tag{49}$$

However, this is not correct for that of the vector field. Consider the special case that the space-time only has the Lorentz Transfomation, thus we have the total variation of the vector field as,

$$A'^{\mu} = \lambda^{\mu}_{\nu}A^{\nu} \Rightarrow \delta_T A_{\mu} = \left(\lambda^{\nu}_{\mu} - g^{\nu}_{\mu}\right) A_{\nu} = \omega^{\nu}_{\mu}A_{\nu} \tag{50}$$

This is what the total variation is for the vector field. For other kinds of space-time transformations, no matter how it changes, the total variation of vector field keep the same form as Eq.(50). On the other hand, what we need in use, is δA_{μ} and $\delta\phi$:

$$\begin{aligned} \delta\phi &= \delta_T\phi - \partial_{\alpha}\phi\delta x^{\alpha} = -\partial_{\alpha}\phi\delta x^{\alpha} \\ \delta A_{\mu} &= \delta_T A_{\mu} - \partial_{\alpha}A_{\mu}\delta x^{\alpha} = \omega^{\alpha}_{\mu}A_{\alpha} - \partial_{\alpha}A_{\mu}\delta x^{\alpha} \end{aligned} \tag{51}$$

Now let us proceed to explore the variation of that of the Action,

$$\delta S = \int d^4x \left\{ \partial_{\mu}\left[\mathcal{L}x^{\beta}g^{\mu}_{\alpha} - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}\phi\right)}x^{\beta}\partial_{\alpha}\phi + c.c. + \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}\left(g^{\beta}_{\nu}A_{\alpha} - \partial_{\alpha}A_{\nu}x^{\beta}\right)\right]\omega^{\alpha}_{\beta}\right\} \tag{52}$$

Which could be further reduced into,

$$\delta S = \int d^4x \left\{ \partial_{\mu}\left[\left(\mathcal{L}g^{\mu}_{\alpha} - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}\phi\right)}\partial_{\alpha}\phi + c.c. - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}\partial_{\alpha}A_{\nu}\right) x_{\beta} + \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}g_{\beta\nu}A_{\alpha}\right]\omega^{\alpha\beta}\right\} \tag{53}$$

The variation of Action again should be froced to be 0 for an arbitrary ω^{α}_{β}. This requires the partial derivative to be zero in Eq.(53). However, one should note that since $\omega^{\mu\nu}$ and $\omega^{\nu\mu}$ or NOT independent, the equation below is NON-zero:

$$\partial_{\mu}\left[\left(\mathcal{L}g^{\mu}_{\alpha} - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}\phi\right)}\partial_{\alpha}\phi + c.c. - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}\partial_{\alpha}A_{\nu}\right) x_{\beta} + \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}g_{\beta\nu}A_{\alpha}\right] \neq 0 \tag{54}$$

Becasue another term, with α, β indices exchanged, to be the opposite value of Eq.(54). Thus we can reach the conclusion that,

$$\begin{aligned} &\partial_{\mu}\left[\left(\mathcal{L}g^{\mu}_{\alpha} - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}\phi\right)}\partial_{\alpha}\phi + c.c. - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}\partial_{\alpha}A_{\nu}\right) x_{\beta} + \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}g_{\beta\nu}A_{\alpha}\right] \\ -\, &\partial_{\mu}\left[\left(\mathcal{L}g^{\mu}_{\beta} - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}\phi\right)}\partial_{\beta}\phi + c.c. - \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}\partial_{\beta}A_{\nu}\right) x_{\alpha} + \frac{\delta\mathcal{L}}{\delta\left(\partial_{\mu}A_{\nu}\right)}g_{\alpha\nu}A_{\beta}\right] = 0 \end{aligned} \tag{55}$$

Recall Eq.(37), the definition of stress tensor due to the translational change of the space-time, $T^{\mu\nu}$, we find:
(1) The first bracket in Eq.(53) corresponds to $-T^{\mu\alpha}$, and $-\tilde{T}^{\mu\alpha}$ respectively (because $\partial_\mu B^{\mu\alpha\nu} = 0$ if $B^{\mu\alpha\nu}$ is antisymmetric in the indices μ, α).
(2) The second term is the extra term which cannot be seen in translational change but can be seen in Lorentz transformation. This is because $\delta_T A_\mu \neq 0$ in Lorentz transformation.
Now it is the time to define the conserved tensor $M^{\mu\alpha\beta}$, satisfying $\partial_\mu M^{\mu\alpha\beta} = 0$,

$$\begin{aligned} M^{\mu\alpha\beta} &= \left(\mathcal{L}g^{\mu\alpha} - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\partial^\alpha\phi + c.c. - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu A_\nu\right)}\partial^\alpha A_\nu\right)x^\beta + \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu A_\nu\right)}g^\beta_\nu A^\alpha \\ &- \left(Lg^{\mu\beta} - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu\phi\right)}\partial^\beta\phi + c.c. - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu A_\nu\right)}\partial^\beta A_\nu\right)x^\alpha - \frac{\delta\mathcal{L}}{\delta\left(\partial_\mu A_\nu\right)}g^\alpha_\nu A^\beta \end{aligned} \tag{56}$$

Plugging in the full form of Lagrangian, and use $\tilde{T}$ instead of T, we get

$$\begin{aligned} M^{\mu\alpha\beta} &= T^{\mu\beta}x^\alpha - T^{\mu\alpha}x^\beta - \left(F^{\mu\beta}A^\alpha - F^{\mu\alpha}A^\beta\right) \\ M^{\mu\alpha\beta} &= \tilde{T}^{\mu\beta}x^\alpha - \tilde{T}^{\mu\alpha}x^\beta \end{aligned} \tag{57}$$

The spatial part of the stress tensor shows the spatial angular momentum of the total angular momentum. The second part is the EM vector field angular momentum. The antisymmetricity of $M^{\mu\alpha\beta}$ implies the symmtricity of $\tilde{T}^{\mu\nu}$.
8. (A)In HW-1 we showed that the lowest energy states can be approximated by freezing the amplitude mode $\rho = \rho_0$, now let us consider the quantum fluctuation of the phase $\omega(x)$ and A_μ. The gauge current is just to take the derivatives on the phase term ω,

$$J^\mu = 2e\rho_0^2\left(\partial^\mu\omega + eA^\mu\right) \tag{58}$$

(B) The total energy corresponds to the Hamiltonian, H,

$$E = \int d^3x\left[\rho_0^2\left(\partial_0\omega + eA_0\right)^2 + \rho_0^2\left(\partial_i\omega + eA_i\right)^2 + V(\rho) + \frac{1}{2}\left(E^2 + B^2\right)\right] \tag{59}$$

(C) The linear momentum is,

$$P^i = \int d^3x\left[2\rho_0^2\left(\partial^0\omega + eA^0\right)\left(\partial^i\omega + eA^i\right) + \left(\vec{E}\times\vec{B}\right)^i\right] \tag{60}$$

9. The analytic continuation to imaginary time of this theory is, to use the Wick rotation $t \to -i\tau$, and $\partial_0 \to i\partial_\tau$. However, under the Wick rotation the inner product of ∂_μ and A^μ between Euclidean and Minkowski space-time should be similar with that of the inner product of x_μ and x^μ, this results in

$$\begin{gathered} s_M^2 = t^2 - x^2 - y^2 - z^2, \quad s_E^2 = -t^2 - x^2 - y^2 - z^2 \\ \partial_\mu A^\mu_M = \partial_0 A_0 - \nabla\cdot\vec{A}, \quad \partial_\mu A^\mu_E = i\partial_\tau \cdot iA_0 - \nabla\cdot\vec{A} = -\partial_\tau A_0 - \nabla\cdot\vec{A} \end{gathered} \tag{61}$$

Plug these two relations back into Eq.(59), we get:

$$E_E = \int d^3x \left[\rho_0^2 \left(\partial_0 \omega + eA_0\right)^2 + \rho_0^2 \left(\partial_i \omega + eA_i\right)^2 + V(\rho) + \frac{1}{2}\left(E^2 + B^2\right)\right] \quad (62)$$

This implies that under the Wick rotation, the vector field changes as

$$A_M^\mu = (\Phi, \vec{A}) \to A_E^\mu = (i\Phi, \vec{A}) \quad (63)$$

Now the energy becomes in the Minkowski space,

$$E_M = \int d^3x \left[-\rho_0^2 \left(\partial_0 \omega + eA_0\right)^2 + \rho_0^2 \left(\partial_i \omega + eA_i\right)^2 + V(\rho) + \frac{1}{2}\left(E^2 + B^2\right)\right] \quad (64)$$

And the path-integral becomes in Minkowski space, $(d\tau = idt)$

$$Z = \int D\phi e^{\frac{i}{\hbar}S} = \int D\phi e^{\frac{i}{\hbar}\int d^4x \left(\rho_0^2(\partial_0\omega + eA_0)^2 - \rho_0^2(\partial_i\omega + eA_i)^2 - V(\rho) - \frac{1}{2}\left(E^2 + B^2\right)\right)} \quad (65)$$

In Minkowski space-time, the path-integral's exponential term is the Lagrangian, while in Euclidean Space-time, it is the Hamiltonian. In Euclidean space-time the dimension is 1 less than that in Minkowski space-time, because the temperature is the time-dimention in Minkowski space-time. Thus if we call the dimension in original space-time – the Minkowski space-time to be D, it is $d + 1 = D$ in Euclidean space-time, i.e., the classical statistical system.