Center of inertia and coordinate transformations in the post-Newtonian charged *n*-body problem in gravitation

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We generalize the field theory propagator by finding a way to make it a function of some additional arbitrary parameters. Thus, it is now possible to obtain Lagrangians (which contain the propagator parameters) from field theory in a more general coordinate system than had previously been possible. We find the n-body (classical) Bażański Lagrangian in this more general coordinate system and we give the relationship between the various coordinate systems by an n-body coordinate transformation involving the propagator parameters. We find the center of inertia for the case of the n-body Baszański Lagrangian in the general coordinate system and find that the potential energy terms $-Gm_im_j/r_{ij}$ and e_ie_j/r_{ij} do *not* in general split equally between particles i and j as they *do* in the case of Bażański coordinates. We also find the center of inertia for the case of the n-body (unchanged) post-Newtonian Lagrangian with parameterized post-Newtonian (PPN) parameters γ and β in standard coordinates, and show that the potential energy terms *do* split equally between a pair of particles.

INTRODUCTION

In a recent paper[1] we started with the two-body Bażański Lagrangian[1,2] (the post-Newtonian Lagrangian for two charged bodies in general relativity) in Bażański coordinates. We then made a coordinate transformation[3] involving two arbitrary dimensionless parameters α_g and α_p to obtain the Lagrangian $\mathscr{L}(\alpha_g,\alpha_p)$ in the new coordinate system. The form of the coordinate transformation[3] was chosen so that the Lagrangian $\mathscr{L}(\alpha_g,\alpha_p)$ would be consistent with the Hamiltonian $\mathscr{H}(\alpha_g,\alpha_p)$ derived from quantum field theory. In deriving $\mathscr{H}(\alpha_g,\alpha_p)$ a graviton propagator[1,4] involving α_g and a photon propagator[1] involving α_p were used. The form of these propagators (first given by Hiida and Okamura[4] for the graviton case) determined the form of the coordinate transformation.

It is now clear to us that the form of the graviton and photon propagators can be generalized by adding additional dimensionless parameters. This will lead to the same additional parameters in the corresponding coordinate transformation.

In Sec. I we shall derive the one-graviton-exchange potential energy term (i.e., term of order G) and the one-photon-exchange potential energy term (i.e., term of order e^2) for two scalar particles where parameters α_g, a_{12g}, and a_{21g} are used in the graviton propagator and parameters α_p, a_{12p}, and a_{21p} are used in the photon propagator; and then in Sec. II we shall give the corresponding two-body coordinate transformations.

In Sec. III we shall give the n-body coordinate transformations and the n-body Lagrangian. They will contain parameters α_{ijg}, a_{ijg} and α_{ijp}, a_{ijp}.

In Sec. IV we shall turn our attention to the topic of center of inertia.[5,6] It is well known[5–7] [for n-body problems involving Darwin (D), Einstein, Infeld, Hoffmann (EIH) or Bażański (B) Lagrangians] that in finding the center of inertia the potential energy terms $-Gm_im_j/r_{ij}$ and e_ie_j/r_{ij} must be split equally between the particles i and j. However, while this $\frac{1}{2}$, $\frac{1}{2}$ split, as we shall call it, holds in Bażański coordinates (same as used by Darwin and EIH) there are certain coordinate systems in which it does *not* hold. We shall show that for the coordinate system introduced in Sec. III the $\frac{1}{2}$, $\frac{1}{2}$ split does not hold in general. We shall also explicitly show what split does occur for this coordinate system. Finally, in Sec. IV we consider the n-body (uncharged) post-Newtonian Lagrangian with parameterized post-Newtonian (PPN) parameters γ and β in the usually given coordinate system and find that the $\frac{1}{2}$, $\frac{1}{2}$ split holds in this case also. In Sec. V we present our conclusions.

I. DERIVATION OF $V_1(r)$ FROM FIELD THEORY

Let us review the most important points in the derivation of the one-graviton exchange interaction as well as the one-photon exchange interaction for the case of two scalar particles. Let e_1, m_1, $\mathbf{P}_1$, E_1, and e_2, m_2, $\mathbf{P}_2$, E_2 denote the charge, rest mass, momentum, and energy of particles 1 and 2 respectively. We also have

$$\mathbf{P}_1=\hbar\mathbf{p},\quad E_1=c\hbar p_0,\quad \lambda_1=m_1c/\hbar,\quad p_\mu^2=-\lambda_1^2, \tag{1}$$

$$\mathbf{P}_2=\hbar\mathbf{q},\quad E_2=c\hbar q_0,\quad \lambda_2=m_2c/\hbar,\quad q_\mu^2=-\lambda_2^2, \tag{2}$$

where p_μ and q_μ are the propagation 4-vectors for particle 1 and 2, respectively. The graviton coupling constant κ is related to Newton's constant of gravitation G and the speed of

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light c by the relation

$$\kappa^2 = 16\pi G/c^4. \quad (3)$$

We shall also use Gaussian electromagnetic units.

Consider the gravitational (or electromagnetic) scattering of two particles of spin 0. Let the initial and final propagation four-vectors for particle 1 be p and p', respectively, and those of particle 2 be q and q', respectively. The quantity $V_1(\mathbf{k})$ is defined in terms of the S matrix as

$$S_2 = (-i/c\hbar V^2)(2\pi)^4\delta(p+q-p'-q') \times a_1^*(\mathbf{p}')a_2^*(\mathbf{q}')V_1(\mathbf{k})a_2(\mathbf{q})a_1(\mathbf{p}), \quad (4)$$

where a_1, a_1^* and a_2, a_2^* denote the annihilation and creation operators for particles 1 and 2, respectively, and the factor V is a volume factor. The results for $V_1(\mathbf{k})$ for the one-graviton exchange and the one-photon exchange interactions are, respectively (see the Appendix),

$$V_{1g}(\mathbf{k}) = -\frac{c^2\hbar^2\kappa^2}{4(p_0'p_0q_0'q_0)^{1/2}}\frac{1}{\mathbf{k}^2-k_0^2}[(p'q')(pq)+(p'q)(pq') - (p'p)(q'q) - \lambda_1^2(q'q) - \lambda_2^2(p'p) - 2\lambda_1^2\lambda_2^2], \quad (5)$$

$$V_{1p}(\mathbf{k}) = \frac{-\pi e_1e_2}{(p_0'p_0q_0'q_0)^{1/2}}\frac{1}{\mathbf{k}^2-k_0^2}[(p'+p)(q'+q)], \quad (6)$$

where $k = p'-p = q-q'$. The quantity $V_1(\mathbf{k})$ can also be defined in terms of the potential energy $V_1(\mathbf{r})$ as

$$V_1(\mathbf{k}) = \int d\mathbf{r}\,\exp(-i\mathbf{p}'\cdot\mathbf{r}_1)\exp(-i\mathbf{q}'\cdot\mathbf{r}_2)V_1(\mathbf{r})\exp(i\mathbf{p}\cdot\mathbf{r}_1) \times\exp(i\mathbf{q}\cdot\mathbf{r}_2), \quad (7)$$

where $\mathbf{r}\equiv\mathbf{r}_1-\mathbf{r}_2$ and $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of particles 1 and 2, respectively. The potential energy $V_1(\mathbf{r})$ is a Hermitian operator and is also momentum dependent [i.e., $V_1(\mathbf{r})\equiv V_1(\mathbf{r},\mathbf{p}_{\rm op},\mathbf{q}_{\rm op})$ where $\mathbf{p}_{\rm op}$ and $\mathbf{q}_{\rm op}$ are operators].

If we are only interested in the classical result, as we are in this paper, the ordering of the factors in $V_1(\mathbf{r},\mathbf{p}_{\rm op},\mathbf{q}_{\rm op})$ makes no difference (i.e., we can neglect delta function terms) and we can thus write Eqs. (5)–(7) as

$$V_{1g}(\mathbf{k}) = -\frac{c^2\hbar^2\kappa^2}{4p_0q_0}\frac{1}{\mathbf{k}^2-k_0^2}[2(pq)^2-\lambda_1^2\lambda_2^2], \quad (8)$$

$$V_{1p}(\mathbf{k}) = -\frac{\pi e_1e_2}{p_0q_0}\frac{1}{\mathbf{k}^2-k_0^2}[4(pq)], \quad (9)$$

$$V_1(\mathbf{k}) = \int d\mathbf{r}\,e^{-i\mathbf{k}\cdot\mathbf{r}}V_1(\mathbf{r}). \quad (10)$$

The inverse of Eq. (10) is

$$V_1(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d\mathbf{k}\,e^{i\mathbf{k}\cdot\mathbf{r}}V_1(\mathbf{k}). \quad (11)$$

In this paper we shall be interested only in the post-Newtonian approximation in which case Eqs. (8) and (9) can be approximated as

$$V_{1g}(\mathbf{k}) = -\frac{c^2\hbar^2\kappa^2\lambda_1\lambda_2}{4\mathbf{k}^2}\left(1+\frac{3}{2}\frac{\mathbf{p}^2}{\lambda_1^2}+\frac{3}{2}\frac{\mathbf{q}^2}{\lambda_2^2} - 4\frac{\mathbf{p}\cdot\mathbf{q}}{\lambda_1\lambda_2}+\frac{k_0^2}{\mathbf{k}^2}\right), \quad (12)$$

$$V_{1p}(\mathbf{k}) = \frac{4\pi e_1e_2}{\mathbf{k}^2}\left(1-\frac{\mathbf{p}\cdot\mathbf{q}}{\lambda_1\lambda_2}+\frac{k_0^2}{\mathbf{k}^2}\right). \quad (13)$$

A. The propagator term

We shall call $1/(\mathbf{k}^2-k_0^2)$ the propagator term and in Eq. (12) and (13) it was expanded as

$$\frac{1}{\mathbf{k}^2-k_0^2} = \frac{1}{\mathbf{k}^2}+\frac{k_0^2}{\mathbf{k}^4}, \quad (14)$$

where

$$\mathbf{k} = \mathbf{p}'-\mathbf{p} = \mathbf{q}-\mathbf{q}', \quad (15)$$

$$k_0 = p_0'-p_0 = q_0-q_0', \quad (16)$$

and

$$p_0'-p_0 = \frac{p_0'^2-p_0^2}{p_0'+p_0} = \frac{\mathbf{p}'^2-\mathbf{p}^2}{p_0'+p_0} = \frac{\mathbf{k}\cdot(\mathbf{p}'+\mathbf{p})}{p_0'+p_0}, \quad (17)$$

$$q_0-q_0' = \frac{q_0^2-q_0'^2}{q_0+q_0'} = \frac{\mathbf{q}^2-\mathbf{q}'^2}{q_0+q_0'} = \frac{\mathbf{k}\cdot(\mathbf{q}+\mathbf{q}')}{q_0+q_0'}. \quad (18)$$

Since we are interested in the classical post-Newtonian result, Eqs. (17) and (18) may be expressed as

$$k_0 = \frac{\mathbf{k}\cdot\mathbf{p}}{\lambda_1} \quad\text{or}\quad k_0 = \frac{\mathbf{k}\cdot\mathbf{q}}{\lambda_2}. \quad (19)$$

The factor k_0^2 which appears in Eqs. (12) and (13) may be written in a symmetrical way as[1,4,8]

$$k_0^2 = (1+4\alpha)\left(\frac{(\mathbf{k}\cdot\mathbf{p})(\mathbf{k}\cdot\mathbf{q})}{\lambda_1\lambda_2}\right) - 2\alpha\left(\frac{(\mathbf{k}\cdot\mathbf{p})^2}{\lambda_1^2}+\frac{(\mathbf{k}\cdot\mathbf{q})^2}{\lambda_2^2}\right). \quad (20)$$

At this stage we must consider $\mathbf{k}$ as an independent variable [i.e., we no longer use Eq. (15)] since we will be using Eq. (11) to obtain $V_1(\mathbf{r})$. For the special case $\alpha = 0$ the resulting Hamiltonian (or Lagrangian) will be in the coordinate system[1,4] of B–EIH–D.

If we go to center-of-mass coordinates[9] (i.e., total momentum equals zero) where $\mathbf{p} = -\mathbf{q}$ we then have, from Eq. (20),

$$k_0^2 = -(1+4\alpha)\left(\frac{(\mathbf{k}\cdot\mathbf{p})^2}{\lambda_1\lambda_2}\right) - 2\alpha\left(\frac{(\mathbf{k}\cdot\mathbf{p})^2}{\lambda_1^2}+\frac{(\mathbf{k}\cdot\mathbf{p})^2}{\lambda_2^2}\right). \quad (21)$$

Thus, $k_0^2 = 0$ in the special case where

$$\frac{1+4\alpha}{\lambda_1\lambda_2}+2\alpha\left(\frac{1}{\lambda_1^2}+\frac{1}{\lambda_2^2}\right) = 0, \quad (22)$$

which implies that[1,10]

$$\alpha = -\frac{\lambda_1\lambda_2}{2(\lambda_1+\lambda_2)^2} = -\frac{m_1m_2}{2(m_1+m_2)^2}. \quad (23)$$

The one-graviton-exchange interaction for particles of various spins in center-of-mass coordinates (with $k_0^2=0$) was first given by Barker, Gupta, and Haracz[11] using Gupta's[12] quantum theory of gravitation.

In Ref. 1 we first suggested that there could be *two* independent α's, an α_p in the photon propagator as well as an α_g in the graviton propagator.

Let us now generalize the form of the propagator term so that k_0^2 will have the form

$$k_0^2 = [1+2\alpha(a_{12}+a_{21})]\left(\frac{(\mathbf{k}\cdot\mathbf{p})(\mathbf{k}\cdot\mathbf{q})}{\lambda_1\lambda_2}\right) - 2\alpha\left(a_{12}\frac{(\mathbf{k}\cdot\mathbf{p})^2}{\lambda_1^2} + a_{21}\frac{(\mathbf{k}\cdot\mathbf{q})^2}{\lambda_2^2}\right), \quad (24)$$

where a_{12} and a_{21} are new dimensionless parameters. It is to be understood that the α's and a's are to have a subscript g when used in the graviton propagator and a subscript p when used in the photon propagator. As we would like our final Hamiltonian or Lagrangian to be in a symmetrical form with respect to particles 1 and 2 we must impose symmetry conditions on $\alpha \equiv \alpha(m_1,m_2)$ and on $a_{12}\equiv a_{12}(m_1,m_2)$ and[13] $a_{21}\equiv a_{21}(m_2,m_1)$ of the form

$$\alpha(m_2,m_1) = \alpha(m_1,m_2), \quad (25)$$

$$a_{21}(m_2,m_1) = a_{12}(m_2,m_1). \quad (26)$$

Note that α of Eq. (23) satisfies Eq. (25).

Let us again go to center-of-mass coordinates in which case

$$k_0^2 = \left[-\frac{1}{\lambda_1\lambda_2} - 2\alpha\left(\frac{a_{12}+a_{21}}{\lambda_1\lambda_2} + \frac{a_{12}}{\lambda_1^2} + \frac{a_{21}}{\lambda_2^2}\right)\right](\mathbf{k}\cdot\mathbf{p})^2. \quad (27)$$

If we want the above result to be the same as for the case when $a_{12}=a_{21}=1$ we must have [from Eqs. (1) and (2) we note that $m_1/m_2 = \lambda_1/\lambda_2$]

$$a_{12}m_2^2 + (a_{12}+a_{21})m_1m_2 + a_{21}m_1^2 = (m_1+m_2)^2. \quad (28)$$

Dividing Eq. (28) by m_1+m_2 gives us

$$a_{12}m_2 + a_{21}m_1 = m_1 + m_2. \quad (29)$$

The solution to Eq. (29) which also satisfies the symmetry condition of Eq. (26) is

$$a_{12} = [(1-a_0)m_1 + a_0m_2]/m_2, \quad (30)$$

$$a_{21} = [(1-a_0)m_2 + a_0m_1]/m_1, \quad (31)$$

where

$$a_0 \equiv a_0(m_1,m_2) = a_0(m_2,m_1). \quad (32)$$

It can easily be shown that Eq. (29) does not restrict the result of Eq. (24) since it is always possible to remove a factor from the a's and absorb it in α in such a way as the resulting a's will satisfy Eq. (29). We shall thus require a_{12} and a_{21} to satisfy Eq. (29). Let us also note that if a_{12} and a_{21} are not mass dependent then $a_{12}=a_{21}=1$ and there will be no generalization of the propagator.

B. Results for $V_{1g}(\mathbf{r})$ and $V_{1p}(\mathbf{r})$

Using the form of k_0^2 given by Eq. (24) in Eqs. (12) and (13) together with Eq. (11) gives us

$$\begin{aligned} V_{1g}(\mathbf{r}) = -\frac{Gm_1m_2}{r}\Bigg[&1 + \left(\frac{3}{2} - \alpha_g a_{12g}\right)\frac{\mathbf{P}_1^2}{m_1^2c^2} + \left(\frac{3}{2} - \alpha_g a_{21g}\right)\frac{\mathbf{P}_2^2}{m_2^2c^2} \\ &+ [\alpha_g(a_{12g}+a_{21g}) - \tfrac{7}{2}]\frac{\mathbf{P}_1\cdot\mathbf{P}_2}{m_1m_2c^2} \\ &- [\tfrac{1}{2} + \alpha_g(a_{12g}+a_{21g})]\frac{(\mathbf{P}_1\cdot\mathbf{r})(\mathbf{P}_2\cdot\mathbf{r})}{m_1m_2c^2r^2} \\ &+ \alpha_g\left(a_{12g}\frac{(\mathbf{P}_1\cdot\mathbf{r})^2}{m_1^2c^2r^2} + a_{21g}\frac{(\mathbf{P}_2\cdot\mathbf{r})^2}{m_2^2c^2r^2}\right)\Bigg], \end{aligned} \quad (33)$$

$$\begin{aligned} V_{1p}(\mathbf{r}) = \frac{e_1e_2}{r}\Bigg[&1 - \alpha_p\left(a_{12p}\frac{\mathbf{P}_1^2}{m_1^2c^2} + a_{21p}\frac{\mathbf{P}_2^2}{m_2^2c^2}\right) \\ &+ [\alpha_p(a_{12p}+a_{21p}) - \tfrac{1}{2}]\frac{\mathbf{P}_1\cdot\mathbf{P}_2}{m_1m_2c^2} \\ &- [\tfrac{1}{2} + \alpha_p(a_{12p}+a_{21p})]\frac{(\mathbf{P}_1\cdot\mathbf{r})(\mathbf{P}_2\cdot\mathbf{r})}{m_1m_2c^2r^2} \\ &+ \alpha_p\left(a_{12p}\frac{(\mathbf{P}_1\cdot\mathbf{r})^2}{m_1^2c^2r^2} + a_{21p}\frac{(\mathbf{P}_2\cdot\mathbf{r})^2}{m_2^2c^2r^2}\right)\Bigg], \end{aligned} \quad (34)$$

where a_{12g}, a_{21g} and a_{12p}, a_{21p} must satisfy Eqs. (29)–(32). Of course subscripts g and p must be added to the a's of these equations for use in Eqs. (33) and (34), respectively.

II. TWO-BODY COORDINATE TRANSFORMATIONS

Let us now consider the two-body coordinate transformations

$$\mathbf{r}_{1B} = \mathbf{r}_1 - \mathbf{r}\left(\alpha_g a_{12g}\frac{Gm_2}{c^2r} - \alpha_p a_{12p}\frac{e_1e_2}{m_1c^2r}\right), \quad (35)$$

$$\mathbf{r}_{2B} = \mathbf{r}_2 + \mathbf{r}\left(\alpha_g a_{21g}\frac{Gm_1}{c^2r} - \alpha_p a_{21p}\frac{e_1e_2}{m_2c^2r}\right), \quad (36)$$

relating the Bażański coordinates $\mathbf{r}_{1B}$, $\mathbf{r}_{2B}$, to the new coordinates $\mathbf{r}_1$, $\mathbf{r}_2$. It can readily be shown that if one starts with the two-body Bażański Lagrangian in Bażański coordinates, and then makes the transformations of Eqs. (35)–(36) to obtain the Lagrangian in the new coordinates, then the corresponding Hamiltonian in the new coordinates will be in agreement with Eqs. (33) and (34). Note that the Hamiltonian will also contain G^2, e^4, and Ge^2 terms which, if derived from field theory, must come from fourth-order S matrix calculations.

From Eqs. (35)–(36) we obtain

$$\mathbf{r}_B=\mathbf{r}\left(1-\alpha_g\frac{GM}{c^2r}+\alpha_p\frac{e_1e_2}{\mu c^2r}\right), \tag{37}$$

where $\mathbf{r}_B=\mathbf{r}_{1B}-\mathbf{r}_{2B}$ and the reduced mass and total mass are given, respectively, by $\mu=m_1m_2/(m_1+m_2)$, and $M=m_1+m_2$. In obtaining Eq. (37) we have made use of Eq. (29), and for this reason Eq. (37) does not depend on the a's.

Let us now consider some special cases of Eqs. (35)–(36).

A. Result with $a_{0g}=a_{0p}=1$

From Eqs. (30) and (31) we obtain $a_{12g}=a_{12p}=1$, and $a_{21g}=a_{21p}=1$, which gives us the transformations[3] corresponding to the propagator with k_0^2 in the form of Eq. (20).

B. Result with $a_{0g}=a_{0p}=\frac{1}{2}$

From Eqs. (30) and (31) we obtain $a_{12g}=a_{12p}=M/2m_2$, and $a_{21g}=a_{21p}=M/2m_1$, which gives us the transformations[14] that were considered in Ref. 1. We rejected these transformations in Ref. 1 since they did *not* correspond to the propagator with k_0^2 in the form of Eq. (20). However, in the light of this present paper they *are* consistent with the field theory results.

C. Result with $a_{0g}=a_{0p}=0$

From Eqs. (30) and (31) we obtain $a_{12g}=a_{12p}=m_1/m_2$, and $a_{21g}=a_{21p}=m_2/m_1$, which is the simplest form (with mass dependence) that the a's can have.

In Ref. 1 we also considered the transformations[15]

$$\mathbf{r}_{1B}=\mathbf{r}_1\left(1-\alpha_g\frac{GM}{c^2r}+\alpha_p\frac{e_1e_2}{\mu c^2r}\right), \tag{38}$$

$$\mathbf{r}_{2B}=\mathbf{r}_2\left(1-\alpha_g\frac{GM}{c^2r}+\alpha_p\frac{e_1e_2}{\mu c^2r}\right), \tag{39}$$

from which we can also obtain Eq. (37). These transformations have some unusual properties[15] and are *not* a special case of Eqs. (35)–(36). We have not found a propagator where Eqs. (38)–(39) would be the corresponding transformations and we are not sure that such a propagator exists.

III. n-BODY RESULTS

We shall now generalize our results to the case of n bodies.

A. n-body coordinate transformations

The n-body generalization of Eqs. (35)–(36) is

$$\mathbf{r}_{iB}=\mathbf{r}_i-\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{r}_{ij}\left(A_{ij}^{g}\frac{Gm_j}{c^2r_{ij}}-A_{ij}^{p}\frac{e_ie_j}{m_ic^2r_{ij}}\right), \tag{40}$$

where $\mathbf{r}_{ij}\equiv\mathbf{r}_i-\mathbf{r}_j$ and (for the sake of brevity) we have set $A_{ij}^{g}\equiv\alpha_{ijg}a_{ijg}$ and $A_{ij}^{p}\equiv\alpha_{ijp}a_{ijp}$. Note that only the α_{ij}'s and a_{ij}'s for which $i\neq j$ are used. The symmetry conditions on $\alpha_{ijg}\equiv\alpha_g(m_i,m_j)$ and on $a_{ijg}\equiv a_{ijg}(m_i,m_j)$ are

$$\alpha_g(m_j,m_i)=\alpha_g(m_i,m_j), \tag{41}$$

$$a_{jig}(m_j,m_i)=a_{ijg}(m_j,m_i), \tag{42}$$

and thus $\alpha_{jig}=\alpha_{ijg}$. Note for the two-body problem there would be only one of the quantities $\alpha_g\equiv\alpha_{12g}\equiv\alpha_{21g}$.

We shall also require that

$$a_{ijg}m_j+a_{jig}m_i=m_i+m_j, \tag{43}$$

which is the n-body generalization of Eq. (29). The solution of Eq. (43) which satisfies the symmetry condition of Eq. (42) is

$$a_{ijg}=[(1-a_{0ijg})m_i+a_{0ijg}m_j]/m_j, \tag{44}$$

where[16]

$$a_{0ijg}\equiv a_{0g}(m_i,m_j)=a_{0g}(m_j,m_i), \tag{45}$$

and thus $a_{0jig}=a_{0ijg}$. For the two-body problem we have $a_{0g}\equiv a_{012g}=a_{021g}$. For α_{ijp} and a_{ijp} we will have a set of equations similar to Eqs. (41)–(45), where the subscript g is replaced by p. Note also that all the α's and a's must be dimensionless.

From Eqs. (40) and (43) we obtain

$$\begin{aligned}\mathbf{r}_{ijB}=\mathbf{r}_{ij}&\left(1-\alpha_{ijg}\frac{GM_{ij}}{c^2r_{ij}}+\alpha_{ijp}\frac{e_ie_j}{\mu_{ij}c^2r_{ij}}\right)\\ &-\sum_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\left[\mathbf{r}_{ik}\left(A_{ik}^{g}\frac{Gm_k}{c^2r_{ik}}-A_{ik}^{p}\frac{e_ie_k}{m_ic^2r_{ik}}\right)\right.\\ &\left.-\mathbf{r}_{jk}\left(A_{jk}^{g}\frac{Gm_k}{c^2r_{jk}}-A_{jk}^{p}\frac{e_je_k}{m_jc^2r_{jk}}\right)\right],\end{aligned} \tag{46}$$

where

$$\mu_{ij}=m_im_j/(m_i+m_j)\quad\text{and}\quad M_{ij}=m_i+m_j. \tag{47}$$

B. n-body Lagrangian

Starting with the n-body Bażański Lagrangian[1,2] in Bażański coordinates and using Eq. (40) gives us the n-body Lagrangian in the new coordinates, as

$$\begin{aligned}\mathscr{L}=&\sum_{i=1}^{n}(-m_ic^2+\tfrac{1}{2}m_iv_i^2+\tfrac{1}{8}m_iv_i^4/c^2)\\ &+\tfrac{1}{2}\sum_{i,j=1}^{n}{}'\left[\frac{Gm_im_j}{r_{ij}}\left(1+(3-2A_{ij}^{g})\frac{v_i^2}{c^2}+(2A_{ij}^{g}-\tfrac{7}{2})\right.\right.\\ &\left.\times\frac{\mathbf{v}_i\cdot\mathbf{v}_j}{c^2}-(\tfrac{1}{2}+2A_{ij}^{g})\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\mathbf{r}_{ij})}{c^2r_{ij}^2}+2A_{ij}^{g}\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})^2}{c^2r_{ij}^2}\right)\end{aligned}$$

$$-\frac{e_i e_j}{r_{ij}}\Big(1 - 2A^p_{ij}\frac{v_i^2}{c^2} + (2A^p_{ij} - \tfrac{1}{2})\frac{\mathbf{v}_i\cdot\mathbf{v}_j}{c^2}$$

$$-(\tfrac{1}{2} + 2A^p_{ij})\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\mathbf{r}_{ij})}{c^2 r_{ij}^2} + 2A^p_{ij}\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})^2}{c^2 r_{ij}^2}\Big)$$

$$+ (\alpha_{ijg} - \tfrac{1}{2})\frac{G^2 m_i m_j M_{ij}}{c^2 r_{ij}^2} + \alpha_{ijp}\frac{e_i^2 e_j^2}{\mu_{ij} c^2 r_{ij}^2}$$

$$+ (1 - \alpha_{ijg} - \alpha_{ijp})\frac{G e_i e_j M_{ij}}{c^2 r_{ij}^2} - \frac{G(e_i^2 m_j + e_j^2 m_i)}{2c^2 r_{ij}^2}\Big]$$

$$+ \sum_{i,j,k=1}^{n}{}' \Big\{ -\frac{G^2 m_i m_j m_k}{2c^2 r_{ij} r_{ik}} + \frac{G e_i e_j m_k}{2c^2}$$

$$\times\Big(\frac{1}{r_{ij} r_{ik}} + \frac{1}{r_{ji} r_{jk}} - \frac{1}{r_{ki} r_{kj}}\Big)$$

$$+ \Big[A^g_{ik}(G^2 m_i m_j m_k - G e_i e_j e_k)$$

$$+ A^p_{ik}\Big(\frac{e_i^2 e_j e_k}{m_i} - G e_i m_j e_k\Big)\Big]\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{c^2 r_{ij}^3 r_{ik}}\Big\}, \tag{48}$$

where Σ' means that no two summation indices are the same. If we set $\alpha_{ijg} = \alpha_{ijp} = 0$ (which implies that $A^g_{ij} = A^p_{ij} = 0$) in Eq. (48) we will obtain the n-body Bażański Lagrangian in *Bażański coordinates*. The rest-energy terms (which do not effect the equations of motion) have been included in the above Lagrangian and are needed for our treatment of the center of inertia in the following section.

IV. CENTER OF INERTIA

Let us start with an n-body Lagrangian of the form $\mathscr{L} = \mathscr{L}(\mathbf{r}_{ij}, \mathbf{v}_k)$, where $\mathscr{L}$ is a scalar in three dimensions. We define, as usual, $\mathbf{P}_i \equiv \partial\mathscr{L}/\partial\mathbf{v}_i$ and $\mathbf{L}_i \equiv \mathbf{r}_i \times \mathbf{P}_i$. The total energy[17]

$$\mathscr{E} = \sum_{i=1}^{n}\mathbf{P}_i\cdot\mathbf{v}_i - \mathscr{L} \tag{49}$$

is conserved since $\mathscr{L}$ is not an explicit function of time. The total momentum

$$\mathbf{P} = \sum_{i=1}^{n}\mathbf{P}_i \tag{50}$$

is conserved since $\mathscr{L}$ is a function of the *differences* in coordinates $\mathbf{r}_{ij}$. The total angular momentum[17]

$$\mathbf{L} = \sum_{i=1}^{n}\mathbf{L}_i \tag{51}$$

is conserved since $\mathscr{L}$ is a scalar in three dimensions. Since $\mathbf{P}_i$ and $\mathbf{L}_i$ are explicitly defined there is no problem in finding $\mathscr{E}$, $\mathbf{P}$, or $\mathbf{L}$. Let us note that, so far, an $\mathscr{E}_i$ where

$$\mathscr{E} = \sum_{i=1}^{n}\mathscr{E}_i \tag{52}$$

has not been defined. If we did have such an $\mathscr{E}_i$ we could then define the center of inertia $\mathbf{r}_{\mathrm{CI}}$ by the definition[18]

$$\mathscr{E}\mathbf{r}_{\mathrm{CI}} \equiv \sum_{i=1}^{n}\mathscr{E}_i\mathbf{r}_i. \tag{53}$$

We also wish to require that

$$(\mathscr{E}/c^2)\mathbf{v}_{\mathrm{CI}} = \mathbf{P}, \tag{54}$$

which will hold if

$$\frac{d}{dt}\Big(\sum_{i=1}^{n}\frac{\mathscr{E}_i}{c^2}\mathbf{r}_i\Big) = \sum_{i=1}^{n}\mathbf{P}_i. \tag{55}$$

We thus must find $\mathscr{E}_i$ such that Eqs. (52) *and* (55) *are satisfied*. In order to show that Eq. (55) is satisfied we will always have to use the equations of motion

$$\dot{\mathbf{P}}_i - \partial\mathscr{L}/\partial\mathbf{r}_i = 0. \tag{56}$$

In order to have $\mathscr{E}_i$ which satisfies Eqs. (52) and (55) (even in the one-body case where $\mathscr{E}_1 = \mathscr{E}$) the Lagrangian must satisfy additional symmetry conditions.[6] We *will not* be concerned with such symmetry conditions in this paper. We *will* be interested in finding (and do find) $\mathscr{E}_i$ which satisfy Eqs. (52) and (55) for some particular Lagrangians.

Let us first look at a very simple case[6] that works out exactly to all orders (in c).

A. Free particle case

The Lagrangian is

$$\mathscr{L} = \sum_{i=1}^{n} - m_i c^2(1 - v_i^2/c^2)^{1/2}, \tag{57}$$

from which it follows that

$$\mathbf{P}_i = m_i\mathbf{v}_i/(1 - v_i^2/c^2)^{1/2}, \tag{58}$$

$$\mathscr{E} = \sum_{i=1}^{n} m_i c^2/(1 - v_i^2/c^2)^{1/2}. \tag{59}$$

We shall choose

$$\mathscr{E}_i = m_i c^2/(1 - v_i^2/c^2)^{1/2}, \tag{60}$$

and note that Eq. (52) is satisfied. We also have

$$\frac{d}{dt}\Big(\frac{\mathscr{E}_i}{c^2}\mathbf{r}_i\Big) = \frac{m_i\mathbf{v}_i}{(1 - v_i^2/c^2)^{1/2}} + \frac{m_i\mathbf{r}_i(\mathbf{v}_i\cdot\mathbf{a}_i)/c^2}{(1 - v_i^2/c^2)^{3/2}}. \tag{61}$$

Since $\mathbf{a}_i = 0$ from the equations of motion, Eq. (55) follows from Eq. (61) after summing.

B. Post-Newtonian case for general relativity (with charge)

For this case we shall use the Lagrangian of Eq. (48). We will have to check that both sides of Eq. (55) are in agreement to order c^{-2}. Thus, for the left-hand side we will need the rest energy and Newtonian kinetic and potential energy terms (but *not* post-Newtonian energy terms), while for the right-hand side we will need the Newtonian and post-Newtonian momentum terms.

Let us try $\mathscr{E}_i$ of the form

$$\mathscr{E}_i = m_i c^2 + \tfrac{1}{2} m_i v_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} \Big(-(\tfrac{1}{2} + b_{ijg}) \frac{G m_i m_j}{r_{ij}} + (\tfrac{1}{2} + b_{ijp}) \frac{e_i e_j}{r_{ij}} \Big), \quad (62)$$

where b_{ijg} and b_{ijp} are to be determined. It follows that

$$\mathscr{E} = \sum_{i=1}^{n} \Big(m_i c^2 + \frac{1}{2} m_i v_i^2 \Big) + \sum_{i,j=1}^{n}{}' \Big(-\frac{1}{2} \frac{G m_i m_j}{r_{ij}} + \frac{1}{2} \frac{e_i e_j}{r_{ij}} \Big), \quad (63)$$

if we require that

$$b_{ijg} = -b_{jig}, \quad b_{ijp} = -b_{jip}. \quad (64)$$

Clearly, Eq. (63) is the energy corresponding to the Lagrangian of Eq. (48) to the order that we need.

Using Eq. (62) in the left-hand side of Eq. (55) and eliminating the acceleration term by using the equations of motion

$$m_i \mathbf{a}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \Big(-\frac{G m_i m_j}{r_{ij}^2} + \frac{e_i e_j}{r_{ij}^2} \Big) \frac{\mathbf{r}_{ij}}{r_{ij}}, \quad (65)$$

we obtain

$$\frac{d}{dt}\Big(\sum_{i=1}^{n} \frac{\mathscr{E}_i}{c^2} \mathbf{r}_i \Big) = \sum_{i=1}^{n} (m_i \mathbf{v}_i + \tfrac{1}{2} m_i v_i^2 \mathbf{v}_i / c^2) + \sum_{i,j=1}^{n}{}' \Big[-\frac{G m_i m_j}{r_{ij}} \Big((\tfrac{1}{2} + b_{ijg}) \frac{\mathbf{v}_i}{c^2} + (\tfrac{1}{2} - b_{ijg}) \frac{(\mathbf{v}_i \cdot \mathbf{r}_{ij})}{c^2 r_{ij}^2} \mathbf{r}_{ij} \Big) + \frac{e_i e_j}{r_{ij}} \Big((\tfrac{1}{2} + b_{ijp}) \frac{\mathbf{v}_i}{c^2} + (\tfrac{1}{2} - b_{ijp}) \frac{(\mathbf{v}_i \cdot \mathbf{r}_{ij})}{c^2 r_{ij}^2} \mathbf{r}_{ij} \Big) \Big]. \quad (66)$$

We next find that $\Sigma_{i=1}^{n} \mathbf{P}_i$ is exactly the same as the right-hand side of Eq. (66) if we put

$$b_{ijg} = \alpha_{ijg}(a_{ijg} - a_{jig}), \quad b_{ijp} = \alpha_{ijp}(a_{ijp} - a_{jip}), \quad (67)$$

which is consistent with Eq. (64).

Thus in the special case of Bażański coordinates (α's $= 0$) or coordinates systems where α's$\neq 0$ but a's $= 1$, the b's $= 0$ and Eq. (62) gives the $\frac{1}{2},\frac{1}{2}$ split (i.e., the potential energy terms $-G m_i m_j / r_{ij}$ and $e_i e_j / r_{ij}$ are split equally between the particles i and j).

On the other hand, for coordinate systems where α's$\neq 0$ and a's$\neq 1$ we do *not* get the $\frac{1}{2},\frac{1}{2}$ split.

C. Post-Newtonian case with γ and β (without charge)

We shall next consider the n-body (uncharged) post-Newtonian Lagrangian with PPN parameters γ and β which can be written as[19,20]

$$\mathscr{L} = \sum_{i=1}^{n} (-m_i c^2 + \tfrac{1}{2} m_i v_i^2 + \tfrac{1}{8} m_i v_i^4 / c^2) + \tfrac{1}{2} \sum_{i,j=1}^{n}{}' \Big[\frac{G m_i m_j}{r_{ij}} \Big(1 + (1 + 2\gamma) \frac{v_i^2}{c^2} - (\tfrac{3}{2} + 2\gamma) \frac{\mathbf{v}_i \cdot \mathbf{v}_j}{c^2} - \frac{1}{2} \frac{(\mathbf{v}_i \cdot \mathbf{r}_{ij})(\mathbf{v}_j \cdot \mathbf{r}_{ij})}{c^2 r_{ij}^2} \Big) + (\tfrac{1}{2} - \beta) \frac{G^2 m_i m_j M_{ij}}{c^2 r_{ij}^2} \Big] + \sum_{i,j,k=1}^{n}{}' \Big((\tfrac{1}{2} - \beta) \frac{G^2 m_i m_j m_k}{c^2 r_{ij} r_{ik}} \Big), \quad (68)$$

in the standard coordinate system (i.e., the Lagrangian becomes the same as EIH Lagrangian in EIH coordinates when $\gamma = \beta = 1$).

If we use $\mathscr{E}_i$ in the form of the $\frac{1}{2},\frac{1}{2}$ split, that is

$$\mathscr{E}_i = m_i c^2 + \tfrac{1}{2} m_i v_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} \Big(-\frac{1}{2} \frac{G m_i m_j}{r_{ij}} \Big), \quad (69)$$

we find that Eq. (55) is satisfied. It should be noted that while $\mathbf{P}_i$ contains γ, $\Sigma_{i=1}^{n} \mathbf{P}_i$ does not. We find that

$$\mathbf{P}_i = m_i \mathbf{v}_i + \tfrac{1}{2} m_i v_i^2 \mathbf{v}_i / c^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} \Big[-\frac{G m_i m_j}{r_{ij}} \Big(-(1 + 2\gamma) \frac{\mathbf{v}_i}{c^2} + (\tfrac{3}{2} + 2\gamma) \frac{\mathbf{v}_j}{c^2} + \frac{1}{2} \frac{(\mathbf{v}_j \cdot \mathbf{r}_{ij})}{c^2 r_{ij}^2} \mathbf{r}_{ij} \Big) \Big], \quad (70)$$

$$\sum_{i=1}^{n} \mathbf{P}_i = \sum_{i=1}^{n} (m_i \mathbf{v}_i + \tfrac{1}{2} m_i v_i^2 \mathbf{v}_i / c^2) + \sum_{i,j=1}^{n}{}' \Big[-\frac{G m_i m_j}{r_{ij}} \Big(\frac{1}{2} \frac{\mathbf{v}_i}{c^2} + \frac{1}{2} \frac{(\mathbf{v}_i \cdot \mathbf{r}_{ij})}{c^2 r_{ij}^2} \mathbf{r}_{ij} \Big) \Big]. \quad (71)$$

Will[21] has shown, for a perfect fluid system, that the $\frac{1}{2},\frac{1}{2}$ split holds in any fully-conservative theory of gravity (these theories contain only the PPN parameters γ and β). However, this result as well as our result depends on the fact that a particular coordinate system has been used.

V. CONCLUSION

We have found a way to generalize the propagator [see Eqs. (14) and (24)] by making it a function of the parameters a_{12} and a_{21} as well as the parameter α which had been included before.[1,4] Thus, it is now possible to directly obtain Hamiltonians from field theory in a wider variety of coordinate systems than had previously been possible. The relationship between the various coordinate systems has been given by our n-body coordinate transformations of Eq. (40).

We have also found the center of inertia for two cases involving post-Newtonian n-body Lagrangians. For the case of the Lagrangian with PPN parameters γ and β in standard coordinates [see Eq. (68)] we found that the $\frac{1}{2},\frac{1}{2}$ split of the potential energy was correct as had previously been found[5,6,7] to be the case for the Bażański, EIH, and Darwin Lagrangians in standard coordinates. However, for the Bażański Lagrangian of Eq. (48), which is in a more general coordinate system, the split was found *not* to be $\frac{1}{2},\frac{1}{2}$ in general, but was found instead to be in accordance with Eqs. (62) and (67).

APPENDIX

In this Appendix we shall fill in some of the intermediate steps leading to Eqs. (5) and (6). Our field theory notation is similar to that of Gupta,[22] except that we are using Gaussian units instead of rationalized Gaussian units. Let U and U be complex scalar fields with mass m_1, charge e_1, and mass m_2, charge e_2, respectively. The interaction terms (using ordered products[22]) with the photon field, A_μ, are[22]

$$:L': = \frac{ie_1}{c\hbar}:\left(\frac{\partial U_1^*}{\partial x_\mu}U_1 - \frac{\partial U_1}{\partial x_\mu}U_1^*\right)A_\mu: - \frac{e_1^2}{c^2\hbar^2}:U_1^*U_1A_\mu^2: + \frac{ie_2}{c\hbar}:\left(\frac{\partial U_2^*}{\partial x_\mu}U_2 - \frac{\partial U_2}{\partial x_\mu}U_2^*\right)A_\mu: - \frac{e_2^2}{c^2\hbar^2}:U_2^*U_2A_\mu^2:, \tag{A1}$$

and the interaction terms with the graviton field, $h_{\mu\nu}$, are[11,12] (to order κ)

$$:L': = -\tfrac{1}{2}\kappa:\left(\frac{\partial U_1^*}{\partial x_\mu}\frac{\partial U_1}{\partial x_\nu} + \frac{\partial U_1^*}{\partial x_\nu}\frac{\partial U_1}{\partial x_\mu} - \delta_{\mu\nu}\frac{\partial U_1^*}{\partial x_\rho}\frac{\partial U_1}{\partial x_\rho} - \delta_{\mu\nu}\lambda_1^2U_1^*U_1\right)h_{\mu\nu}:$$

$$-\tfrac{1}{2}\kappa:\left(\frac{\partial U_2^*}{\partial x_\mu}\frac{\partial U_2}{\partial x_\nu} + \frac{\partial U_2^*}{\partial x_\nu}\frac{\partial U_2}{\partial x_\mu} - \delta_{\mu\nu}\frac{\partial U_2^*}{\partial x_\rho}\frac{\partial U_2}{\partial x_\rho} - \delta_{\mu\nu}\lambda_2^2U_2^*U_2\right)h_{\mu\nu}:. \tag{A2}$$

The contractions are[12,22,23]

$$A_\mu^{\cdot}(x)A_\nu^{\cdot}(x') = -4\pi i c\hbar\delta_{\mu\nu}D_F(x-x'), \tag{A3}$$

$$h_{\mu\nu}^{\cdot}(x)h_{\lambda\rho}^{\cdot}(x') = -ic\hbar(\delta_{\mu\lambda}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\lambda} - \delta_{\mu\nu}\delta_{\lambda\rho})D_F(x-x'), \tag{A4}$$

where

$$D_F(x-x') = \lim_{\epsilon\to+0}\frac{1}{(2\pi)^4}\int dk\, e^{ik(x-x')}\frac{1}{k^2 - i\epsilon}. \tag{A5}$$

We also have[22]

$$S_2 = \frac{-1}{2c^2\hbar^2}\int dx\int dx'\, T[:H'(x)::H'(x'):], \tag{A6}$$

where $:H':$ may be replaced[22,24] by $-:L':$. Using Eqs. (A1) and (A2) in Eq. (A6) we get, respectively, the one-photon and one-graviton exchange results

$$S_{2p} = \frac{e_1e_2}{c^4\hbar^4}\int dx\int dx':\left(\frac{\partial U_1^*(x)}{\partial x_\mu}U_1(x) - \frac{\partial U_1(x)}{\partial x_\mu}U_1^*(x)\right)A_\mu^{\cdot}(x)\left(\frac{\partial U_2^*(x')}{\partial x'_\nu}U_2(x') - \frac{\partial U_2(x')}{\partial x'_\nu}U_2^*(x')\right)A_\nu^{\cdot}(x'):, \tag{A7}$$

$$S_{2g} = -\frac{\kappa^2}{4c^2\hbar^2}\int dx\int dx':\left(\frac{\partial U_1^*(x)}{\partial x_\mu}\frac{\partial U_1(x)}{\partial x_\nu} + \frac{\partial U_1^*(x)}{\partial x_\nu}\frac{\partial U_1(x)}{\partial x_\mu} - \delta_{\mu\nu}\frac{\partial U_1^*(x)}{\partial x_\rho}\frac{\partial U_1(x)}{\partial x_\rho} - \delta_{\mu\nu}\lambda_1^2U_1^*(x)U_1(x)\right)h_{\mu\nu}^{\cdot}(x)$$

$$\times\left(\frac{\partial U_2^*(x')}{\partial x'_\alpha}\frac{\partial U_2(x')}{\partial x'_\beta} + \frac{\partial U_2^*(x')}{\partial x'_\beta}\frac{\partial U_2(x')}{\partial x'_\alpha} - \delta_{\alpha\beta}\frac{\partial U_2^*(x')}{\partial x'_\lambda}\frac{\partial U_2(x')}{\partial x'_\lambda} - \delta_{\alpha\beta}\lambda_2^2U_2^*(x')U_2(x')\right)h_{\alpha\beta}^{\cdot}(x'):. \tag{A8}$$

Using Eqs. (A3), (A4), and (A5) in Eq. (A7) and (A8) along with[22,25]

$$U_1(x) = (c\hbar/2p_0V)^{1/2}a_1(\mathbf{p})e^{ipx}, \tag{A9}$$

$$U_1^*(x) = (c\hbar/2p'_0V)^{1/2}a_1^*(\mathbf{p}')e^{-ip'x}, \tag{A10}$$

$$U_2(x') = (c\hbar/2q_0V)^{1/2}a_2(\mathbf{q})e^{iqx'}, \tag{A11}$$

$$U_2^*(x') = (c\hbar/2q'_0V)^{1/2}a_2^*(\mathbf{q}')e^{-iq'x'}, \tag{A12}$$

and then integrating gives us [after comparing with Eq. (4)] the results of Eqs. (6) and (5).

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[8]Sometimes it is convenient to set $x = 4\alpha + 1$ as is done in Refs. 1 and 4.
[9]See Appendix of Ref. 1. Note also that center of mass is *not* the same thing as center of inertia according to the way we define these quantities. The position $\mathbf{r}_{CM}$ is defined for a two-body problem ($\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_{CM} = \nu_1\mathbf{r}_1 + \nu_2\mathbf{r}_2$ with $\nu_1 + \nu_2 = 1$; which gives us $\mathbf{P} = \nu_2\mathbf{P}_1 - \nu_1\mathbf{P}_2$, $\mathbf{P}_{CM} = \mathbf{P}_1 + \mathbf{P}_2$) in which case $\dot{\mathbf{v}}_{CM} \neq 0$, $\dot{\mathbf{P}}_{CM} = 0$. The position $\mathbf{r}_{CI}$ is defined for the n-body problem (see Sec. IV of this paper) in which case $\dot{\mathbf{v}}_{CI} = 0$.
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[13] Note that $a_{21}\not\equiv a_{21}(m_1,m_2)$.
[14] See Eqs. (3b) and (4b) in Appendix of Ref. 1.
[15] See Eqs. (3a) and (4a) and discussion in Appendix of Ref. 1.
[16] In Sec. IV of Ref. 1 we had an α_g without *ij* indices and had thought of this quantity as being a totally symmetric function of *all* the masses. We prefer the procedure of this paper which is *more* general in the sense of the *ij* indices, but *less* general in that it involves only the *two* masses m_i and m_j. In this paper one could make the following generalization: Let α_{ijg} and a_{0ijg} be replaced, respectively, by $\alpha_{ijg}\bar{\alpha}_g$ and $a_{0ijg}\bar{a}_g$ where $\bar{\alpha}_g$ and $\bar{a}_g$ are totally symmetric functions of *all* the masses. Everything said here also holds for p replacing g.
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