POSITION OPERATORS FOR SYSTEMS EXHIBITING THE SPECIAL RELATIVISTIC RELATION BETWEEN MOMENTUM AND VELOCITY

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We have previously shown [1] that if the position operator is defined as in ref. [2], the movement of the *mean* position of a free particle obeys the classical equation $\mathbf{v} = \mathbf{P}/P_0$ where P_0 is the total energy, including the rest mass. Conversely, it will be demonstrated here that the validity of this equation implies that, for spinless particles, the position operator is that of ref. [2]. For spin 1/2 particles, however, another choice is also possible (eq. (7)). The corresponding value of the orbital angular momentum in the latter case is unity, whereas for the state of ref. [2] it is zero.

1. Possible definitions of the position operator. In a recent communication [1], we have shown that, for an elementary system, the choice of position operator defined in ref. [2], $\mathbf{q}(t)$ say, leads, for an arbitrary positive energy state, to the classical connection between the expectation values of positions and momenta (in our units $\hbar = c = 1$):

$$\langle \mathbf{q}(t)\rangle = \langle \mathbf{q}(0)\rangle + t\langle \mathbf{v}\rangle, \tag{1}$$

where [2] (we use the Heisenberg picture and, as variables of the wave function, the components of the momentum $\mathbf{P}$)

$$\mathbf{q}(t) = \mathrm{i}\,\mathrm{e}^{\mathrm{i}P_0 t}\,[\nabla_P - \mathbf{P}/2P_0^2]\,\mathrm{e}^{-\mathrm{i}P_0 t}, \tag{2}$$

$$\mathbf{v} = (\mathbf{P}/P_0), \tag{3}$$

$$P_0 = (m^2+P^2)^{1/2} \equiv (m^2+\mathbf{P}^2)^{1/2}. \tag{4}$$

The state vector of the state localized at time t at the point $\mathbf{r}$ is $\mathrm{e}^{\mathrm{i}P_0 t - \mathrm{i}\mathbf{P}\cdot\mathbf{r}}P_0^{1/2}\,\xi(s)$, where $\xi(s)$ is a $2s+1$ valued arbitrary function of the spin coordinate s.

Eq. (1) is valid for an arbitrary (positive energy) state of the system. The relations of the quantum-mechanical momentum operators to the velocity operators is the one postulated by the special theory of relativity. The position operators $\mathbf{q}(t)$ are defined by eq. (2) for all systems of non-zero mass and arbitrary spin s, as well as systems of zero mass and $s = 0$ and $1/2$.

We turn now to the question of whether the position operator of eq. (2) is unique. Our discussion will be limited to the cases of $s = 0$ and $1/2$. A priori, it is clear that we could add to $\mathbf{q}(t)$ any self-adjoint vector operator which commutes with P_0 and still have eq. (3) fulfilled. For $s = 0$, the only possible vector is $\mathbf{P}\,f(P^2)$ but this is not acceptable if it is postulated that the position operator be invariant under time reversal. For $s = 1/2$, $\xi(s)$ is two-valued, and other possibilities which must be considered are $f(P^2)\,(\boldsymbol{\sigma}\cdot\mathbf{P})\mathbf{P}$, $f(P^2)\,\boldsymbol{\sigma}\times\mathbf{P}$, and $f(P^2)\,\boldsymbol{\sigma}$, where the components of $\boldsymbol{\sigma}$ denote the Pauli spin matrices. Recalling that under parity transformation

$$\boldsymbol{\sigma}\to\boldsymbol{\sigma},\quad \mathbf{P}\to-\mathbf{P},\quad \mathbf{q}\to-\mathbf{q}, \tag{5}$$

and that under time reversal

$$\boldsymbol{\sigma}\to-\boldsymbol{\sigma},\quad \mathbf{P}\to-\mathbf{P},\quad \mathbf{q}\to\mathbf{q}, \tag{6}$$

we note that the first and third possibilities are not invariant under time reversal. On the other hand, the second possibility behaves properly under both time reversal and parity transformation. If we add the further postulate that the components of the new position operator, to be denoted by $\mathbf{Q}$, commute with each other, as do those of $\mathbf{q}$, we are led to the conclusion that the only

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operator, apart from the $\boldsymbol{q}$ of eq. (2), which satisfies our conditions is, as can be shown by a little computation,

$$Q = q + (P \times \sigma)/P^2. \tag{7}$$

The state vector ψ_0 which represents the state localized at time $t = 0$ at the origin of the coordinate system obeys the equation $Q(0)\psi_0 = 0$. It then follows from eqs. (2) and (7) that

$$(\mathrm{i}\nabla_P - \mathrm{i}P/2P_0^2 + (P \times \sigma)/P^2)\psi_0 = 0. \tag{8}$$

The solution of eq. (8) can be obtained, from the state vector of the state localized at the origin if the position operator of eq. (2) is accepted, by applying to it the operator $\phi = (\boldsymbol{\sigma} \cdot \boldsymbol{P})/P$. In fact, one can easily verify by means of the identity

$$(P \times \sigma)(P \cdot \sigma) = -\mathrm{i}P^2\sigma + \mathrm{i}(P \cdot \sigma)P, \tag{9}$$

that

$$\psi_0 = \phi P_0^{1/2} = (\sigma \cdot P/P)P_0^{1/2} \tag{10}$$

satisfies eq. (8).

Using eq. (8) we can obtain the localized wave function in coordinate space, viz.,

$$\Psi_0(r) = (2\pi)^{-3/2} \int \psi(P,s) \mathrm{e}^{\mathrm{i}r\cdot P}\, \mathrm{d}P/P_0$$

$$= (2\pi)^{-3} \int \phi\, \mathrm{e}^{\mathrm{i}r\cdot P}\, \mathrm{d}P/P_0^{1/2}. \tag{11}$$

In the case where ϕ is replaced by unity, we have the localized wave function $\psi_{\mathrm{NW}}(r)$ of ref. [2]. The latter is a well-behaved function of r which goes as e^{-mr} for large r and as $r^{-5/2}$ for small r. For the $\Psi(r)$ of eq. (11), one also finds that it is proportional to $r^{-5/2}$ for $m = 0$. In the case of $m \neq 0$, one also verifies that $\Psi(r)$ behaves regularly.

We turn now to the question of orthogonality. If we subject the localized state of ref. [2], $P_0^{1/2}$, to any finite spatial displacement it becomes orthogonal to the original state. Using the fact that $(\boldsymbol{\sigma} \cdot \boldsymbol{P})^2 = P^2$, it follows immediately that the state $\psi(P,s)$ has the same property. However, it may be verified that if we subject $P_0^{1/2}$ to any finite displacement the resulting state is not orthogonal to $\psi_0(P,s)$. A feature shared by both types of states is that if we subject either to a finite temporal displacement, one finds a finite probability for the system's position to be outside the light-cone which originated at the initial position of the particle.

This follows already from the remarks of Fleming [3] and Hegerfeldt [4].

2. The angular momenta and the relation between the two position operator definitions. If one replaces in the usual orbital angular momentum operators

$$\mathcal{L}_{ij} = P_j r_i - P_i r_j, \tag{12}$$

the $\boldsymbol{r}$ by the $\boldsymbol{q}$ of eq. (2) (with $t = 0$), which corresponds to the original definition of the position operator, one obtains the usual

$$\mathcal{L}_{ij} = \mathrm{i}(P_j\, \partial/\partial P_i - P_i\, \partial/\partial P_j), \tag{12a}$$

since the term additional to the ∇_P term in eq. (2) commutes with the P_i. If one uses for the $\boldsymbol{r}$ the $\boldsymbol{Q}$ given by eq. (7), one obtains a more complicated expression,

$$\mathcal{L}_{ij} = \mathrm{i}\left(P_j \frac{\partial}{\partial P_i} - P_i \frac{\partial}{\partial P_j}\right) + \sum_k \left[\frac{P\cdot\sigma}{P^2} P_k - \sigma_k\right]\epsilon_{ijk}, \tag{13}$$

where ϵ_{ijk} is 1 if ijk is an even, and -1 if it is an odd permutation of 1,2,3. Because of the complicated nature of eq. (13), it is questionable whether one should define an orbital angular momentum in this case. It is worth observing, however, that if eq. (12a) is adopted, the orbital angular momentum of the state localized at the origin is 0 for the original definition of this localized state $(P_0^{1/2})$, it is 1 with the definition (eq. (10)) considered here. We remark that the investigations of ref. [2] were restricted to $l = 0$ states so that, in that context, the position operator obtained there is unique.

Let us observe, finally, that the operator $\phi = (\boldsymbol{\sigma} \cdot \boldsymbol{P})/P$ which transforms the original expression for the localized state into the one here considered, is both self-adjoint and unitary, $\phi^2 = 1$. Since ϕ commutes with all displacement operators, and since every state can be considered as a linear combination of spatially displaced ψ_0, and since the value of Q for such displaced states is equal to the displacement, $\boldsymbol{Q}$ of eq. (7) can be obtained from $\boldsymbol{q}$ by transformation with ϕ:

$$Q = \phi q \phi^{-1} = \phi q \phi. \tag{14}$$

This can be verified, of course, directly also. It then also follows, considering that $\boldsymbol{P}$ commutes with ϕ, that for any function $f(P,q)$,

$$f(P,Q) = \phi f(P,q)\phi. \tag{14a}$$

In fact eq. (13) can be verified more quickly by means of eq. (14a) than directly.

As to the total angular momentum $\boldsymbol{J} = \mathcal{L} + (1/2)\boldsymbol{\sigma}$, however, this remains unchanged by the transformation with ϕ:

$$\phi \boldsymbol{J} \phi^{-1} = \phi \boldsymbol{J} \phi = \boldsymbol{J}, \tag{15}$$

as can be verified directly using the explicit form of ϕ given in eq. (10). Actually, eq. (15) is the consequence of the postulates which led to the choice of ϕ: it was assumed to be a "scalar operator", i.e. an operator which leaves the transformation properties, with respect to rotations, of the original localized states $P_0^{1/2} \times \xi(s)$ unchanged. Hence, if R is the quantum mechanical operator of a rotation which transforms $P_0^{1/2}\xi(s)$ into $P_0^{1/2}\xi'(s)$,

$$R\, P_0^{1/2}\, \xi(s) = P_0^{1/2}\, \xi'(s), \tag{16}$$

it will also transform $\phi P_0^{1/2}\, \xi(s)$ into $\phi P_0^{1/2}\, \xi'(s)$,

$$R\, \phi P_0^{1/2}\, \xi(s) = \phi P_0^{1/2}\, \xi'(s). \tag{16a}$$

Comparison of this with eq. (16) shows that

$$R\, \phi P_0^{1/2}\, \xi(s) = \phi R\, P_0^{1/2}\, \xi(s). \tag{17}$$

We also require that ϕ be invariant under translations

$$T(\boldsymbol{a})\phi = \phi\, T(\boldsymbol{a}), \tag{17a}$$

where T is the displacement operator and $\boldsymbol{a}$ is the displacement. It follows from these and from $R\, T(\boldsymbol{a}) R^{-1} = T(R\boldsymbol{a})$ that

$$\begin{aligned} R\phi\, T(\boldsymbol{a}) P_0^{1/2}\, \xi(s) &= R\, T(\boldsymbol{a}) R^{-1} \cdot R\, \phi P_0^{1/2}\, \xi(s) \\ &= T(R\boldsymbol{a}) \cdot \phi R\, P_0^{1/2}\, \xi(s) = \phi T(R\boldsymbol{a}) R\, P_0^{1/2}\, \xi(s) \\ &= \phi R\, T(\boldsymbol{a}) P_0^{1/2}\, \xi(s). \end{aligned} \tag{18}$$

However, since every state can be considered as a linear combination of spatially displaced states $T(\boldsymbol{a}) P_0^{1/2}\, \xi(s)$, we conclude from eq. (18) that

$$R\phi = \phi R, \tag{19}$$

for *every* state. Hence, since $\boldsymbol{J}$ is the infinitesimal generator for the rotation group, eq. (15) also follows. Thus, from eqs. (14) and (15), we see that as far as purely spatial operations are concerned, the two definitions of the position operator and of the localized states, are identical in the sense that they can be obtained from each other by a unitary transformation which does not affect the spatial invariance operators. This is not true, however, for "boosts"; the operators of these do not commute with ϕ so that the two definitions of the localized states do have different consequences for instance if we consider the position from the point of view of two frames of reference which are in motion with respect to each other. Under a "boost" transformation a localized state does not remain localized but transforms into a linear combination of spatially displaced localized states, the linear combination being different for the two cases of localization.

3. Expressions for the current for the two localizations. Another question of interest relates to the direction of the current. First of all, it may be verified that the continuity equation

$$\boldsymbol{\nabla} \cdot \boldsymbol{j} + \delta\rho/\delta t = 0, \tag{20}$$

is satisfied if one defines a current density $\boldsymbol{j}$ and a probability density ρ as follows:

$$\boldsymbol{j}(\boldsymbol{r},t) = (2\pi)^{-3} \iint \psi^*(\boldsymbol{P},s)\, e^{i(\boldsymbol{P}\cdot\boldsymbol{r} - P_0 t)} \frac{\boldsymbol{P}+\boldsymbol{P}'}{P_0 + P_0'} \times \psi(\boldsymbol{P}',s) e^{-i(\boldsymbol{P}'\cdot\boldsymbol{r} - P_0' t)} \frac{d\boldsymbol{P}}{P_0}\frac{d\boldsymbol{P}'}{P_0'}, \tag{21}$$

and

$$\rho(\boldsymbol{r},t) = (2\pi)^{-3} \iint \psi^*(\boldsymbol{P},s)\, e^{i(\boldsymbol{P}\cdot\boldsymbol{r} - P_0' t)} \times \psi(\boldsymbol{P}',s)\, e^{-i(\boldsymbol{P}'\cdot\boldsymbol{r} - P_0' t)} \frac{d\boldsymbol{P}}{P_0}\frac{d\boldsymbol{P}'}{P_0'}. \tag{22}$$

Now writing

$$\boldsymbol{J}(t) = \int \boldsymbol{j}(\boldsymbol{r},t)\, d\boldsymbol{r}, \tag{23}$$

we obtain

$$\boldsymbol{J} = \boldsymbol{\mathfrak{v}}. \tag{24}$$

Thus, we have shown that, for spin 1/2 particles, the momentum, the velocity, and the current satisfy the classical equation for the two states discussed above and for no others.

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