ON THE RELATION BETWEEN MOMENTUM AND VELOCITY FOR ELEMENTARY SYSTEMS

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We investigate how the quantum mechanical position operator can be defined for elementary systems so that the connection, postulated by the special theory of relatively, between velocity (the time derivative of the position) and momentum remains valid.

The relation between momentum $\boldsymbol{P}$ and velocity $\boldsymbol{v}$ in relativistic classical mechanics is the well-known formula

$$\boldsymbol{v} = \boldsymbol{P}/P_0 , \tag{1}$$

where (in our units $c = \hbar = 1$)

$$P_0 = (m^2 + \boldsymbol{P}^2)^{1/2} . \tag{2}$$

The situation in quantum mechanics is not immediately obvious because although $\boldsymbol{P}$ has a natural definition – it is the generator of spatial translations of the state vector [1, 2] – this is not so for $\boldsymbol{v}$. For an elementary system (the definition [3] of which is broader than that of an elementary particle) of definite mass m, it was shown [4], using invariance principles only, that eq. (1) holds in quantum mechanics if the velocity is defined as $\Delta\boldsymbol{r}/\Delta t$, with the time interval Δt so long that the initial and final uncertainties of the position are of no consequence. If this is the case, it is not necessary to define these positions exactly i.e. the position operator does not have to be specified fully. It is our purpose here to extend these considerations to include infinitesimal time intervals. In this case, of course, the "velocity" does depend on the definition of the position, that is on the form chosen for the position operator and we'll show that eq. (1) becomes valid if the position operator has the form given in ref. [3].

Localized states exist [3] for all systems of non-zero mass and arbitrary spin s, as well as systems of zero mass and $s = 0$ and $1/2$. Whereas the momentum-velocity relation which we derive is true for all such systems, our procedure will be more transparent if we confine our attention to positive energy Klein–Gordon particles ($s = 0$) since the same argument holds in the case of arbitrary s. In the former case the momentum-space wave function of the only state which is localized at the origin, at time $t = 0$, is, according to ref. [3],

$$\psi = (2\pi)^{-3/2} P_0^{1/2} . \tag{3}$$

and it follows that the expectation value, at $t = 0$, of the position operator $\boldsymbol{q}$ is

$$\langle \boldsymbol{q}(0)\rangle = (2\pi)^{-3} \int \left(\int e^{i\boldsymbol{r}\cdot\boldsymbol{P}} P_0^{1/2} \Phi(\boldsymbol{P}) \frac{d\boldsymbol{P}}{P_0} \right)^* \times \boldsymbol{r} \left(\int e^{i\boldsymbol{r}\cdot\boldsymbol{P}} P_0^{1/2} \Phi(\boldsymbol{P}) \frac{d\boldsymbol{P}}{P_0} \right) d\boldsymbol{r}$$

$$= \int \frac{d\boldsymbol{P}}{P_0} \Phi^*(\boldsymbol{P})\, i \left[\nabla_P - \frac{\boldsymbol{P}}{2P_0^2} \right] \Phi(\boldsymbol{P}) , \tag{4}$$

where $\Phi(\boldsymbol{P})$ is normalized so that

$$\int \frac{d\boldsymbol{P}}{P_0} |\Phi(\boldsymbol{P})|^2 = 1 . \tag{5}$$

The calculation leading to the rhs of eq. (4) has been given previously [3]. Now since the time displacement operator in momentum space is simply multiplication with $e^{-iP_0 t}$ it follows that

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$$\langle \boldsymbol{q}(t)\rangle = (2\pi)^{-3}\int\left(\int e^{i\boldsymbol{r}\cdot\boldsymbol{P}-iP_0 t}\,\Phi(\boldsymbol{P})\frac{d\boldsymbol{P}}{P_0^{1/2}}\right)^*$$

$$\times\, \boldsymbol{r}\left(\int e^{i\boldsymbol{r}\cdot\boldsymbol{P}-iP_0 t}\,\Phi(\boldsymbol{P})\frac{d\boldsymbol{P}}{P_0^{1/2}}\right)d\boldsymbol{r}\,. \tag{6}$$

Thus, using eq. (2), we obtain

$$\langle q(\boldsymbol{t})\rangle = \langle \boldsymbol{q}(0)\rangle + t\int \frac{d\boldsymbol{P}}{P_0}\,\Phi^*(\boldsymbol{P})\,(\boldsymbol{P}/P_0)\,\Phi(\boldsymbol{P})\,. \tag{7}$$

Thus

$$\langle \boldsymbol{q}(t)\rangle = \langle \boldsymbol{q}(0)\rangle + t\langle \boldsymbol{v}\rangle\,, \tag{8}$$

where $\boldsymbol{v} = (\boldsymbol{P}/P_0)$.

Thus eq. (1) holds for quantum mechanical operators, whose expectation values obey eq. (8), where $\boldsymbol{q}(t)$ is the position operator at time t, with no restriction on the size of the time interval t. Of course, the non-causal feature of such position operators, which was pointed out by Fleming [5] and Hegerfeldt [6], is not eliminated by the present considerations. In fact, Hegerfeldt has given an elegant proof of a general feature of a relativistic theory viz. that causality will be violated if one insists on strict localizability.

References

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