Post-Newtonian two-body and n-body problems with electric charge in general relativity

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Starting with the Bażański two-body post-Newtonian Lagrangian with electric charge in general relativity, we construct a coordinate transformation (not involving center-of-mass coordinates) with two arbitrary parameters and obtain a Hamiltonian which is in agreement with one derived from quantum field theory. The field theory Hamiltonian corresponds to using an arbitrary parameter x_p in the photon propagator as well as an arbitrary parameter x_g in the graviton propagator. These results are also generalized to the case of n bodies. The condition for static balance $e_i = \pm G^{1/2} m_i$ is found to hold both for the exact Reissner–Nordstrøm "one-body" problem and for the post-Newtonian n-body problem. An alternate condition for static balance $e_i = \pm(G m_1 m_2)^{1/2}$ is found to hold for the post-Newtonian two-body problem. The precession of the perihelion for the post-Newtonian two-body problem is given along with four special cases, one of which is the two-body generalization of the "one-body" special relativity result of Sommerfeld. Post-Newtonian two-body equations of motion (in center-of-mass coordinates) with the condition of static balance are also examined.

INTRODUCTION

Bażański[1,2] has given the two-body post-Newtonian equations of motion[1] and Lagrangian[2] for two charged bodies in general relativity. His Lagrangian contains both the potential-energy terms of the Einstein–Infeld–Hoffman[3] Lagrangian (G, Gvv, and G^2 terms) and the potential-energy terms of the Darwin[4] Lagrangian (e^2 and e^2vv terms) as well as some additional mixed potential-energy terms (Ge^2 terms).

In Sec. I we write the Bażański Lagrangian and Hamiltonian in the more convenient notation[5] of Landau and Lifshitz.[6]

We then make a coordinate transformation from the coordinate system used by Bażański, Einstein–Infeld–Hoffmann, and Darwin to a new arbitrary coordinate system characterized by two arbitrary dimensionless parameters α_g and α_p in the transformation equations. We then obtain (in the new coordinate system) the Bażański Lagrangian and Hamiltonian which contain the parameters α_g and α_p. The coordinate transformation used is new, in two respects, compared to what has been done before by Hiida and Okamura[7] and by ourselves.[8] First, the transformation is made before going to center-of-mass coordinates and, second, it contains the additional parameter α_p which means that the gravitational and electromagnetic aspects of this paper are treated in a similar manner.

We then show how one can derive the potential-energy terms containing α_g and α_p in the Bażański Hamiltonian from quantum field theory and mention which terms have (and which have not) been so derived.

Next, we introduce center-of-mass coordinates in our Lagrangian. Then, after making the large-mass large-charge approximation, we compare our result with the Reissner–Nordstrøm[9] Lagrangian.

In Sec. II we discuss the conditions of static balance[5] where the electric and gravitational forces cancel out when the two particles are at rest.

In Sec. III we give the post-Newtonian equation of motion in a center-of-mass coordinate system and then find the precession of the orbit (perihelion precession) which agrees with the special cases of Robertson[10] and Sommerfeld.[11] Solutions to the post-Newtonian equations of motion for the two-body problem in center-of-mass coordinates, satisfying the condition of static balance are found.

In Sec. IV we generalize our results to the case of n-bodies and in Sec. V we present our conclusions.

I. LAGRANGIAN AND HAMILTONIAN

The two-body post-Newtonian Lagrangian[2] and equations of motion[1] for the case of charged particles in general relativity have been given by Bażański. The Lagrangian in Bażański (B) coordinates [same coordinates as used by Einstein–Infeld–Hoffmann[3] (EIH) and Darwin[4] (D)] can be written as

$$\begin{aligned} L(\mathbf{r}_{1B}, \mathbf{v}_{1B}; \mathbf{r}_{2B}, \mathbf{v}_{2B}) &= \tfrac{1}{2}m_1 v_{1B}^2 + \tfrac{1}{2}m_2 v_{2B}^2 + \tfrac{1}{8}m_1 v_{1B}^4/c^2 + \tfrac{1}{8}m_2 v_{2B}^4/c^2 \\ &+ \frac{Gm_1m_2}{r_B}[1 + \tfrac{3}{2}(v_{1B}^2 + v_{2B}^2)/c^2 - \tfrac{7}{2}(\mathbf{v}_{1B}\cdot\mathbf{v}_{2B})/c^2 \\ &- \tfrac{1}{2}(\mathbf{v}_{1B}\cdot\mathbf{r}_B)(\mathbf{v}_{2B}\cdot\mathbf{r}_B)/c^2 r_B^2] - \frac{e_1e_2}{r_B}[1 - \tfrac{1}{2}(\mathbf{v}_{1B}\cdot\mathbf{v}_{2B})/c^2 \\ &- \tfrac{1}{2}(\mathbf{v}_{1B}\cdot\mathbf{r}_B)(\mathbf{v}_{2B}\cdot\mathbf{r}_B)/c^2 r_B^2] - \frac{G^2m_1m_2(m_1+m_2)}{2c^2r_B^2} \\ &+ \frac{Ge_1e_2(m_1+m_2)}{c^2r_B^2} - \frac{G(e_1^2m_2 + e_2^2m_1)}{2c^2r_B^2}, \end{aligned} \quad (1)$$

where c and G are the speed of light and gravitational constant respectively, $\mathbf{r}_B=\mathbf{r}_{1B}-\mathbf{r}_{2B}$ and thus $\mathbf{v}_B=\mathbf{v}_{1B}-\mathbf{v}_{2B}$. By the usual standard procedure we obtain the Hamiltonian

$$\mathcal{H}(\mathbf{r}_{1B},\mathbf{P}_{1B};\mathbf{r}_{2B},\mathbf{P}_{2B})$$
$$=\frac{P_{1B}^2}{2m_1}+\frac{P_{2B}^2}{2m_2}-\frac{P_{1B}^4}{8m_1^3c^2}-\frac{P_{2B}^4}{8m_2^3c^2}-\frac{Gm_1m_2}{r_B}\times\left[1+\frac{3}{2}\left(\frac{P_{1B}^2}{m_1^2c^2}+\frac{P_{2B}^2}{m_2^2c^2}\right)-\frac{7}{2}\,\frac{(\mathbf{P}_{1B}\cdot\mathbf{P}_{2B})}{m_1m_2c^2}-\frac{1}{2}\,\frac{(\mathbf{P}_{1B}\cdot\mathbf{r}_B)(\mathbf{P}_{2B}\cdot\mathbf{r}_B)}{m_1m_2c^2r_B^2}\right]+\frac{e_1e_2}{r_B}\left[1-\frac{1}{2}\,\frac{(\mathbf{P}_{1B}\cdot\mathbf{P}_{2B})}{m_1m_2c^2}-\frac{1}{2}\,\frac{(\mathbf{P}_{1B}\cdot\mathbf{r}_B)(\mathbf{P}_{2B}\cdot\mathbf{r}_B)}{m_1m_2c^2r_B^2}\right]+\frac{G^2m_1m_2(m_1+m_2)}{2c^2r_B^2}-\frac{Ge_1e_2(m_1+m_2)}{c^2r_B^2}+\frac{G(e_1^2m_2+e_2^2m_1)}{2c^2r_B^2}. \quad (2)$$

A. Coordinate transformation

We shall now make the coordinate transformation (see Appendix)

$$\mathbf{r}_{1B}=\mathbf{r}_1-\mathbf{r}\left(\alpha_g\frac{Gm_2}{c^2r}-\alpha_p\frac{e_1e_2}{m_1c^2r}\right), \quad (3)$$

$$\mathbf{r}_{2B}=\mathbf{r}_2+\mathbf{r}\left(\alpha_g\frac{Gm_1}{c^2r}-\alpha_p\frac{e_1e_2}{m_2c^2r}\right), \quad (4)$$

which implies that

$$\mathbf{r}_B=\mathbf{r}\left(1-\alpha_g\frac{GM}{c^2r}+\alpha_p\frac{e_1e_2}{\mu c^2r}\right), \quad (5)$$

$$\mathbf{v}_{1B}=\mathbf{v}_1-\left(\alpha_g\frac{Gm_2}{c^2r}-\alpha_p\frac{e_1e_2}{m_1c^2r}\right)\left[\mathbf{v}-\frac{(\mathbf{v}\cdot\mathbf{r})\mathbf{r}}{r^2}\right], \quad (6)$$

$$\mathbf{v}_{2B}=\mathbf{v}_2+\left(\alpha_g\frac{Gm_1}{c^2r}-\alpha_p\frac{e_1e_2}{m_2c^2r}\right)\left[\mathbf{v}-\frac{(\mathbf{v}\cdot\mathbf{r})\mathbf{r}}{r^2}\right], \quad (7)$$

$$\mathbf{P}_{1B}=\mathbf{P}_1+\left(\alpha_g\frac{Gm_2}{c^2r}-\alpha_p\frac{e_1e_2}{m_1c^2r}\right)\left[\mathbf{P}_1-\frac{(\mathbf{P}_1\cdot\mathbf{r})\mathbf{r}}{r^2}\right]-\left(\alpha_g\frac{Gm_1}{c^2r}-\alpha_p\frac{e_1e_2}{m_2c^2r}\right)\left[\mathbf{P}_2-\frac{(\mathbf{P}_2\cdot\mathbf{r})\mathbf{r}}{r^2}\right], \quad (8)$$

$$\mathbf{P}_{2B}=\mathbf{P}_2+\left(\alpha_g\frac{Gm_1}{c^2r}-\alpha_p\frac{e_1e_2}{m_2c^2r}\right)\left[\mathbf{P}_2-\frac{(\mathbf{P}_2\cdot\mathbf{r})\mathbf{r}}{r^2}\right]-\left(\alpha_g\frac{Gm_2}{c^2r}-\alpha_p\frac{e_1e_2}{m_1c^2r}\right)\left[\mathbf{P}_1-\frac{(\mathbf{P}_1\cdot\mathbf{r})\mathbf{r}}{r^2}\right], \quad (9)$$

where α_g and α_p are arbitrary dimensionless parameters, and $\mu=m_1m_2/(m_1+m_2)$ and $M=m_1+m_2$ are the reduced mass and total mass, respectively.

The Hamiltonian of Eq. (2) in the new coordinates is

$$\mathcal{H}(\alpha_g,\alpha_p)=\mathcal{H}_0+V_{1(\mathrm{EIH})}(\alpha_g)+V_{2(\mathrm{EIH})}(\alpha_g)+V_{1(\mathrm{D})}(\alpha_p)+V_{2(\mathrm{D})}(\alpha_p)+V_{2(\mathrm{B})}(\alpha_g,\alpha_p), \quad (10)$$

where

$$\mathcal{H}_0=\frac{P_1^2}{2m_1}+\frac{P_2^2}{2m_2}-\frac{P_1^4}{8m_1^3c^2}-\frac{P_2^4}{8m_2^3c^2}, \quad (11)$$

$$V_{1(\mathrm{EIH})}(\alpha_g)=-\frac{Gm_1m_2}{r}\left\{1+(\tfrac{3}{2}-\alpha_g)\left(\frac{P_1^2}{m_1^2c^2}+\frac{P_2^2}{m_2^2c^2}\right)+(2\alpha_g-\tfrac{7}{2})\frac{(\mathbf{P}_1\cdot\mathbf{P}_2)}{m_1m_2c^2}-(\tfrac{1}{2}+2\alpha_g)\frac{(\mathbf{P}_1\cdot\mathbf{r})(\mathbf{P}_2\cdot\mathbf{r})}{m_1m_2c^2r^2}+\alpha_g\left[\frac{(\mathbf{P}_1\cdot\mathbf{r})^2}{m_1^2c^2r^2}+\frac{(\mathbf{P}_2\cdot\mathbf{r})^2}{m_2^2c^2r^2}\right]\right\}, \quad (12)$$

$$V_{2(\mathrm{EIH})}(\alpha_g)=(\tfrac{1}{2}-\alpha_g)\frac{G^2m_1m_2(m_1+m_2)}{c^2r^2}, \quad (13)$$

$$V_{1(\mathrm{D})}(\alpha_p)=\frac{e_1e_2}{r}\left\{1-\alpha_p\left(\frac{P_1^2}{m_1^2c^2}+\frac{P_2^2}{m_2^2c^2}\right)+(2\alpha_p-\tfrac{1}{2})\frac{(\mathbf{P}_1\cdot\mathbf{P}_2)}{m_1m_2c^2}-(\tfrac{1}{2}+2\alpha_p)\frac{(\mathbf{P}_1\cdot\mathbf{r})(\mathbf{P}_2\cdot\mathbf{r})}{m_1m_2c^2r^2}+\alpha_p\left[\frac{(\mathbf{P}_1\cdot\mathbf{r})^2}{m_1^2c^2r^2}+\frac{(\mathbf{P}_2\cdot\mathbf{r})^2}{m_2^2c^2r^2}\right]\right\}, \quad (14)$$

$$V_{2(\mathrm{D})}(\alpha_p)=-\alpha_p\frac{e_1^2e_2^2}{\mu c^2r^2}, \quad (15)$$

$$V_{2(\mathrm{B})}(\alpha_g,\alpha_p)=(\alpha_g+\alpha_p-1)\frac{Ge_1e_2(m_1+m_2)}{c^2r^2}+\frac{G(e_1^2m_2+e_2^2m_1)}{2c^2r^2}. \quad (16)$$

The coordinate system is B—EIH—D if both $\alpha_g=0$ and $\alpha_p=0$. In a coordinate system where $\alpha_g=\frac{1}{2}$ we have $V_{2(\mathrm{EIH})}(\frac{1}{2})=0$, while for a coordinate system where $\alpha_p=0$ we have $V_{2(\mathrm{D})}(0)=0$.

The special cases of the Einstein—Infeld—Hoffman Hamiltonian (pure gravitation, i.e., $e_1=e_2=0$) and the Darwin Hamiltonian (pure electromagnetism, i.e., $G=0$) can be written, respectively, as

$$\mathcal{H}_{\mathrm{EIH}}(\alpha_g)=\mathcal{H}_0+V_{1(\mathrm{EIH})}(\alpha_g)+V_{2(\mathrm{EIH})}(\alpha_g), \quad (17)$$

$$\mathcal{H}_{\mathrm{D}}(\alpha_p)=\mathcal{H}_0+V_{1(\mathrm{D})}(\alpha_p)+V_{2(\mathrm{D})}(\alpha_p). \quad (18)$$

The Lagrangian corresponding to Eq. (10) can be written as

$$\mathcal{L}(\alpha_g,\alpha_p)=\mathcal{L}_0-V_{1(\mathrm{EIH})}(\alpha_g)-V_{2(\mathrm{EIH})}(\alpha_g)-V_{1(\mathrm{D})}(\alpha_p)-V_{2(\mathrm{D})}(\alpha_p)-V_{2(\mathrm{B})}(\alpha_g,\alpha_p), \quad (19)$$

where

$$\mathcal{L}_0=\tfrac{1}{2}m_1v_1^2+\tfrac{1}{2}m_2v_2^2+\tfrac{1}{8}m_1v_1^4/c^2+\tfrac{1}{8}m_2v_2^4/c^2, \quad (20)$$

and $m_1\mathbf{v}_1$ and $m_2\mathbf{v}_2$ replace $\mathbf{P}_1$ and $\mathbf{P}_2$, respectively, in the potential energy terms of Eqs. (12) and (14).

B. Hamiltonian from quantum field theory

The potential-energy terms $V_{1(\mathrm{EIH})}(\alpha_g)$ and $V_{2(\mathrm{EIH})}(\alpha_g)$ can be derived from the one-graviton exchange interaction and the two-graviton exchange interaction,[12] respectively. The potential-energy terms $V_{1(\mathrm{D})}(\alpha_p)$ and $V_{2(\mathrm{D})}(\alpha_p)$ can be derived from the one-photon exchange interaction and the two-photon exchange interaction,[12] respectively. The potential-energy term $V_{2(\mathrm{B})}(\alpha_g,\alpha_p)$ can be derived from the one-graviton one-photon exchange interaction.[12]

Both the graviton and the photon propagators are proportional to $1/(\mathbf{k}^2-k_0^2)$, [see Eqs. (12)—(18) of Ref. 8], where k_0^2 can be written in a form that contains an arbitrary dimensionless parameter x. For the graviton case let this parameter be x_g and for the photon case let this parameter be x_p. It is convenient to introduce two other dimensionless parameters α_g and α_p which are given by

$$\alpha_g=-\tfrac{1}{4}(1-x_g)\quad\text{and}\quad\alpha_p=-\tfrac{1}{4}(1-x_p). \quad (21)$$

Thus, the potential-energy terms derived from quantum field theory will contain two arbitrary parameters α_g and α_p. The proper interpretation of the Hamiltonians with arbitrary parameters α_g and α_p is that the Hamiltonians are related to each other by the coordinate transformation given by Eqs. (3) and (4) whose form was carefully chosen to be consistent with the field theory results.

Iwasaki[13] gave the field theory derivation for $V_{1(\mathrm{EIH})}(0)$ and $V_{2(\mathrm{EIH})}(0)$, while Hiida and Okamura[7] gave the field theory derivation [see their Eqs. (1.3) and (3.20)] for $V_{1(\mathrm{EIH})}(\alpha_g)$ and $V_{2(\mathrm{EIH})}(\alpha_g)$ which agree with our Eqs. (12) and (13). We have verified that the field theory derivation for $V_{1(\mathrm{EIH})}(\alpha_g)$ and $V_{1(\mathrm{D})}(\alpha_p)$ is in agreement with Eqs. (12) and (14). To our knowledge, a quantum field theory derivation of $V_{2(\mathrm{D})}(\alpha_p)$ and $V_{2(\mathrm{B})}(\alpha_g, \alpha_p)$ has not been made.

C. Center-of-mass coordinates

Going to center-of-mass coordinates we put $\mathbf{P} = \mathbf{P}_1 = -\mathbf{P}_2$ in Eq. (10) to obtain

$$\begin{aligned}
H(\alpha_g, \alpha_p; \mathbf{r}, \mathbf{P}) &= \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) P^2 - \frac{1}{8}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right)\frac{P^4}{c^2} \\
&- \frac{Gm_1m_2}{r}\left\{1 + \left[\frac{1}{2} + \left(\frac{3}{2} - \alpha_g\right)\frac{M}{\mu}\right]\frac{P^2}{m_1m_2c^2}\right. \\
&\left. + \left(\frac{1}{2} + \alpha_g\frac{M}{\mu}\right)\frac{(\mathbf{P}\cdot\mathbf{r})^2}{m_1m_2c^2r^2}\right\} + \frac{e_1e_2}{r} \\
&\times\left\{1 + \left(\frac{1}{2} - \alpha_p\frac{M}{\mu}\right)\frac{P^2}{m_1m_2c^2} + \left(\frac{1}{2} + \alpha_p\frac{M}{\mu}\right)\right. \\
&\left.\times\frac{(\mathbf{P}\cdot\mathbf{r})^2}{m_1m_2c^2r^2}\right\} + \left(\frac{1}{2} - \alpha_g\right)\frac{G^2\mu M^2}{c^2r^2} - \alpha_p\frac{e_1^2e_2^2}{\mu c^2r^2} \\
&+ (\alpha_g + \alpha_p - 1)\frac{Ge_1e_2M}{c^2r^2} + \frac{G(e_1^2m_2 + e_2^2m_1)}{2c^2r^2}. \qquad (22)
\end{aligned}$$

Equation (22) also could have been obtained by starting with Eq. (2), going to center-of-mass coordinates, and then making the transformation of Eq. (5) which implies that

$$\mathbf{v}_B = \mathbf{v} - \left(\alpha_g\frac{GM}{c^2r} - \alpha_p\frac{e_1e_2}{\mu c^2r}\right)\left[\mathbf{v} - \frac{(\mathbf{v}\cdot\mathbf{r})\,\mathbf{r}}{r^2}\right], \qquad (23)$$

$$\mathbf{P}_B = \mathbf{P} + \left(\alpha_g\frac{GM}{c^2r} - \alpha_p\frac{e_1e_2}{\mu c^2r}\right)\left[\mathbf{P} - \frac{(\mathbf{P}\cdot\mathbf{r})\,\mathbf{r}}{r^2}\right]. \qquad (24)$$

The pure gravitational part of Eq. (22) [G, GPP, and G^2 terms] has been given by Hiida and Okamura[7] and by ourselves.[8] The one-graviton exchange part of Eq. (22) [G and GPP terms] with $\alpha_g = -\frac{1}{2}\mu/M$ has been derived by Barker, Gupta, and Haracz[14] using Gupta's[15] quantum theory of gravitation.

The Lagrangian corresponding to Eq. (22) can be written as

$$\begin{aligned}
L(\alpha_g, \alpha_p; \mathbf{r}, \mathbf{v}) &= \tfrac{1}{2}\mu v^2 + \tfrac{1}{8}(1 - 3\mu/M)\frac{\mu v^4}{c^2} + \frac{G\mu M}{r}\left[1 + \left(\frac{3}{2} - \alpha_g + \frac{1}{2}\frac{\mu}{M}\right)\frac{v^2}{c^2}\right. \\
&\left. + \left(\alpha_g + \frac{1}{2}\frac{\mu}{M}\right)\frac{(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^2}\right] - \frac{e_1e_2}{r}\left[1 + \left(-\alpha_p + \frac{1}{2}\frac{\mu}{M}\right)\frac{v^2}{c^2}\right. \\
&\left. + \left(\alpha_p + \frac{1}{2}\frac{\mu}{M}\right)\frac{(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^2}\right] + \left(\alpha_g - \frac{1}{2}\right)\frac{G^2\mu M^2}{c^2r^2} + \alpha_p\frac{e_1^2e_2^2}{\mu c^2r^2} \\
&+ (1 - \alpha_g - \alpha_p)\frac{Ge_1e_2M}{c^2r^2} - \frac{G(e_1^2m_2 + e_2^2m_1)}{2c^2r^2}. \qquad (25)
\end{aligned}$$

In the coordinate system where $\alpha_g = \alpha_p = -\mu/2M$ we can write Eq. (25) as

$$\begin{aligned}
L(\alpha_g = \alpha_p = -\mu/2M; \mathbf{r}, \mathbf{v}) &= \tfrac{1}{2}\mu v^2 + G'\mu M/r + \tfrac{1}{8}k_1\mu v^4/c^2 \\
&+ \tfrac{3}{2}k_2G'\mu Mv^2/c^2r - \tfrac{1}{2}k_3G'^2\mu M^2/c^2r^2, \qquad (26)
\end{aligned}$$

where

$$G' = G - e_1e_2/m_1m_2, \qquad (27)$$

$$k_1 = 1 - 3\mu/M, \qquad (28)$$

$$k_2 = 1 + \tfrac{2}{3}\mu/M + Z_a, \qquad (29)$$

$$k_3 = 1 + \mu/M + Z_b + Z_aZ_b - Z_a^2, \qquad (30)$$

and

$$Z_a = \frac{e_1e_2}{G'm_1m_2}, \qquad (31)$$

$$Z_b = \frac{e_1^2m_2 + e_2^2m_1}{G'm_1m_2M}. \qquad (32)$$

As the precession of the orbit (see Sec. III) is independent of the parameters α_g and α_p it is convenient (for the derivation of the precession of the orbit) to choose the Lagrangian in as simple a form as possible.[8] By setting $\alpha_g = \alpha_p = -\mu/2M$, we eliminate the $(\mathbf{v}\cdot\mathbf{r})^2$ potential energy terms in Eq. (25). We can also eliminate the v^2 potential energy terms in Eq. (25) by setting $\alpha_g = \frac{3}{2} + \frac{1}{2}\mu/M$ and $\alpha_p = \frac{1}{2}\mu/M$.

D. Large-mass, large-charge approximation

Applying the large-mass approximation ($m_2 \gg m_1$) and the large-charge approximation ($e_2 \gg e_1$) with the condition $e_2^2m_1 \gg e_1^2m_2$ to Eq. (25) we obtain the so-called "one-body" Lagrangian (there are actually two bodies) which is

$$\begin{aligned}
L(\alpha_g, \alpha_p; \mathbf{r}, \mathbf{v}) &= \tfrac{1}{2}m_1v^2 + \tfrac{1}{8}m_1\frac{v^4}{c^2} + \frac{Gm_1m_2}{r}\left[1 + \left(\tfrac{3}{2} - \alpha_g\right)\frac{v^2}{c^2}\right. \\
&\left. + \alpha_g\frac{(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^2}\right] - \frac{e_1e_2}{r}\left[1 - \alpha_p\frac{v^2}{c^2} + \alpha_p\frac{(\mathbf{v}\cdot\mathbf{r})^2}{c^2r^2}\right] \\
&+ \left(\alpha_g - \frac{1}{2}\right)\frac{G^2m_1m_2^2}{c^2r^2} + \alpha_p\frac{e_1^2e_2^2}{m_1c^2r^2} \\
&+ (1 - \alpha_g - \alpha_p)\frac{Ge_1e_2m_2}{c^2r^2} - \frac{Ge_2^2m_1}{2c^2r^2}. \qquad (33)
\end{aligned}$$

The Lagrangian for a test charged particle (mass m_1 and charge e_1) in the field of a heavy large-charged particle (mass m_2 and charge e_2), where again $m_2 \gg m_1$ and $e_2 \gg e_1$ and $e_2^2m_1 \gg e_1^2m_2$, is given by

$$L = -m_1c[-g_{00}c^2 - g_{ij}\dot{x}^i\dot{x}^j]^{1/2} - e_1A_0. \qquad (34)$$

The Reissner–Nordstrøm[9] solution for g_{00}, g_{ij}, and A_0 in this "one-body" Lagrangian is

$$g_{00} = -\left(1 - \frac{2Gm_2}{c^2r} + \frac{Ge_2^2}{c^4r^2}\right), \qquad (35)$$

$$g_{ij} = \delta_{ij} - \left(1 + \frac{1}{g_{00}}\right)\frac{x^ix^j}{r^2}, \qquad (36)$$

$$A_0 = e_2/r. \qquad (37)$$

The post-Newtonian expansion of Eq. (34) (apart from the rest energy term $-m_1c^2$) is Eq. (33) with $\alpha_g=1$ and $\alpha_p=0$. The coordinates are Schwarzschild coordinates.[8]

II. CONDITION OF STATIC BALANCE

For the Lagrangians of Eqs. (25) and (34) (which are in the center-of-mass system), we can write Lagrange's equations as

$$\mathbf{F}=\dot{\mathbf{P}}, \tag{38}$$

where

$$\mathbf{F}\equiv\frac{\partial L}{\partial \mathbf{r}} \quad\text{and}\quad \mathbf{P}\equiv\frac{\partial L}{\partial \mathbf{v}}. \tag{39}$$

If we now set[5]

$$e_i=\pm G^{1/2}m_i, \quad i=1,2 \tag{40}$$

(the notation is meant to imply that the + sign holds for all i or the − sign holds for all i) we find that $(L)_{\mathbf{v}=0}\equiv 0,\ -m_1c^2$ for Eqs. (25) and (34), respectively, and hence, $(\mathbf{F})_{\mathbf{v}=0}\equiv 0$. Thus, if the particles are at rest they will remain at rest. We note that if $\mathbf{v}$ is not zero the force will not be zero. The condition for static balance[5] of Eq. (40) holds for both the exact "one-body" problem of Eq. (34) and the post-Newtonian two-body problem of Eq. (25).

Let us now look more carefully at the post-Newtonian two-body problem of Eq. (25). In order to have static balance for all r the static $1/r$ terms and the static $1/r^2$ terms must independently cancel out in the Lagrangian of Eq. (25). We thus must have

$$e_1e_2=Gm_1m_2, \tag{41}$$

$$e_1^2m_2+e_2^2m_1=2e_1e_2(m_1+m_2)-Gm_1m_2(m_1+m_2). \tag{42}$$

Note that the $1/r^2$ terms proportional to α_g and α_p cancel out due to Eq. (41). Using Eq. (41) in Eq. (42) we obtain

$$m_1e_2(e_2-e_1)=m_2e_1(e_2-e_1), \tag{43}$$

which gives us the solution of Eq. (40) and also another solution for static balance, namely,

$$e_i=\pm(Gm_1m_2)^{1/2}, \quad i=1,2. \tag{44}$$

In the special case where $m_1=m_2$ the two solutions of Eq. (40) and (44) become the same.

It should be noted that Eq. (44) is not a solution to the exact "one-body" problem of Eq. (34). This is as expected since we must have $e_2\gg e_1$ and $m_2\gg m_1$ for Eq. (34) to be valid.

We shall return to our discussion of static balance in Sec. IV.

III. EQUATION OF MOTION AND PRECESSION OF THE ORBIT

Lagrange's equations of motion for the Lagrangian of Eq. (26) may be written as

$$\dot{\mathbf{v}}+\frac{G'M\mathbf{r}}{r^3}=\frac{G'M}{c^2r^3}\left[4k_4G'M\mathbf{r}/r-k_5v^2\mathbf{r}+4k_6(\mathbf{v}\cdot\mathbf{r})\mathbf{v}\right], \tag{45}$$

where

$$k_4=\tfrac{3}{4}k_2+\tfrac{1}{4}k_3=1+\tfrac{3}{4}\mu/M+\tfrac{3}{4}Z_a+\tfrac{1}{4}(Z_b+Z_aZ_b-Z_a^2), \tag{46}$$

$$k_5=\tfrac{3}{2}k_2-\tfrac{1}{2}k_1=1+\tfrac{5}{2}\mu/M+\tfrac{3}{2}Z_a, \tag{47}$$

$$k_6=\tfrac{1}{4}k_1+\tfrac{3}{4}k_2=1-\tfrac{1}{4}\mu/M+\tfrac{3}{4}Z_a. \tag{48}$$

The secular results for the precession of the orbit can be written as[8]

$$\mathring{E}_{av}=0, \quad \dot{\mathbf{L}}_{av}=\mathbf{\Omega}^*\times\mathbf{L}, \quad \mathbf{A}_{av}=\mathbf{\Omega}^*\times\mathbf{A}, \tag{49}$$

where $E=\mu(\frac{1}{2}v^2-G'M/r)$, $\mathbf{L}=\mu\mathbf{r}\times\mathbf{v}$, and $\mathbf{A}=\mu[\mathbf{v}\times(\mathbf{r}\times\mathbf{v})-G'M\mathbf{r}/r]$ are the Newtonian energy, orbital angular momentum, and Runge–Lenz vector, respectively, and

$$\mathbf{\Omega}^*=\frac{(4k_6-2k_4+k_5)\,G'M\bar{\omega}}{c^2a(1-e^2)}\,\mathbf{n}, \tag{50}$$

where a is the semimajor axis, e is the eccentricity, $\bar{\omega}$ is the average orbital angular velocity, and $\mathbf{n}$ is a unit vector in the $\mathbf{L}$ direction. To insure a bound orbit we must have $G'>0$. The result for $\mathbf{\Omega}^*$ can be cast into different forms by using the relations

$$\frac{L/\mu}{a^2(1-e^2)^{1/2}}=\left(\frac{G'M}{a^3}\right)^{1/2}=\frac{2\pi}{T}=\bar{\omega}, \tag{51}$$

where T is the orbital period.

Unfortunately, the expression

$$\begin{aligned}4k_6-2k_4+k_5&=\tfrac{1}{2}k_1+3k_2-\tfrac{1}{2}k_3\\&=(1+Z_a)(3-\tfrac{1}{2}Z_b)+\tfrac{1}{2}Z_a^2\end{aligned} \tag{52}$$

contained in $\mathbf{\Omega}^*$ cannot be expressed in a simpler form.

However, we do have four special cases in which the form of Eq. (50) and (52) can be put in a very simple form.

A. First case: $e_1=e_2=0$

We then have $G'=G$, $Z_a=Z_b=0$, $4k_6-2k_4+k_5=3$ and thus

$$\mathbf{\Omega}^*=\frac{3GM\bar{\omega}}{c^2a(1-e^2)}\,\mathbf{n}, \tag{53}$$

which is the result of Robertson.[8,10]

B. Second case: $G=0$

We then have $G'=-e_1e_2/m_1m_2$, $Z_a=-1$, $Z_b=-(e_1^2m_2+e_2^2m_1)/e_1e_2M$, $4k_6-2k_4+k_5=\frac{1}{2}$ and thus

$$\mathbf{\Omega}^*=\frac{|e_1e_2|\bar{\omega}/\mu}{2c^2a(1-e^2)}\,\mathbf{n}, \tag{54}$$

where $e_1e_2<0$ because the orbit must be bound. Equation (54) is the two-body generalization of the "one-body" Sommerfeld[11] result and reduces to the Sommerfeld result under the large mass approximation $m_2\gg m_1$.

C. Third case: $e_1=0$

We then have $G'=G$, $Z_a=0$, $Z_b=e_2^2/Gm_2M$, $4k_6-2k_4+k_5=3-e_2^2/2Gm_2M$ and thus

$$\mathbf{\Omega}^*=\frac{(3GM-e_2^2/2m_2)\,\bar{\omega}}{c^2a(1-e^2)}\,\mathbf{n}. \tag{55}$$

We also have a similar result for the case $e_2 = 0$.

D. Fourth case: $e_i = \pm(-1)^i G^{1/2} m_i$ or $e_i = \pm(-1)^i (Gm_1 m_2)^{1/2}$

In these cases both the Newtonian gravitational and electric forces are equal and attractive. We then have $G' = 2G$, $Z_a = -\frac{1}{2}$, $Z_b = \frac{1}{2}$, $4k_6 - 2k_4 + k_5 = \frac{3}{2}$ and thus

$$\mathbf{\Omega}^* = \frac{3GM\overline{\omega}}{c^2 a(1-e^2)}\mathbf{n}. \tag{56}$$

The results of the first and fourth cases as given by Eqs. (53) and (56) are only superficially the same since $G'M = \overline{\omega}^2 a^3$ [see Eq. (51)] is different in each of these two cases.

E. Equations of motion with $e_i = \pm G^{1/2} m_i$ or $e_i = \pm (Gm_1 m_2)^{1/2}$

Using any one of these conditions of static balance which imply $G' = 0$ and $G'Z_a = G'Z_b = G$, in Eq. (45) we obtain

$$\dot{\mathbf{v}} = \frac{GM}{c^2 r^3}\left[-\tfrac{3}{2}v^2\mathbf{r} + 3(\mathbf{v}\cdot\mathbf{r})\mathbf{v}\right], \tag{57}$$

which shows the acceleration $\dot{\mathbf{v}}$ will be zero when $\mathbf{v}$ is zero. Equation (57) has a solution for a bound circular orbit with any velocity ($v \ll c$ to be consistent with the post-Newtonian approximation) at $r = 3GM/2c^2$. However, Eq. (57) being a post-Newtonian approximation is *not* valid for r of the order of GM/c^2.

Equation (57) is in a form similar to the equation of motion of a photon (neglecting spin effects) in the gravitational field of the sun which can be written as

$$\dot{\mathbf{v}} = \frac{Gm_\odot}{c^2 r^3}\left[-2c^2\mathbf{r} + 4(\mathbf{v}\cdot\mathbf{r})\mathbf{v}\right], \tag{57p}$$

which gives the well-known deflection angle of

$$\theta_p = 4Gm_\odot/c^2 R, \tag{58p}$$

where $m_\odot$ is the mass of the sun and R is the distance of closest approach. Hence, the deflection angle for the similar problem with Eq. (57) would be

$$\theta = 3GM/c^2 R, \tag{58}$$

with the conditions that $R \gg GM/c^2$ and $v \ll c$.

IV. GENERALIZATION TO n BODIES

The post-Newtonian n-body Lagrangian with electric charge has also been given by Bażański.[16] The n-body generalization of the two-body coordinate transformation of Eqs. (3) and (4) is

$$\mathbf{r}_{iB} = \mathbf{r}_i - \sum_{\substack{j=1\\ i\neq i}}^{n} \mathbf{r}_{ij}\left(\alpha_g \frac{Gm_j}{c^2 r_{ij}} - \alpha_p \frac{e_i e_j}{m_i c^2 r_{ij}}\right), \tag{59}$$

where $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ and thus $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$. Transforming the Bażański Lagrangian[16] which is a function of $\mathbf{r}_{iB}$ and $\mathbf{v}_{iB}$ into the new variables $\mathbf{r}_i$ and $\mathbf{v}_i$ we obtain

$$\begin{aligned}
L(n\text{-body};\alpha_g,\alpha_p) &= \sum_{i=1}^{n}\left\{\tfrac{1}{2}m_i v_i^2 + \tfrac{1}{8}m_i v_i^4/c^2\right\} + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\left\{\frac{Gm_i m_j}{r_{ij}}\left[1 - \frac{(\mathbf{v}_i\cdot\mathbf{v}_j)}{2c^2} - \frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\mathbf{r}_{ij})}{2c^2 r_{ij}^2} + \left(\frac{3}{2} - \alpha_g\right)\frac{v_{ij}^2}{c^2}\right.\right.\\
&\quad\left. + \alpha_g\frac{(\mathbf{v}_{ij}\cdot\mathbf{r}_{ij})^2}{c^2 r_{ij}^2}\right] - \frac{e_i e_j}{r_{ij}}\left[1 - \frac{(\mathbf{v}_i\cdot\mathbf{v}_j)}{2c^2} - \frac{(\mathbf{v}_i\cdot\mathbf{r}_{ij})(\mathbf{v}_j\cdot\mathbf{r}_{ij})}{2c^2 r_{ij}^2} - \alpha_p\frac{v_{ij}^2}{c^2} + \alpha_p\frac{(\mathbf{v}_{ij}\cdot\mathbf{r}_{ij})^2}{c^2 r_{ij}^2}\right]\\
&\quad\left. + \left(\alpha_g - \frac{1}{2}\right)\frac{G^2 m_i m_j M_{ij}}{c^2 r_{ij}^2} + \alpha_p\frac{e_i^2 e_j^2}{\mu_{ij}c^2 r_{ij}^2} + (1-\alpha_g-\alpha_p)\frac{Ge_i e_j M_{ij}}{c^2 r_{ij}^2} - \frac{G(e_i^2 m_j + e_j^2 m_i)}{2c^2 r_{ij}^2}\right\} + \sum_{i=1}^{n}\sum_{j=i+1}^{n}\sum_{k=j+1}^{n}\\
&\quad\times\left\{\frac{G^2 m_i m_j m_k}{c^2}\left[-\frac{1}{r_{ij}r_{ik}} - \frac{1}{r_{ji}r_{jk}} - \frac{1}{r_{ki}r_{kj}} + \alpha_g\left(\frac{1}{r_{ij}^2} + \frac{1}{r_{ik}^2}\right)\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}r_{ik}} + \alpha_g\left(\frac{1}{r_{ji}^2} + \frac{1}{r_{jk}^2}\right)\frac{\mathbf{r}_{ji}\cdot\mathbf{r}_{jk}}{r_{ji}r_{jk}}\right.\right.\\
&\quad\left. + \alpha_g\left(\frac{1}{r_{ki}^2} + \frac{1}{r_{kj}^2}\right)\frac{\mathbf{r}_{ki}\cdot\mathbf{r}_{kj}}{r_{ki}r_{kj}}\right] + \alpha_p\left[\frac{e_i^2 e_j e_k}{m_i c^2}\left(\frac{1}{r_{ij}^2} + \frac{1}{r_{ik}^2}\right)\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}r_{ik}} + \frac{e_i e_j^2 e_k}{m_j c^2}\left(\frac{1}{r_{ji}^2} + \frac{1}{r_{jk}^2}\right)\frac{\mathbf{r}_{ji}\cdot\mathbf{r}_{jk}}{r_{ji}r_{jk}}\right.\\
&\quad\left. + \frac{e_i e_j e_k^2}{m_k c^2}\left(\frac{1}{r_{ki}^2} + \frac{1}{r_{kj}^2}\right)\frac{\mathbf{r}_{ki}\cdot\mathbf{r}_{kj}}{r_{ki}r_{kj}}\right] + \frac{Gm_i e_j e_k}{c^2}\left[-\frac{1}{r_{ij}r_{ik}} + \frac{1}{r_{ji}r_{jk}} + \frac{1}{r_{ki}r_{kj}} - (\alpha_g+\alpha_p)\left(\frac{1}{r_{ij}^2} + \frac{1}{r_{ik}^2}\right)\frac{\mathbf{r}_{ij}\cdot\mathbf{r}_{ik}}{r_{ij}r_{ik}}\right]\\
&\quad + \frac{Ge_i m_j e_k}{c^2}\left[+\frac{1}{r_{ij}r_{ik}} - \frac{1}{r_{ji}r_{jk}} + \frac{1}{r_{ki}r_{kj}} - (\alpha_g+\alpha_p)\left(\frac{1}{r_{ji}^2} + \frac{1}{r_{jk}^2}\right)\frac{\mathbf{r}_{ji}\cdot\mathbf{r}_{jk}}{r_{ji}r_{jk}}\right]\\
&\quad\left. + \frac{Ge_i e_j m_k}{c^2}\left[+\frac{1}{r_{ij}r_{ik}} + \frac{1}{r_{ji}r_{jk}} - \frac{1}{r_{ki}r_{kj}} - (\alpha_g+\alpha_p)\left(\frac{1}{r_{ki}^2} + \frac{1}{r_{kj}^2}\right)\frac{\mathbf{r}_{ki}\cdot\mathbf{r}_{kj}}{r_{ki}r_{kj}}\right]\right\},
\end{aligned} \tag{60}$$

where $M_{ij} = m_i + m_j$ and $\mu_{ij} = m_i m_j / M_{ij}$. To obtain the above Lagrangian in B—EIH—D coordinates (the particular form given by Bażański[16]) one merely has to set $\alpha_g = \alpha_p = 0$. The above type of summation is not quite as compact as the form given by Bażański[16] but it does have the nice feature of avoiding permutations in the various combinations so that there is much less summing to do. For example, for the case of $n = 3$ there are only three i, j combinations in the double sum and one i, j, k combination in the triple sum. In the triple sum the terms proportional to $G^2 m_1 m_2 m_3$ agree with the field theory result of Hiida and Okamura[7] [see their Eqs. (3.3), (3.8), and (3.18)]. The velocity dependent terms in Eq. (60) are combined in the manner of Khan and O'Connell[5] while the velocity dependent terms in Eq. (1) are combined in the manner of Landau and Lifshitz.[6]

A. Condition of static balance

The generalization of Eq. (40) to n bodies is

$$e_i = \pm G^{1/2} m_i, \quad i = 1, \ldots, n. \tag{61}$$

Note that Eq. (44) cannot be generalized to n bodies. Using Eq. (61) in Eq. (60) we obtain

$$\begin{aligned} L(e_k = \pm G^{1/2} m_k, \text{ all } k) &= \sum_{i=1}^{n} \{\tfrac{1}{2} m_i v_i^2 + \tfrac{1}{8} m_i v_i^4/c^2\} + \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{G m_i m_j}{r_{ij}} \\ &\times \left[(\tfrac{3}{2} - \alpha_g + \alpha_p) \frac{v_{ij}^2}{c^2} + (\alpha_g - \alpha_p) \frac{(\mathbf{v}_{ij} \cdot \mathbf{r}_{ij})^2}{c^2 r_{ij}^2} \right]. \end{aligned} \tag{62}$$

We also have

$$\mathbf{F}_i = \dot{\mathbf{P}}_i, \tag{63}$$

where

$$\mathbf{F}_i \equiv \frac{\partial L}{\partial \mathbf{r}_i} \quad \text{and} \quad \mathbf{P}_i \equiv \frac{\partial L}{\partial \mathbf{v}_i}. \tag{64}$$

In the case where all the velocities are equal ($\mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_n$) we have

$$\mathbf{F}_i(e_k = \pm G^{1/2} m_k, \text{ all } k;\ \mathbf{v}_1 = \mathbf{v}_2 = \cdots = \mathbf{v}_n) = 0, \tag{65}$$

and thus the n bodies will move with constant velocity. A similar argument for the two-body problem (with $\alpha_g = \alpha_p = 0$) has been given by Khan and O'Connell.[5]

Das[17] has shown that $\sigma(\mathbf{r}) = \pm G^{1/2} \rho(\mathbf{r})$ [where $\sigma(\mathbf{r})$ is charge density and $\rho(\mathbf{r})$ is mass density] leads to the exact static solutions of Weyl,[18] Curzon,[19] Majumdar,[20] and Papapetrou.[21] We conclude that the static solution of Eq. (61) is exact to all orders. We do not know whether or not the static solution of Eq. (44) is exact beyond the post-Newtonian approximation. For exact stationary solutions see Refs. 22 and 23 and for exact solutions with magnetic charge see Refs. 24 and 25.

V. CONCLUSION

Starting with the Bażański post-Newtonian two body Lagrangian, we have shown that it is possible to construct a coordinate transformation (before going to center-of-mass coordinates) containing two arbitrary parameters α_g and α_p in such a manner as to produce a Hamiltonian consistent with that derived from quantum field theory and containing the same arbitrary parameters. In particular, we have introduced the new parameter x_p in the photon propagator which is treated in the same manner as the graviton propagator[7,8] with x_g. Thus, the electromagnetic and gravitational aspects of this paper are treated symmetrically. We have also generalized the above results to n bodies.

The condition for static balance,[5] $e_i = \pm G^{1/2} m_i$ has been examined and found to hold for the Reissner—Nordstrøm "one-body" problem and also for the post-Newtonian n-body problem. We also know that this solution is exact to all orders. For the post-Newtonian two-body problem an alternate condition for static balance $e_i = \pm (G m_1 m_2)^{1/2}$ has been found. We do not know if this condition is exact beyond the post-Newtonian approximation. *It would be interesting to check this conditions in the post-post-Newtonian two-body problem.*

We have found the precession of the perihelion for the post-Newtonian two-body problem with charge and have looked at four special cases: The first case agrees with the result of Robertson[10] and the large-mass approximation of the second case agrees with the result of Sommerfeld.[11]

We have also looked at the post-Newtonian two-body equations of motion, where the condition of static balance has been used, and found a solution mathematically similar to the well-known one for the gravitational deflection of light by the sun.

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APPENDIX

When we first sought to find the correct form of Eqs. (3) and (4) we knew from previous work[7,8] that Eq. (5) had to be in the form

$$\mathbf{r}_B = \mathbf{r} \left(1 - \alpha_g \frac{GM}{c^2 r} + \alpha_p \frac{e_1 e_2}{\mu c^2 r}\right). \tag{5}$$

We first tried the transformations

$$\mathbf{r}_{1B} = \mathbf{r}_1 \left(1 - \alpha_g \frac{GM}{c^2 r} + \alpha_p \frac{e_1 e_2}{\mu c^2 r}\right), \tag{3a}$$

$$\mathbf{r}_{2B} = \mathbf{r}_2 \left(1 - \alpha_g \frac{GM}{c^2 r} + \alpha_p \frac{e_1 e_2}{\mu c^2 r}\right), \tag{4a}$$

which are consistent with Eq. (5). However, besides being *inconsistent* with the field theory results (see Sec. I, part B), the Bażański Lagrangian in these new coordinates $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{v}_1$, $\mathbf{v}_2$ had a peculiar form which could *not* be expressed as a function of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}$. If we now were to change to a new coordinate system where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $\mathbf{r}_{CM} = \nu_1 \mathbf{r}_1 + \nu_2 \mathbf{r}_2$ with $\nu_1 + \nu_2 = 1$ ($\mathbf{r}_1 = \mathbf{r}_{CM} + \nu_2 \mathbf{r}$, $\mathbf{r}_2 = \mathbf{r}_{CM} - \nu_1 \mathbf{r}$ in inverse form) the Lagrangian $L(\mathbf{r}, \mathbf{v}; \mathbf{r}_{CM}, \mathbf{v}_{CM})$ would not be cyclic for the coordinate $\mathbf{r}_{CM}$. The corresponding momentum relations are $\mathbf{P} = \nu_2 \mathbf{P}_1 - \nu_1 \mathbf{P}_2$, $\mathbf{P}_{CM} = \mathbf{P}_1 + \mathbf{P}_2$ ($\mathbf{P}_1 = \nu_1 \mathbf{P}_{CM} + \mathbf{P}$, $\mathbf{P}_2 = \nu_2 \mathbf{P}_{CM} - \mathbf{P}$ in inverse form). Since this particular Lagrangian contains the coordinate $\mathbf{r}_{CM}$, its canonical momentum $\mathbf{P}_{CM}$ does *not* satisfy the relation $\dot{\mathbf{P}}_{CM} = 0$. However, $\mathbf{P}_{CM}$ *does* satisfy the relation

$$\dot{\mathbf{P}}_{CM} = \left(\alpha_g \frac{GM}{c^2 r} - \alpha_p \frac{e_1 e_2}{\mu c^2 r}\right) \frac{(\mathbf{v} \cdot \mathbf{r})}{r^2} \mathbf{P}_{CM}, \tag{A1}$$

so that we can still introduce center-of-mass coordinates where $\mathbf{P}_{CM} = 0$.

We next tried the transformations

$$\mathbf{r}_{1B} = \mathbf{r}_1 - \tfrac{1}{2} \mathbf{r} \left(\alpha_g \frac{GM}{c^2 r} - \alpha_p \frac{e_1 e_2}{\mu c^2 r}\right), \tag{3b}$$

$$\mathbf{r}_{2B} = \mathbf{r}_2 + \tfrac{1}{2} \mathbf{r} \left(\alpha_g \frac{GM}{c^2 r} - \alpha_p \frac{e_1 e_2}{\mu c^2 r}\right), \tag{4b}$$

which are again consistent with Eq. (5). While the Lagrangian resulting from these transformations *could* be expressed as a function of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}$, it was, nevertheless, *inconsistent* with the field theory results.

The correct transformations, Eqs. (3) and (4), which are consistent with Eq. (5), were next tried. The Lagrangian resulting from these transformations *can* be expressed as a function of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{r}$ and *is* consistent with the field theory results.

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