Phys 582 – General Field Theory
Problem Set No.1 Solutions

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August 29, 2011

1 The Landau Theory of Phase Transitions as a Classical Field Theory

1. Due to the free energy density ε, the free energy is:

$$F = \int \varepsilon d^3x = \int \left(\frac{1}{2} \left(\nabla \phi(\vec{x}) \right)^2 + U\left(\phi(\vec{x}) \right) \right) d^3x \tag{1}$$

To find the saddle point of this free energy, we set the variation of free energy to be zero:

$$\delta F = \int \left(\frac{\delta \varepsilon}{\delta \nabla \phi} \delta \nabla \phi + \frac{\delta \varepsilon}{\delta \phi} \delta \phi \right) d^3x = \int \left(-\nabla \left(\frac{\delta \varepsilon}{\delta \nabla \phi} \right) + \frac{\delta \varepsilon}{\delta \phi} \right) \delta \phi d^3x \tag{2}$$

Here we have used: (1) $\nabla \delta \phi = \delta \nabla \phi$; (2) the variation vanishes at the boundaries. The saddle-point equation is, to let the variation of ε to be 0, i.e.,

$$-\nabla \left(\frac{\delta \varepsilon}{\delta \nabla \phi} \right) + \frac{\delta \varepsilon}{\delta \phi} = 0 \Rightarrow -\nabla^2 \phi + \frac{\delta U}{\delta \phi} = 0 \tag{3}$$

Plugging in the form of $U(\phi)$ in this problem, we have:

$$-\nabla^2 \phi + m_0^2 \phi + \frac{\lambda_4}{3!} \phi^3 + \frac{\lambda_6}{5!} \phi^5 = 0 \tag{4}$$

2. The constant field ϕ implies that $\nabla \phi = 0$. Eq. (4) reduces into:

$$m_0^2 \phi + \frac{\lambda_4}{3!} \phi^3 + \frac{\lambda_6}{5!} \phi^5 = 0 \Rightarrow \bar{\phi}^2 = 0 \quad \text{or} \quad \bar{\phi}^2 = -10 \frac{\lambda_4}{\lambda_6} \left(1 \mp \sqrt{1 - \frac{6 \mathrm{m}_0^2 \lambda_6}{5 \lambda_4^2}} \right) \tag{5}$$

Since $\lambda_4 < 0, \lambda_6 > 0$ and the field $\bar{\phi}$ is assumed to be a real field, imaginary $\bar{\phi}$ is meaningless. This implies that for $T < T_0$ and $T > T_0$ the number of meaningful solutions is different. What is more, above T_0 also exists a critical transition temperature $T^* > T_0$ that, the "nontrivial free energy" is lower than "trivial free energy": $F(\bar{\phi} \neq 0) < F(\bar{\phi} = 0)$. The system tends to be in the lower free energy state, equavalently speaking, the symmetry breaking state. The critical temperature can be determined by $F(\bar{\phi}_c \neq 0) = F(\bar{\phi} = 0)$. Therefore we need to consider the corresponding free energy density:

$$\varepsilon\left(\bar{\phi}_c \neq 0\right) = \varepsilon(\bar{\phi} = 0) \Rightarrow \left(m_0^2 + \frac{\lambda_4}{12}\bar{\phi}_c^2 + \frac{\lambda_6}{360}\bar{\phi}_c^4\right)\bar{\phi}_c^2 = 0 \tag{6}$$

$\phi_c^2 \neq 0$ due to $T^* > T_0 \Rightarrow m_0^2 > 0$, we can solve the above equation as:

$$m_0^2 + \frac{\lambda_4}{12}\bar{\phi}_c^2 + \frac{\lambda_6}{360}\bar{\phi}_c^4 = 0 \Rightarrow \bar{\phi}_c^2 = 180\left(-\frac{\lambda_4}{12\lambda_6} \pm \sqrt{\frac{\lambda_4^2}{144\lambda_6^2} - \frac{m_0^2}{90\lambda_6}}\right) \tag{7}$$

On the other hand, recall the nontrivial solution in Eq.(5), we have:

$$\bar{\phi}_c^2 = 180\left(-\frac{\lambda_4}{12\lambda_6} \pm \sqrt{\frac{\lambda_4^2}{144\lambda_6^2} - \frac{m_0^2}{90\lambda_6}}\right) = -10\frac{\lambda_4}{\lambda_6}\left(1 \mp \sqrt{1 - \frac{6m_0^2\lambda_6}{5\lambda_4^2}}\right) \tag{8}$$

Let us denote $\frac{m_0^2\lambda_6}{\lambda_4^2} = x$ and NOTE: the above equation corresponds to 4 equations!

$$\frac{1}{2} \pm \sqrt{\frac{9}{4} - \frac{18}{5}x} = \pm\sqrt{1 - \frac{6}{5}x} \quad \text{and} \quad \frac{1}{2} \pm \sqrt{\frac{9}{4} - \frac{18}{5}\mathrm{x}} = \mp\sqrt{1 - \frac{6}{5}\mathrm{x}} \tag{9}$$

Although there are 4 equations, the Physical solution (i.e., real field solution and lower free energy requirements) is only one:

$$x = \frac{5}{8} \Rightarrow T^* = \frac{5}{8}\frac{\lambda_4^2}{a\lambda_6} + T_0 \tag{10}$$

At the transition temperature $T = T^* + \epsilon \to T = T^* - \epsilon$, the mean-field solution switches from $\bar{\phi}^2 = 0 \to \bar{\phi}^2 = -\frac{15\lambda_4}{\lambda_6}$: this is the incontinuous phase transition, the first-order phase transition.

Up to now, we still havn't decided which $\bar{\phi}^2$ of the two solutions in Eq.(5) should be the real ground state. In the two states, the mean-field free energy is given by:

$$\varepsilon = \frac{-\lambda_4^3}{\lambda_6^2}\left(\frac{5}{9} + \frac{5}{3}\frac{m_0^2\lambda_6}{\lambda_4^2} \mp \frac{5}{9}\sqrt{1 - \frac{6m_0^2\lambda_6}{5\lambda_4^2}}\right)\text{for}\bar{\phi}^2 = \frac{-10\lambda_4}{\lambda_6}\left(1 \mp \sqrt{1 - \frac{6\mathrm{m}_0^2\lambda_6}{5\lambda_4^2}}\right) \tag{11}$$

Of course the minus-sign free-energy is smaller, which corresponds to the mean-field solution also with the minus sign. The plot for $U(\bar{\phi})$ to $\bar{\phi}$ is from fig.1 to fig.6.

3. Now the four-point coupling constant $\lambda_4 > 0$,

$$T > T_0, m_0^2 > 0 \Rightarrow \min(\varepsilon) = 0 \quad \text{when} \quad \bar{\phi}^2 = 0$$

$$T < T_0, m_0^2 < 0 \Rightarrow \min(\varepsilon) < 0 \quad \text{when} \quad \bar{\phi}^2 = 10\frac{\lambda_4}{\lambda_6}\left(-1 + \sqrt{1 - \frac{6\mathrm{m}_0^2\lambda_6}{5\lambda_4^2}}\right) \tag{12}$$

This implies that the transition temperature occurs at temperature $T = T_0$. At $T = T_0$ the mean-field solution is exactly 0. Therefore the order parameter, $\bar{\phi}$ is continuous. This is the second-order phase transition. (Moreover, you can prove the derivative of order parameter to temperature is discontinuous, of order $(T - T_0)^{-1}$).
4. Recall Eq.(10) we have the expression for $\lambda_4 < 0$ gives the behavior of the phase boundary as a function of $T^* - T_0$, which is the first order phase transition.

$$\lambda_4 = -\sqrt{\frac{8a\lambda_6}{5}\left(T^* - T_0\right)} \tag{13}$$

For $\lambda_4 > 0$, the phase boundary always occurs at $T = T_0$, and it corresponds to the second order phase transition. The plot for the boundary is fig.9.

2 Scalar Electrodynamics

1. For a local transiformation,

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \Rightarrow D'_\mu = \partial_\mu + ieA_\mu + ie\partial_\mu \Lambda \tag{14}$$

The complex scalar field potential terms are of course invariant:

$$|\phi'(x)|^2 = \phi'^*(x)\phi'(x) = \phi^*(x)e^{ie\Lambda(x)}\phi(x)e^{-ie\Lambda(x)} = \phi^*(x)\phi(x) = |\phi(x)|^2$$
$$\left(|\phi'(x)|^2\right)^2 = \left(|\phi(x)|^2\right)^2 \tag{15}$$

The EM field stress tensor transforms like:

$$F'^{\mu\nu} = \partial^\mu A'^\nu - \partial^\nu A'^\mu = \partial^\mu \left(A^\nu + \partial^\nu \Lambda\right) - \partial^\nu \left(A^\mu + \partial^\mu \Lambda\right) = F^{\mu\nu} \tag{16}$$

Finally the scalar-gauge field dynamic term transforms as:

$$D'_\mu \phi'(x) = \left(\partial_\mu + ieA_\mu + ie\partial_\mu \Lambda\right)\left(\phi(x)e^{-ie\Lambda(x)}\right) = D_\mu \phi(x) e^{-ie\Lambda(x)}$$
$$\Rightarrow \left(D'_\mu \phi'\right)^* \left(D'^\mu \phi'\right) = \left(D_\mu \phi e^{-ie\Lambda}\right)^* \left(D^\mu \phi e^{-ie\Lambda}\right) = \left(D_\mu \phi\right)^* \left(D^\mu \phi\right) \tag{17}$$

In conclusion, the total Lagrangian Density is invariant under gauge transformation.
2. The classical equations of motion comes from the Lagrange Equation,

$$\frac{\delta \mathcal{L}}{\delta \phi} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi}\right) = 0, \frac{\delta \mathcal{L}}{\delta \phi^*} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu \phi^*}\right) = 0, \frac{\delta \mathcal{L}}{\delta A_\nu} - \partial_\mu \left(\frac{\delta \mathcal{L}}{\delta \partial_\mu A_\nu}\right) = 0 \tag{18}$$

Solving these three equations, we obtain the more ugly form:

$$-\frac{1}{2}\partial^2 \phi^* + ieA^\mu \partial_\mu \phi^* + \frac{1}{2}ie\left(\partial_\mu A^\mu\right)\phi^* + \frac{1}{2}e^2 A^2 \phi^* - \frac{m^2}{2}\phi^* - \frac{\lambda}{12}|\phi|^2 \phi^* = 0$$
$$-\frac{1}{2}\partial^2 \phi - ieA^\mu \partial_\mu \phi - \frac{1}{2}ie\left(\partial_\mu A^\mu\right)\phi + \frac{1}{2}e^2 A^2 \phi - \frac{m^2}{2}\phi - \frac{\lambda}{12}|\phi|^2 \phi = 0$$
$$\frac{1}{2}\left(\partial^\nu \phi\right)^* ie\phi - \frac{1}{2}(ie\phi^*)\partial^\nu \phi + e^2 A^\nu \phi^* \phi + \partial_\mu F^{\mu\nu} = 0 \tag{19}$$

However, due to the newly defined $D_\mu = \partial_\mu + ieA_\mu$ and $D^\mu = \partial^\mu + ieA^\mu$, one can expect the simplified version of the above equations by using D_μ:

$$\begin{aligned} -\frac{1}{2}\left(D^\mu D_\mu \phi\right)^* - \frac{m^2}{2}\phi^* - \frac{\lambda}{12}|\phi|^2\phi^* = 0 \\ -\frac{1}{2}D^\mu D_\mu \phi - \frac{m^2}{2}\phi - \frac{\lambda}{12}|\phi|^2\phi = 0 \\ \frac{1}{2}ie[\phi\left(D^\nu\phi\right)^* - \phi^*\left(D^\nu\phi\right)] + \partial_\mu F^{\mu\nu} = 0 \end{aligned} \tag{20}$$

3. The canonical momentum is:

$$\Pi = \frac{\delta\mathcal{L}}{\delta\partial_0\phi} \Rightarrow H = \frac{\delta\mathcal{L}}{\delta\partial_0\phi}\phi - \mathcal{L} \tag{21}$$

The scalar field and EM field momentums are:

$$\begin{aligned} \Pi = \frac{1}{2}\left(\partial_0\phi^* - ieA_0\phi^*\right) = \frac{1}{2}\left(D_0\phi\right)^* \\ \Pi^* = \frac{1}{2}\left(\partial_0\phi + ieA_0\phi\right) = \frac{1}{2}\left(D_0\phi\right) \\ \Pi^i = F^{i0} \end{aligned} \tag{22}$$

The Hamiltonian Density by definition, is

$$\begin{aligned} h &= \Pi\partial_0\phi + \Pi^*\partial_0\phi^* + \Pi^\mu\partial_0 A_\mu - \mathcal{L} \\ &= \frac{1}{2}\left(D_0\phi\right)^*\partial_0\phi + \frac{1}{2}\left(D_0\phi\right)\partial_0\phi^* + F^{\mu 0}\partial_0 A_\mu - \mathcal{L} \\ &= \left(D_0\phi\right)^*\left(D_0\phi\right) + \frac{1}{2}\left(D_0\phi\right)^*\left(-ieA_0\right)\phi + \frac{1}{2}D_0\phi\left(ieA_0\right)\phi^* + E_i\partial_0 A_i - \mathcal{L} \\ &= \frac{1}{2}\left(D_0\phi\right)^*\left(D_0\phi\right) + \frac{1}{2}\left(D_i\phi\right)^*\left(D_i\phi\right) + \frac{1}{2}ieA_0\left[\phi^*\left(D_0\phi\right) - \phi\left(D_0\phi\right)^*\right] \\ &\quad + U(\phi) + E_i\partial_0 A_i + \frac{1}{4}F^2 \end{aligned} \tag{23}$$

Note from Eq.(19): the Conserved Quantity, Charge, is the special case of that of the third equation of Eq.(19), and note: $D^0\phi = 2\Pi^*$:

$$ie\left(\phi\Pi - \phi^*\Pi^*\right) + \partial_\mu F^{\mu 0} = 0 \tag{24}$$

Therefore we have:

$$-\int d^3x \frac{ieA_0}{2}\left(\phi\Pi - \phi^*\Pi^*\right) = \int d^3x A_0\partial_\mu F^{\mu 0} = -\int d^3x\left(\partial_\mu A_0\right)F^{\mu 0} \tag{25}$$

Where we have dropped the boundary term. Combining this back into Eq.(23) with the 3rd and 5th terms, we have:

$$\begin{aligned} \int d^3x\left(\frac{ieA_0}{2}[\phi^*\left(D^0\phi\right) - \phi\left(D^0\phi\right)^*] + F^{\mu 0}\partial_0 A_\mu\right) &= \int d^3x\left(\partial_0 A_\mu - \partial_\mu A_0\right)F^{\mu 0} \\ &= \int d^3x F_{0\mu}F^{\mu 0} = \int d^3x E_i^2 \end{aligned} \tag{26}$$

Use this conclusion back into Eq.(23), with $\frac{1}{4}F^2$, we finally have,

$$H = \int d^3x \left(\frac{1}{2}\Pi^*\Pi + \frac{1}{2}\left(D_i\phi\right)^*\left(D_i\phi\right) + U(\phi) + \frac{1}{2}\left(E^2 + B^2\right) \right) \tag{27}$$

This, together with Gauss's Law, Eq.(24), gives the Hamiltonian of the system. Let us have a look at the third equation in Eq.(22), Π^i. In the case of the gauge field, there is no canonical momentum conjugate to A_0 which is why it is a Lagrange multiplier field that enforces a constraint, Gauss' Law. This is why Professor Fradkin suggest me not to use the notation Π^μ but to use Π^i since $\Pi^0 = 0$.

4. Using ρ, θ instead of ϕ^*, ϕ,

$$D_\mu\phi = \left(\partial_\mu\rho + i\rho\left(\partial_\mu\theta + eA_\mu\right)\right)e^{i\theta} \tag{28}$$

From Eq.(19),

$$\begin{aligned} \partial^2\rho - \rho\left(\partial_\mu\theta + eA_\mu\right)^2 + m_0^2\rho + \frac{\lambda}{6}\rho^3 = 0 \\ \rho\left(\partial^2\theta + e\partial^\mu A_\mu\right) + \left(\partial^\mu\rho\right)\left(2\partial_\mu\theta + eA_\mu\right) = 0 \\ e\rho\left(\partial^\nu\theta + eA^\nu\right) + \partial_\mu F^{\mu\nu} = 0 \end{aligned} \tag{29}$$

For the London Gauge $\theta = 0$, the equations of motions are reduced into:

$$\begin{aligned} \partial^2\rho - \rho\left(eA_\mu\right)^2 + m_0^2\rho + \frac{\lambda}{6}\rho^3 = 0 \\ \rho\partial^\mu A_\mu + A_\mu\partial^\mu\rho = 0 \\ e^2\rho A^\nu + \partial_\mu F^{\mu\nu} = 0 \end{aligned} \tag{30}$$

The Lagrangian Density is,

$$\mathcal{L} = \frac{1}{2}\partial^\mu\rho\partial_\mu\rho + \frac{1}{2}e^2A^2\rho^2 - \frac{1}{2}m_0^2\rho^2 - \frac{\lambda}{4!}\rho^4 - \frac{1}{4}F^2 \tag{31}$$

5. If $m_0^2 < 0$, then $-\frac{1}{2}m_0^2 > 0$. For the case $\rho = \bar{\rho}$ the effective Lagrangian Density for London Gauge is reduced from Eq.(31) that, we set $\partial_\mu\rho$ to be 0,

$$\mathcal{L} = \left(\frac{1}{2}e^2A^2\rho^2 - \frac{1}{2}m_0^2\rho^2 - \frac{\lambda}{4!}\rho^4\right) - \frac{1}{4}F^2 \tag{32}$$

The equations of motion in Eq.(30) gives the minimization of the Lagrangian Density,

$$\rho = \sqrt{\frac{6}{\lambda}\left(e^2A^2 + |m_0^2|\right)}, \quad \partial_\mu A^\mu = 0, \quad A_\mu = 0 \Rightarrow \bar{\rho} = \sqrt{\frac{6}{\lambda}|m_0^2|} \tag{33}$$

Fix the classical solution, and plug back into our Lagrangian Density, Eq.(32),

$$\mathcal{L} = \frac{3|m_0^2|}{2\lambda}\left(2e^2A^2 + |m_0^2|\right) - \frac{1}{4}F^2 = \mathcal{L}_0 + \frac{1}{2}m_{\text{ph}}^2A^2 - \frac{1}{4}F^2 \tag{34}$$

Where $m_{\text{ph}}^2 = \bar{\rho}e^2$. Now the equations of motion for the fluctuation fields are,

$$\frac{\delta\mathcal{L}}{\delta A_\nu} - \partial_\mu \left(\frac{\delta\mathcal{L}}{\delta\partial_\mu A_\nu} \right) = 0 \Rightarrow m_{\text{ph}}^2 A^\nu + \partial_\mu F^{\nu\mu} = 0 \tag{35}$$

From the previous equations of motion that, $\partial_\mu A^\mu = 0$ in Eq.(33), we have $\partial_\mu F^{\nu\mu}$ to be, $\partial_\mu \left(\partial^\nu A^\mu - \partial^\mu A^\nu\right) = -\partial^2 A^\nu$. Thus Eq.(35) is reduced into:

$$m_{\text{ph}}^2 A^\nu - \partial^2 A^\nu = 0 \tag{36}$$

Which is the famous Klein-Gordon equation. Here, the $m_{\text{ph}}^2 = \bar{\rho}e^2$ is the effective photon mass.

3 The Dirac Equation

1. Two ways to approach this problem. First is to find the algebra properties of γ^μ matrices:

$$\left(\gamma^0\right)^+ = \gamma^0, \quad \left(\gamma^{1,2,3}\right)^+ = -\gamma^{1,2,3} \tag{37}$$

With these relations the Dirac Eqaution $(i\partial\!\!\!/ - m)\,\psi = 0$ transforms as:

$$\gamma^0 \left[(i\partial\!\!\!/ - m)\,\psi\right]^+ = 0 \Rightarrow \gamma^0\psi^+ \left[-i\left(\overleftarrow{\partial\!\!\!/}\right)^+ - m\right] = 0 \tag{38}$$

Since,

$$\begin{aligned} (\partial_0)^+ = \partial_0 &\Rightarrow \left(\partial_0\gamma^0\right)^+ = \partial_0\gamma^0 \\ (\partial_{1,2,3})^+ = -\partial_{1,2,3} &\Rightarrow \left(\partial_{1,2,3}\gamma^{1,2,3}\right)^+ = \partial_{1,2,3}\gamma^{1,2,3} \\ &\Rightarrow (\partial\!\!\!/)^+ = \partial\!\!\!/ \end{aligned} \tag{39}$$

Therefore, the Dirac Equation can be written as,

$$\gamma^0\psi^+ \left(-i\overleftarrow{\partial\!\!\!/} - m\right) = 0 \Rightarrow \bar{\psi}\left(-i\overleftarrow{\partial\!\!\!/} - m\right) = 0 \tag{40}$$

Now let us take care of the 4-current. To prove it is conserved, what we need is to prove $\partial_\mu j^\mu = 0$ (below we have used Eq.(38) and the Dirac Equation):

$$\partial_\mu j^\mu = \partial_\mu \left(\bar{\psi}\gamma^\mu\psi\right) = \left(\partial_\mu\bar{\psi}\right)\gamma^\mu\psi + \bar{\psi}\gamma^\mu\left(\partial_\mu\psi\right) = \left(\partial\!\!\!/\bar{\psi}\right)\psi + \bar{\psi}\left(\partial\!\!\!/\psi\right) = 0 \tag{41}$$

Another way for this problem, is to use the Lagrangian Density $\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi$, and use the equations of motion,

$$\partial_\mu\left(\frac{\delta L}{\delta\partial_\mu\psi}\right) - \frac{\delta L}{\delta\psi} = 0 \qquad \partial_\mu\left(\frac{\delta L}{\delta\partial_\mu\bar{\psi}}\right) - \frac{\delta L}{\delta\bar{\psi}} = 0 \tag{42}$$

We can also reach the same conclusion.
2. If the spinor satisfies Dirac Equation, $(i\partial\!\!\!/ - m)\psi = 0$,

$$(i\partial\!\!\!/ + m)(i\partial\!\!\!/ - m)\psi = 0 \Rightarrow \left(i\partial\!\!\!/ \cdot i\partial\!\!\!/ - m^2\right)\psi = 0 \tag{43}$$

The first term above turns out to be:

$$\begin{aligned} \partial\!\!\!/ \cdot \partial\!\!\!/ &= \partial_\mu \partial_\nu \gamma^\mu \gamma^\nu = \partial_\mu \partial_\nu \left(\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]\right) \\ \partial\!\!\!/ \cdot \partial\!\!\!/ &= \partial_\nu \partial_\mu \gamma^\nu \gamma^\mu = \partial_\nu \partial_\mu \left(\frac{1}{2}\{\gamma^\nu, \gamma^\mu\} + \frac{1}{2}[\gamma^\nu, \gamma^\mu]\right) \end{aligned} \tag{44}$$

Sum these two equations up, use the symmetric property for $\{\gamma^\mu, \gamma^\nu\}$ and anti-symmetric for $[\gamma^\mu, \gamma^\nu]$,

$$2\partial\!\!\!/ \cdot \partial\!\!\!/ = \partial_\mu \partial_\nu \{\gamma^\mu, \gamma^\nu\} = 2\partial_\mu \partial_\nu g^{\mu\nu} = 2\left(\partial_0^2 - \partial_1^2 - \partial_2^2 - \partial_3^2\right) = 2\partial^2 \tag{45}$$

Therefore, the spinor satisfies Klein-Gordon Equation:

$$\left(\partial^2 + m^2\right)\psi = 0 \tag{46}$$

3. (a) Use the properties of gamma matrices,

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu] = g^{\mu\nu} - i\sigma^{\mu\nu} \tag{47}$$

Thus the inner product of $A\!\!\!/ B\!\!\!/$ is,

$$A\!\!\!/ B\!\!\!/ = A_\mu B_\nu \gamma^\mu \gamma^\nu = A_\mu B_\nu \left(g^{\mu\nu} - i\sigma^{\mu\nu}\right) = A \cdot B - i\sigma_{\mu\nu} A^\mu B^\nu \tag{48}$$

(b) The trace take over the spinor indices, thus

$$Tr\left(A\!\!\!/ B\!\!\!/\right) = Tr\left(A_\mu B_\nu g^{\mu\nu} - i\sigma_{\mu\nu} A^\mu B^\nu\right) = A_\mu B_\nu g^{\mu\nu} Tr\left(1_4\right) = 4A \cdot B \tag{49}$$

(c) Use $\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = g^{\mu\nu}$,

$$\gamma^\lambda \gamma^\mu \gamma_\lambda = \gamma^\lambda \gamma^\mu \gamma^\lambda g_{\lambda\lambda} = 2g^{\lambda\mu} \gamma^\lambda g_{\lambda\lambda} - \gamma^\mu \gamma^\lambda \gamma^\lambda g_{\lambda\lambda} \tag{50}$$

Since

$$\gamma^\mu \gamma^\lambda \gamma^\lambda g_{\lambda\lambda} = 4\gamma^\mu, \quad g^{\lambda\mu} g_{\lambda\lambda} = g^\mu_\lambda = \delta^\mu_\lambda \tag{51}$$

Therefore we have

$$\gamma^\lambda \gamma^\mu \gamma_\lambda = 2\gamma^\mu - 4\gamma^\mu = -2\gamma^\mu \tag{52}$$

4 Transformation Properties of Field Bilinears in the Dirac Theory

(a) The spinor transformation is, $\psi'(x') = S(\Lambda)\psi(x)$, with $S(\Lambda) = \exp\left(-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)$, therefore,

$$S(\Lambda)^{+} = \left[\exp\left(-\frac{\mathrm{i}}{4}\sigma_{\mu\nu}\omega^{\mu\nu}\right)\right]^{+} = \exp\left(\frac{\mathrm{i}}{4}\left(\omega^{\mu\nu}\right)^{+}\sigma^{+}_{\mu\nu}\right) = \exp\left(\frac{\mathrm{i}}{4}\omega^{\mu\nu}\sigma^{+}_{\mu\nu}\right) \quad (53)$$

Note $\left(\omega^{\mu\nu}\right)^{+} = \omega^{\mu\nu}$ becasue $\omega^{\mu\nu}$ is a number, and '+' has nothing to do with a number. Therefore, what we want to prove is,

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\left(\gamma^0\right)^{-1}S^{+}\gamma^0 S\psi(x) = \bar{\psi}(x)\gamma^0 S^{+}\gamma^0 S\psi(x) \quad (54)$$

To prove the $\gamma^0 S^{+}\gamma^0 S = I$, use the Taylor Series,

$$\gamma^0 S^{+}\gamma^0 = \gamma^0\left[\sum_{n=0}^{\infty}\left(\frac{i}{4}\omega^{\mu\nu}\sigma^{+}_{\mu\nu}\right)^{n}\right]\gamma^0 = \sum_{n=0}^{\infty}\left(\frac{i}{4}\omega^{\mu\nu}\gamma^0\sigma^{+}_{\mu\nu}\gamma^0\right)^{n} \quad (55)$$

Since $\left(\sigma_{\mu\nu}\right)^{+} = -\frac{i}{2}\left[\gamma^{\nu+},\gamma^{\mu+}\right]$, and $\left(\gamma^0\right)^{+} = \gamma^0, \left(\gamma^{1,2,3}\right)^{+} = -\gamma^{1,2,3}$, we can prove the identity,

$$\gamma^0\left(\gamma^{\nu+}\gamma^{\mu+} - \gamma^{\mu+}\gamma^{\nu+}\right)\gamma^0 = -\left[\gamma^{\mu},\gamma^{\nu}\right] \Rightarrow \gamma^0\sigma^{+}_{\mu\nu}\gamma^0 = \sigma_{\mu\nu} \quad (56)$$

Therefore we reach the conclusion that,

$$\gamma^0 S^{+}\gamma^0 = S^{-1} \Rightarrow \bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x) \quad (57)$$

(b) Continue from above,

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \bar{\psi}(x)S^{-1}\gamma_5 S\psi(x) = \bar{\psi}(x)S^{-1}i\gamma^0\gamma^1\gamma^2\gamma^3 S\psi(x) \quad (58)$$

Use $S^{-1}\gamma^{\mu}S\left(\Lambda^{-1}\right)^{\nu}_{\mu} = \gamma^{\nu}$, or, equavalently $S^{-1}\gamma^{\mu}S = \gamma^{\nu}\Lambda^{\mu}_{\nu}$,

$$\begin{aligned}
S^{-1}i\gamma^0\gamma^1\gamma^2\gamma^3 S &= i\left(S^{-1}\gamma^0 S\right)\left(S^{-1}\gamma^1 S\right)\left(S^{-1}\gamma^2 S\right)\left(S^{-1}\gamma^3 S\right) \\
&= i\left(\gamma^{\alpha}\Lambda^0_{\alpha}\right)\left(\gamma^{\beta}\Lambda^1_{\beta}\right)\left(\gamma^{\lambda}\Lambda^2_{\lambda}\right)\left(\gamma^{\delta}\Lambda^3_{\delta}\right) \\
&= i\gamma^{\alpha}\gamma^{\beta}\gamma^{\lambda}\gamma^{\delta}\Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\lambda}\Lambda^3_{\delta} \\
&= i\epsilon^{\alpha\beta\lambda\delta}\gamma^0\gamma^1\gamma^2\gamma^3\Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\lambda}\Lambda^3_{\delta} \\
&= \gamma^5\epsilon^{\alpha\beta\lambda\delta}\Lambda^0_{\alpha}\Lambda^1_{\beta}\Lambda^2_{\lambda}\Lambda^3_{\delta} = \gamma^5\mathrm{det}\Lambda
\end{aligned} \quad (59)$$

This is what we need for this proof.

$$\bar{\psi}'(x')\gamma_5\psi'(x') = \bar{\psi}(x)\gamma_5\psi(x)\mathrm{det}\Lambda \quad (60)$$

(c) This is just the same as the proof in (b) problem:

$$\bar{\psi}'(x')\gamma^{\mu}\psi'(x') = \bar{\psi}(x)S^{-1}\gamma^{\mu}S\psi(x) = \bar{\psi}(x)\gamma^{\nu}\Lambda^{\mu}_{\nu}\psi(x) \quad (61)$$

(d) Again duplicate the process of (b),

$$\bar{\psi}'(x')\gamma_5\gamma^\mu\psi'(x') = \bar{\psi}(x)\left(S^{-1}\gamma_5 S\right)\left(S^{-1}\gamma^\mu S\right)\psi(x) = \bar{\psi}(x)\gamma_5\gamma^\nu\psi(x)\Lambda^\mu_\nu \det\Lambda \quad (62)$$

(e) The proof for a combination of γ^μ matrices is the same as that of one γ^μ matrix:

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = \bar{\psi}(x)\left(S^{-1}\frac{i}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)S\right)\psi(x)$$
$$= \bar{\psi}(x)\left(\frac{i}{2}\left(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha\right)\right)\psi(x)\Lambda^\mu_\alpha\Lambda^\nu_\beta = \bar{\psi}(x)\sigma^{\mu\nu}\psi(x)\Lambda^\mu_\alpha\Lambda^\nu_\beta \quad (63)$$

5 Chiral Symmetry

1. Using the new form of slash operators in Chiral Representation, since:

$$\not{\partial} = \partial_0\gamma^0 - \vec{\partial}\cdot\vec{\gamma} \quad (64)$$

The new Dirac Equation is written as:

$$\begin{pmatrix} -mI & -i\partial_0 I - i\vec{\sigma}\cdot\vec{\partial} \\ -i\partial_0 I + i\vec{\sigma}\cdot\vec{\partial} & -mI \end{pmatrix}\begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \quad (65)$$

or, equavalently,

$$-m\phi - \left(i\partial_0 I + i\vec{\sigma}\cdot\vec{\partial}\right)\chi = 0$$
$$-m\chi - \left(i\partial_0 I - i\vec{\sigma}\cdot\vec{\partial}\right)\phi = 0 \quad (66)$$

2. The massless Dirac Eqaution with $m = 0$, reduces to:

$$\left(\partial_0 I + \vec{\sigma}\cdot\vec{\partial}\right)\chi = 0$$
$$\left(\partial_0 I - \vec{\sigma}\cdot\vec{\partial}\right)\phi = 0 \quad (67)$$

Hence ϕ and χ decouples. Let us denote $\chi = (\chi_1, \chi_2)^{\mathrm{T}}$ and $\phi = (\phi_1, \phi_2)^{\mathrm{T}}$,

$$\begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} -\partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & -\partial_0 - \partial_3 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \quad (68)$$

Plugging into the plane wave solution, i.e., $\chi_i = \chi_{i0}e^{ik_\mu x^\mu}$, $\phi_i = \phi_{i0}e^{ip_\nu x^\nu}$, the above two matrices are:

$$\begin{pmatrix} k_0 - k_3 & -k_1 + ik_2 \\ -k_1 - ik_3 & k_0 + k_3 \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$
$$\begin{pmatrix} -p_0 - p_3 & -p_1 + ip_2 \\ -p_1 - ip_2 & -p_0 + p_3 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = 0 \quad (69)$$

Solving these two matrix equations, we get,

$$k_0^2 = \vec{k}^2, \quad p_0^2 = \vec{p}^2 \tag{70}$$

Therefore the dispersion relation is $k_0 = \omega = \pm|\vec{k}|, p_0 = \omega = \pm|\vec{p}|$. The relations between two components of that of 2-spinors are:

$$\chi_2 = \frac{k_0 - k_3}{k_1 - ik_2}\chi_1, \quad \phi_2 = \frac{-p_0 - p_3}{p_1 - ip_2}\phi_1 \tag{71}$$

By definition ϕ lives in the upper part of the spinor wave function, and γ_5 matrix acting on it resulting the $+1$ eigenvalue, and hence ϕ wave has the $+1$ chirality. Similar argument shows χ has the -1 chirality.
3. First of all let us calculate what $e^{i\gamma_5\theta}$ is (to use Taylor Expansion):

$$e^{i\gamma_5\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}(\gamma_5)^n = \sum_{n=2k}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} + \sum_{n=2k+1}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!}\gamma_5 = \cos\theta + i\gamma_5 \sin\theta \tag{72}$$

Write in the matrix form:

$$e^{i\gamma_5\theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \tag{73}$$

(a) ϕ and χ transforms as:

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} e^{i\theta}\phi \\ e^{-i\theta}\chi \end{pmatrix} \tag{74}$$

(b) By definition $\bar{\psi} = \psi^+\gamma^0$, $\psi'^+ = \psi^+ e^{-i\gamma_5\theta}$, we get $\bar{\psi}' = \bar{\psi}\gamma^0 e^{-i\gamma_5\theta}\gamma^0 = \bar{\psi}e^{i\gamma_5\theta}$.
(c) Using the conclusions obtained above, $\bar{\psi}'\psi' = \bar{\psi}e^{2i\gamma_5\theta}\psi$;

$$\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}e^{i\gamma_5\theta}\gamma^\mu e^{i\gamma_5\theta}\psi \tag{75}$$

Since,

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \gamma^\mu \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \gamma^\mu \tag{76}$$

We find the quantity is conserved, $\bar{\psi}'\gamma^\mu\psi' = \bar{\psi}\gamma^\mu\psi$.
(d) The form of Dirac Equation is written in Eq.(64). If $m \neq 0$, the coupling between ϕ and χ is non-zero. However, under a Chiral Transformation ϕ and χ rotate in the opposite angle, Dirac Equation cannot be invariant. Recall Eq.(64), if the Dirac Eqaution acts on the transformed spinor wave function,

$$\begin{pmatrix} -m_{11}I & -i\partial_0 I - i\vec{\sigma}\cdot\vec{\partial} \\ -i\partial_0 I + i\vec{\sigma}\cdot\vec{\partial} & -m_{22}I \end{pmatrix} \begin{pmatrix} e^{i\theta}\phi \\ e^{-i\theta}\chi \end{pmatrix} = 0$$
$$\Rightarrow \begin{pmatrix} -m_{11}Ie^{2i\theta} & -i\partial_0 I - i\vec{\sigma}\cdot\vec{\partial} \\ -i\partial_0 I + i\vec{\sigma}\cdot\vec{\partial} & -m_{22}Ie^{-2i\theta} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = 0 \tag{77}$$

The mass term breaks the Chiral Symmetry, the newly defined mass is,

$$m \begin{pmatrix} Ie^{-2i\theta} & 0 \\ 0 & Ie^{2i\theta} \end{pmatrix} = me^{-2i\gamma_5\theta} \tag{78}$$

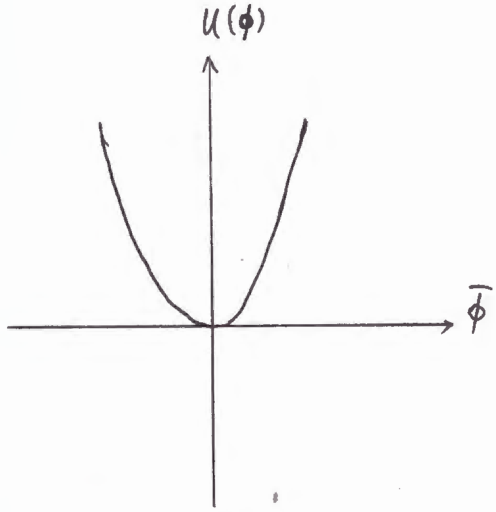

Fig. 1. $\chi > \frac{5}{6}$

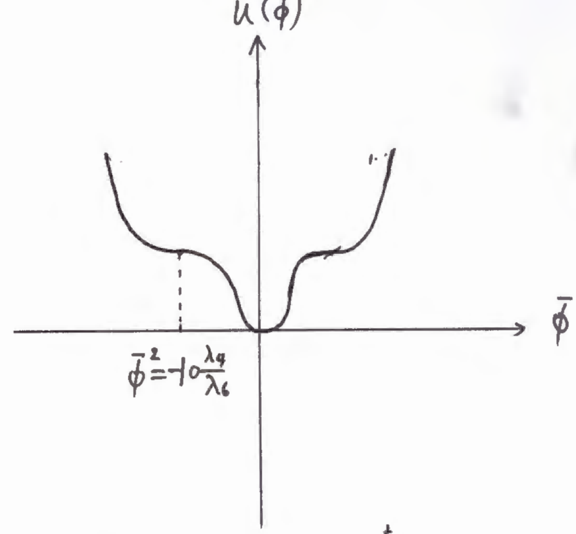

Fig 2. $\chi = \frac{5}{6}$

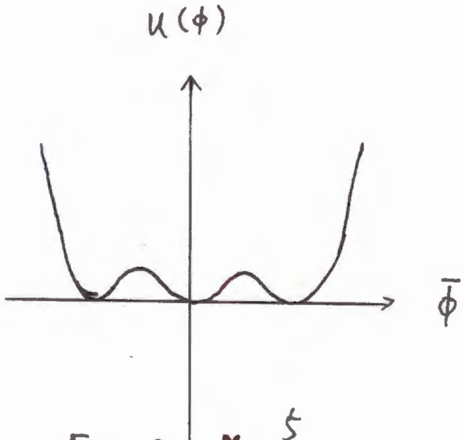

Fig 3. $\chi = \frac{5}{8}$

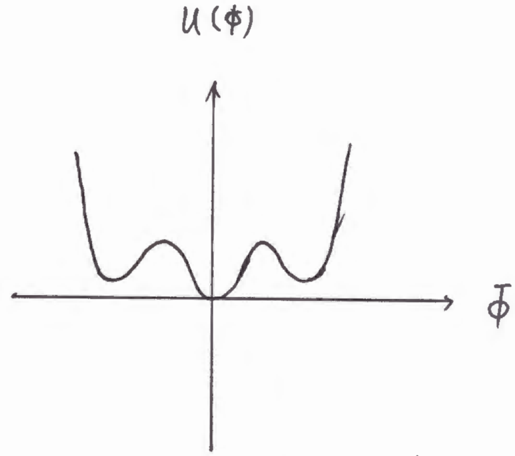

Fig 4. $\frac{5}{6} > \chi > \frac{5}{8}$

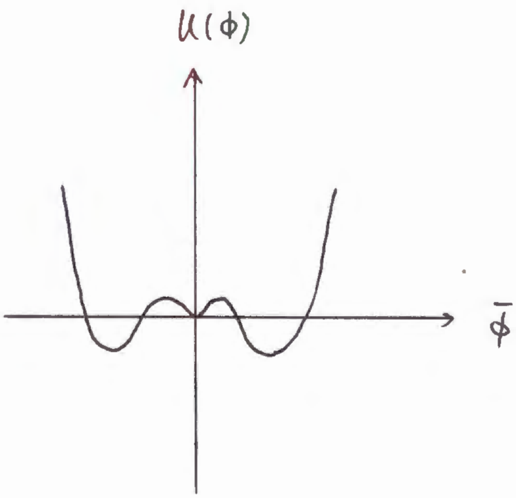

Fig 5. $\frac{5}{8} > \chi > 0$

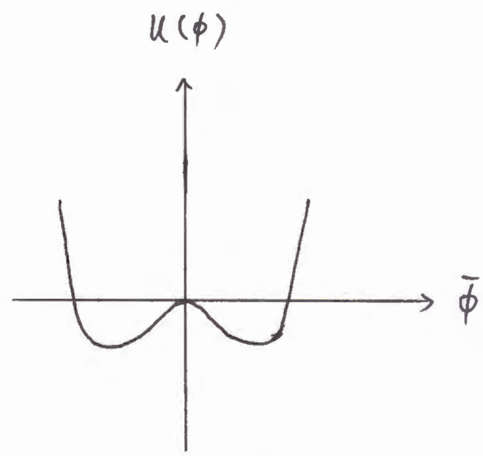

Fig 6. $\chi < 0$

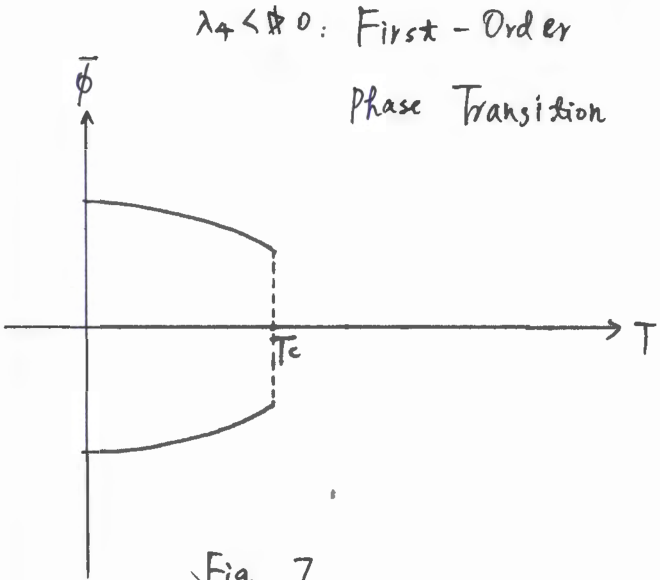

Fig 7.

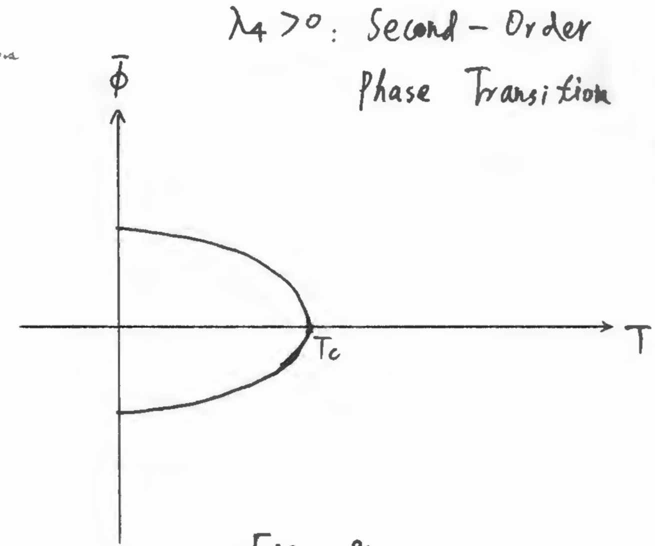

Fig. 8.

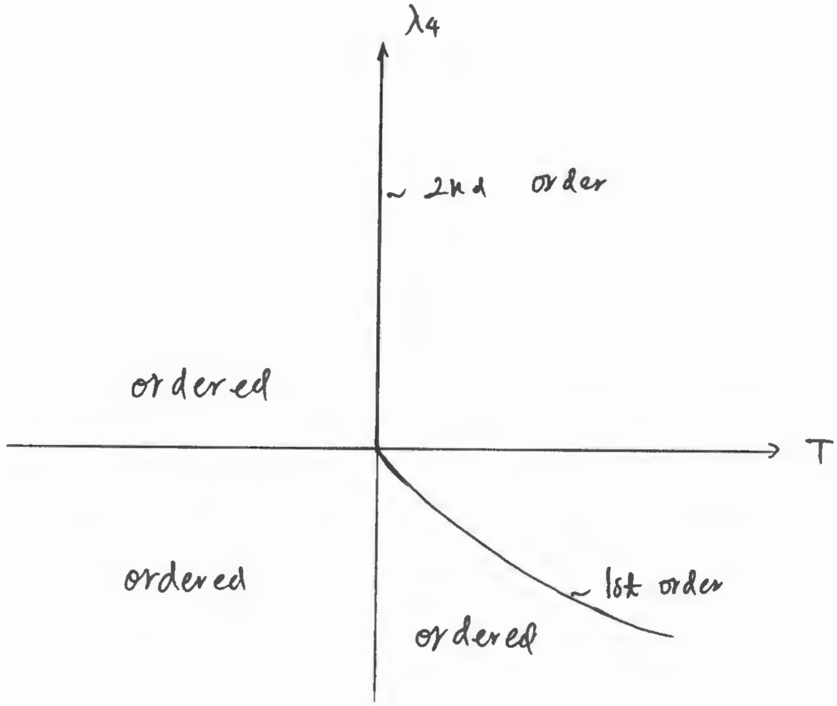

Fig. 9.