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Two-body problems—A unified, classical, and simple treatment of spin-orbit effects

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We show how the spin-orbit contribution to the two-body Hamiltonian, to order c^{-2}, may be derived classically, in a simple but rigorous manner, from a knowledge of just the corresponding contributions for the one-body problem. As an illustration of the generality of our results, we consider, as particular cases, the electromagnetic and gravitational interactions and demonstrate how the usual results follow directly.

The Hamiltonian, to order c^{-2}, for the electromagnetic interaction between two charged bodies, of arbitrary masses and spins, was derived by Breit[1] in 1929, and the corresponding gravitational interaction by Barker and O'Connell[2] in 1975. Modern derivations of the Breit interaction use field-theory techniques[3] and this was also the basis of the approach of Barker and O'Connell. A key element which emerges from the work of the latter authors is that one can write down a Hamiltonian which describes both microscopic and macroscopic interactions[2,4]—from the gravitational interaction of elementary spinning particles with arbitrary spin values to the gravitational interaction of massive rotating celestial bodies.[5] The results in the latter case have now been verified by a variety of purely classical techniques.[6-9] However, since all of these derivations are rather lengthy, compared to the derivation when one particle is heavy ("one-body" problem), we feel it is worthwhile to show that the spin-orbit contribution to the two-body Hamiltonian, to order c^{-2}, may be derived classically, in a simple but rigorous manner, from a knowledge of just the corresponding contributions for the one-body problem.

Let m_i, $\vec{r}_i$, $\vec{v}_i$, $\vec{P}_i$, and $\vec{S}^{(i)}$ $(i=1,2)$ denote the masses, coordinates, velocities, canonical momenta, and spins of bodies 1 and 2, respectively. In addition, let e_1 and $-e_2$ denote the corresponding charges (the choice of charges with opposite signs is purely for convenience). Then the spin-orbit interaction of body 2 in its own rest system may be written as

$$U_2' = \frac{y}{m_2 c^2 r}\, \vec{S}^{(2)}\cdot(\vec{r}\times\vec{v})\,\frac{dV}{dr}\,, \tag{1}$$

where $\vec{r}\equiv\vec{r}_1-\vec{r}_2$, $\vec{v}\equiv\vec{v}_1-\vec{v}_2$, V is the static nonrelativistic potential $[V=-(e_1e_2/r)$ for the electromagnetic interaction and $V=-(Gm_1m_2/r)$ for the gravitational interaction], and y is a parameter which characterizes the interaction. For the electromagnetic interaction,[10] as is well known,[10] $y=g/2$, whereas for the Einstein gravitational interaction $y=2$, both for the interaction of elementary particles[4] and macroscopic bodies.[4,11,12]

We now consider the spin-orbit interaction energy, due to the spin of body 2, in the center-of-momentum system $(\vec{P}_1=-\vec{P}_2=\vec{P})$. It is simply

$$U_2 = U_2' + U_2^T\,, \tag{2}$$

where U_2^T, the Thomas interaction term[10] associated with body 2, is given explicitly by

$$U_2^{(T)} = \frac{1}{2c^2}\,\vec{S}^{(2)}\cdot(\vec{a}_2\times\vec{v}_2)\,, \tag{3}$$

where $\vec{a}_2$ is the acceleration of body 2. Thus

$$U_2^{(T)} = -\frac{1}{2m_2c^2r}\,\vec{S}^{(2)}\cdot(\vec{r}\times\vec{v}_2)\,\frac{dV}{dr}\,. \tag{4}$$

Now, to lowest order, $\vec{v}_2=\vec{P}_2/m_2$ and

$$\vec{P}=\mu\vec{v}\,, \tag{5}$$

where $\mu=m_1m_2/(m_1+m_2)$. It follows, from Eqs. (1), (2), (4), and (5), that

$$U_2 = \frac{1}{m_1m_2c^2r}\,\frac{dV}{dr}\left[y\left(1+\frac{m_1}{m_2}\right)-\frac{1}{2}\,\frac{m_1}{m_2}\right]\vec{S}^{(2)}\cdot\vec{L}\,, \tag{6}$$

where $\vec{L}=\vec{r}\times\vec{P}$ is the total angular momentum of the two-body system. Similarly,

$$U_1 = \frac{1}{m_1m_2c^2r}\,\frac{dV}{dr}\left[y\left(1+\frac{m_2}{m_1}\right)-\frac{1}{2}\,\frac{m_2}{m_1}\right]\vec{S}^{(1)}\cdot\vec{L}\,. \tag{7}$$

Now, to order c^{-2}, the spin-orbit coupling interaction involves only lowest order in the appropriate coupling constant. This linearity in the spin-orbit contribution implies that the sum

$$U=U_1+U_2 \tag{8}$$

is thus the total spin-orbit interaction for the two-body system, in the center-of-momentum coordinate system. In particular, for the electromagnetic $[U\equiv U_e$ and we write $g\equiv 2(1+\kappa)$, where κ is the anomalous magnetic moment] and gravitational interactions $(U\equiv U_g)$ we obtain

$$U_e=\left(\frac{e_1e_2}{m_1m_2}\right)\frac{1}{c^2r^3}\left\{\left[(\kappa+1)+(\kappa+\tfrac{1}{2})\frac{m_1}{m_2}\right]\vec{S}^{(2)}\cdot\vec{L}+(1\leftrightarrow 2)\right\} \tag{9}$$

and

$$U_g=G\,\frac{1}{c^2r^3}\left[\left(2+\frac{3}{2}\,\frac{m_1}{m_2}\right)\vec{S}^{(2)}\cdot\vec{L}+(1\leftrightarrow 2)\right], \tag{10}$$

in agreement with the results of Breit,[1] and Barker and O'Connell,[2] respectively.

In the case of arbitrary theories of gravitation[13] $[U\equiv U_g(\gamma)]$, $y=\gamma+1$, where γ is the familiar parametrized-post-Newtonian parameter. Hence

$$U_g(\gamma)=G\,\frac{1}{c^2r^3}\left\{\left[(\gamma+1)+(\gamma+\tfrac{1}{2})\frac{m_1}{m_2}\right]\vec{S}^{(2)}\cdot\vec{L}+(1\rightarrow 2)\right\}, \tag{11}$$

again in agreement with previously obtained results.[6,13]

A comment is necessary on the use of the U_i^T $(i=1,2)$ terms for the gravitational interactions. These Thomas terms[10] arise from the use of the Lorentz transformation and one might object that such a transformation is no longer appropriate in a curved space. However, while this is so in general, we are working only to order c^{-2}, and thus it becomes clear that we may ignore gravitational corrections to the Lorentz transformation.

Finally, we wish to emphasize that Eqs. (6) to (8) are applicable to *any* static potential—for example, to the Yukawa potential, to the Hulthén potentials, and to combined potentials (the latter is of interest for the treatment of charged black holes).

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[10] J. D. Jackson, *Classical Electrodynamics*, 2nd edition (Wiley, New York, 1975), pp. 543–546. We denote the gyromagnetic ratio by g.

[11] See Ref. 9. This derivation is purely classical. It is based on an analysis of the motion a nonspinning test body of mass m_1 in the gravitational field of a stationary body of mass m_2 and rotational angular momentum $\vec{S}^{(2)}$. To the order required, the metric of body 2 is simply the well-known Lense-Thirring metric.

[12] One of us (Lai-Him Chan, unpublished) has derived the result $y=2$ from an analysis of the geodesic equations of motion of body 1 in the stationary field due to body 2. Using techniques analogous to those used in electrodynamics it follows that $U_2'=-\vec{\mu}_g^{(2)}\cdot\vec{B}_g^{(2)}$, where $\vec{\mu}_g^{(2)}=\vec{S}^{(2)}/2c$ is the effective "gravitational magnetic moment" of body 2 and $\vec{B}_g^{(1)}=4(\vec{r}\times\vec{v})(dV/dr)/cr$ is the effective "gravitational magnetic field" due to the motion of body 1, as seen from the rest frame of body 2.

[13] B. M. Barker and R. F. O'Connell, Phys. Rev. D 14, 861 (1976).