CENTER OF INERTIA FOR THE POST-NEWTONIAN n-BODY PROBLEM IN GRAVITATION WITH PPN PARAMETERS γ AND β

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We find the center of inertia for the case of the post-Newtonian n-body lagrangian (in standard coordinates) with PPN parameters γ and β, and find that the potential energy term $-Gm_im_j/r_{ij}$ splits equally between the particles i and j.

It is well known [1–4] that in finding the center of inertia for post-Newtonian n-body problems, involving Darwin or Einstein–Infeld–Hoffmann (EIH) or Bazański lagrangians (in standard coordinates [5]), the potential energy terms $-Gm_im_j/r_{ij}$ and e_ie_j/r_{ij} must be split equally between the particles i and j.

We shall show that this $\frac{1}{2},\frac{1}{2}$ split, as we shall call it, holds also for the case of the post-Newtonian n-body lagrangian (in standard coordinates) with parameterized post-Newtonian (PPN) parameters γ and β. This lagrangian can be written as [6,7]

$$\mathcal{L}=\sum_{i=1}^{n}\left(-m_ic^2+\tfrac{1}{2}m_iv_i^2+\tfrac{1}{8}m_iv_i^4/c^2\right)$$

$$+\tfrac{1}{2}\sum_{i,j=1}^{n}{}' \left\{\frac{Gm_im_j}{r_{ij}}\left[1+(1+2\gamma)\frac{v_i^2}{c^2}-(\tfrac{3}{2}+2\gamma)\frac{\mathbf{v}_i\cdot\mathbf{v}_j}{c^2}\right.\right.$$

$$\left.\left.-\tfrac{1}{2}\frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\mathbf{r}_{ij})}{c^2r_{ij}^2}+(\tfrac{1}{2}-\beta)\frac{G^2m_im_j(m_i+m_j)}{c^2r_{ij}}\right]\right\}$$

$$+\sum_{i,j,k=1}^{n}{}'\left\{(\tfrac{1}{2}-\beta)\frac{G^2m_im_jm_k}{c^2r_{ij}r_{ik}}\right\}, \qquad (1)$$

where $\mathbf{r}_{ij}\equiv\mathbf{r}_i-\mathbf{r}_j$ and Σ' means that no two summation indices are the same. By standard coordinates we mean that the above lagrangian becomes the same as the EIH lagranian in EIH coordinates when $\gamma=\beta=1$. The rest energy terms (which do not effect the equations of motion) have been included in the above lagrangian since they are needed for the treatment of the center of inertia.

We define, as usual, $\mathbf{P}_i\equiv\partial\mathcal{L}/\partial\mathbf{v}_i$ and $\mathbf{L}_i\equiv\mathbf{r}_i\times\mathbf{P}_i$. Then, the total energy $\mathcal{E}$, the total momentum $\mathbf{P}$, and the total angular momentum $\mathbf{L}$ will beconserved [8], where

$$\mathcal{E}=\sum_{i=1}^{n}\mathbf{P}_i\cdot\mathbf{v}_i-\mathcal{L}, \qquad (2)$$

$$\mathbf{P}=\sum_{i=1}^{n}\mathbf{P}_i, \qquad (3)$$

$$\mathbf{L}=\sum_{i=1}^{n}\mathbf{L}_i. \qquad (4)$$

The center of inertia $\mathbf{r}_{\mathrm{CI}}$ is defined by the relation

$$\mathcal{E}r_{\rm CI} \equiv \sum_{i=1}^{n} \mathcal{E}_i r_i \ , \tag{5}$$

where

$$\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}_i \ . \tag{6}$$

We also demand that

$$(\mathcal{E}/c^2)\boldsymbol{v}_{\rm CI} = \boldsymbol{P} \ , \tag{7}$$

which will hold if

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{n} \frac{\mathcal{E}_i}{c^2} r_i\right] = \sum_{i=1}^{n} \boldsymbol{P}_i \ . \tag{8}$$

We thus must find $\mathcal{E}_i$ such that eqs. (6) and (8) are satisfied.

We will have to check that both sides of eq. (8) are in agreement to order c^{-2}. Thus, for the left hand side we will need the rest energy and Newtonian kinetic and potential energy terms (but *not* the post-Newtonian energy terms) while for the right hand side we will need the Newtonian and post-Newtonian momentum terms.

If we use $\mathcal{E}_i$ in the form of the $\frac{1}{2}, \frac{1}{2}$ split, that is

$$\mathcal{E}_i = m_i c^2 + \tfrac{1}{2} m_i v_i^2 + \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-\tfrac{1}{2}\frac{G m_i m_j}{r_{ij}}\right), \tag{9}$$

in the left hand side of eq. (8) and eliminate the acceleration terms by using the equations of motion

$$m_i \boldsymbol{a}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \left(-\frac{G m_i m_j}{r_{ij}^3} \boldsymbol{r}_{ij}\right), \tag{10}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{i=1}^{n} \frac{\mathcal{E}_i}{c^2} r_i\right] = \sum_{i=1}^{n} (m_i \boldsymbol{v}_i + \tfrac{1}{2} m_i v_i^2 \boldsymbol{v}_i/c^2)$$

$$+ \sum_{i,j=1}^{n}{}' \left\{-\frac{G m_i m_j}{r_{ij}}\left[\tfrac{1}{2}\frac{\boldsymbol{v}_i}{c^2} + \tfrac{1}{2}\frac{(\boldsymbol{v}_i \cdot \boldsymbol{r}_{ij})}{c^2 r_{ij}^2} \boldsymbol{r}_{ij}\right]\right\} . \tag{11}$$

We next find that

$$\boldsymbol{P}_i = m_i \boldsymbol{v}_i + \tfrac{1}{2} m_i v_i^2 \boldsymbol{v}_i/c^2$$

$$+ \sum_{\substack{j=1 \\ j\neq i}}^{n} \left\{-\frac{G m_i m_j}{r_{ij}}\left[-(1+2\gamma)\frac{\boldsymbol{v}_i}{c^2} + (\tfrac{3}{2}+2\gamma)\frac{\boldsymbol{v}_j}{c^2}\right.\right.$$

$$\left.\left. + \tfrac{1}{2}\frac{(\boldsymbol{v}_j \cdot \boldsymbol{r}_{ij})}{c^2 r_{ij}^2} \boldsymbol{r}_{ij}\right]\right\}, \tag{12}$$

so that $\Sigma_{i=1}^{n} \boldsymbol{P}_i$ is exactly the same as the right hand side of eq. (11) and, thus, eq. (8) is satisfied. It should be noted that while $\boldsymbol{P}_i$ contains γ, $\Sigma_{i=1}^{n} \boldsymbol{P}_i$ does not.

In conclusion, we would like to point out that the $\frac{1}{2}, \frac{1}{2}$ split holds only for certain coordinate systems. We have found [9] coordinate systems where something other than the $\frac{1}{2}, \frac{1}{2}$ split occurs.

References

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