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Contact Interactions in the Einstein and Einstein-Cartan-Sciama-Kibble (ECSK) Theories of Gravitation

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I show that spin-spin contact interactions occur in a quantum version of Einstein's conventional theory, which are of the same order of magnitude as those occurring in Einstein-Cartan-Sciama-Kibble (ECSK) theory. Thus, all the features which were thought to be peculiar to the ECSK theory—singularity behavior, possibility of stopping gravitational collapse, and so on—actually occur in Einstein's theory. In addition, I examine a contact term (independent of spin orientations and analogous to the Darwin term of quantum electrodynamics) which always gives rise to repulsive gravitation forces.

Perhaps the most theoretically appealing alternative to Einstein's general theory of relativity is the ECSK (Einstein-Cartan[1]-Sciama[2,3]-Kibble[4]) theory. This is basically a generalization of Einstein's theory in which the spin of matter, as well as its mass, plays a dynamical role. This is achieved by the introduction of a nonsymmetric affine connection $\Gamma_{ij}{}^{k}$ (while retaining a symmetric metric tensor). The non-Riemannian part of the connection is the so-called contortion tensor $K_{ij}{}^{k}$, which couples to the spin angular momentum density $\tau_{k}{}^{ji}$.

The nonsymmetric part of the connection is the Cartan torsion tensor $S_{ij}{}^{k}$, so called in recognition of the fact that Cartan was the first to introduce the concept of torsion of space-time. However, Cartan did not develop a detailed theory but this is not surprising when one recalls that Cartan's work preceded the introduction of the spin of an elementary particle by Goudsmit and Uhlenbeck.[5] The dynamical role of spin was clearly recognized for the first time by Sciama.[2] In essence, Sciama pointed out that an expression for the canonical spin angular momentum is obtained by varying the matter Lagrangian density $\mathcal{L}_M$ with respect to the contortion tensor $K_{ij}{}^{k}$ analogous to the derivation of the canonical energy-momentum tensor σ_{ij} from the variation of $\mathcal{L}_M$ with respect to the metric tensor g_{ij}.

The ECSK field equations were derived by Sciama[2,3] and Kibble[4] using gauge theory. Recent interest in the theory, particularly in its geometric aspects, has been spearheaded by the work of Hehl and Trautman and their respective associates, which is summarized in the recent exhaustive review of Hehl *et al.*[6]

One of the most striking features of the ECSK theory is the prediction that[6] "...in addition to all the gravitational interactions between particles that occur in general relativity...there is a new, very weak, universal spin contact interaction of gravitational origin." This may be expressed mathematically by saying that, to first order in k ($k \equiv 8\pi G/c^4$, where G is the Newtonian gravitational constant), the non-Riemannian contribution to the total Lagrangian is

$$\Delta\mathcal{L} = \tfrac{1}{2}\tau_{k}{}^{ji}K_{ij}{}^{k} = k\{-\tfrac{1}{2}\tau_{ijk}\tau^{ijk} + \tau_{ijk}\tau^{jki} + \tau_{ik}{}^{k}\tau^{il}{}_{l}\}. \qquad (1)$$

However—and this is a point not generally appreciated—a quantum version of conventional general relativity[7-10] also includes a spin contact interaction between two elementary spinning particles. In the case of the interaction of n particles, each with spin $\frac{1}{2}\hbar\vec{\sigma}$, the spin contact contribution to the Lagrangian in Einstein theory, $\Delta\mathcal{L}_E^{(1)}$ say, is[7]

$$\Delta\mathcal{L}_E^{(1)} = -\frac{\pi}{3} m c^2 r_s \lambda_c^2 \sum_{\substack{a=1\\a>b}} \sum_{b=1} \vec{\sigma}_a \cdot \vec{\sigma}_b \, \delta(\vec{r}_a - \vec{r}_b), \quad (2)$$

where $r_s \equiv 2Gm/c^2$ is the Schwarzschild radius and $\lambda_c \equiv \hbar/mc$ is the Compton wavelength.

The question I wish to examine is the relationship between $\Delta\mathcal{L}$ and $\Delta\mathcal{L}_E^{(1)}$. Now, in ECSK theory, matter is treated as a continuum—σ_{ij} and τ_{ijk} give the distribution of energy-momentum and spin, respectively, over space-time. Thus, our first task will be to convert $\Delta\mathcal{L}$ into discrete particle-particle interaction form.

I define the spin tensor density of rank 2, τ_{ij}, in the usual way, viz.[11]

$$\tau_{ij}{}^k = \tau_{ij} u^k, \quad (3)$$

where u_i is the four-velocity, with the subsidiary condition[12]

$$\tau_{ij} u^j = 0. \quad (4)$$

Next, I set

$$\tau^{ij} = \epsilon^{ijkl} S_k u_l, \quad (5)$$

where S_k is the spin four-vector density. Equations (4) and (5) imply that

$$S_i u_i \equiv S \cdot u = 0. \quad (6)$$

I can introduce a contortion vector K_i in a similar manner. Then, after some algebra, I find that $\Delta\mathcal{L}$ may be written in the simple form

$$\Delta\mathcal{L} = -S \cdot K = -k S \cdot S. \quad (7)$$

I am still treating matter as a continuum. However, as with Trautman[13] and Hehl *et al.*,[6] I can consider the continuum to consist of elementary particles, each of spin $\frac{1}{2}$. Hence, I am led to write

$$S(\vec{r}) = \tfrac{1}{2}\hbar \textstyle\sum_a \sigma_a \, \delta(\vec{r} - \vec{r}_a), \quad (8)$$

where "a" refers to a summation over particles. Now consider the "rest" picture where $S = (\vec{S}, 0)$. Here

$$S \cdot S = |\vec{S}|^2 = \tfrac{1}{4}\hbar^2 \textstyle\sum_a \sum_b \vec{\sigma}_a \cdot \vec{\sigma}_b \, \delta(\vec{r} - \vec{r}_a) \delta(\vec{r} - \vec{r}_b), \quad (9)$$

where $\vec{\sigma}_a$ and $\vec{\sigma}_b$ refer to the Pauli spinors of particles a and b.

The infinite self-interaction terms (corresponding to $a = b$) occurring in the summation may be removed by renormalization, with the result that

$$\Delta\mathcal{L} = -\tfrac{1}{2} k \hbar^2 \sum_{a>b} \sum_b \vec{\sigma}_a \cdot \vec{\sigma}_b \, \delta(\vec{r} - \vec{r}_b). \quad (10)$$

Comparing Eqs. (2) and (10), I conclude that

$$\Delta\mathcal{L}_E^{(1)} = \tfrac{1}{6} \Delta\mathcal{L}. \quad (11)$$

Hence, the magnitude of the spin contact interaction terms inherent in Einstein's theory are one-sixth of the corresponding terms in ECSK theory.[14] However, the important point I wish to emphasize is that if Einstein's theory is *the* correct theory of gravitation, then such spin contact terms do occur with all the attendant consequences which were thought to be peculiar to the ECSK theory —particularly those pertaining to the singularity behavior and the possibility of stopping gravitational collapse.[13,6]

On reflection, the occurrence of such a spin contact term in Einstein's theory should not be too surprising because, as I have remarked previously,[7] a similar term occurs in quantum electrodynamics—it is the so-called Fermi term, which is responsible for the hyperfine splitting of the ground state of the hydrogen atom.

In addition to $\Delta\mathcal{L}_E^{(1)}$, there are also noncontact spin-spin as well as spin-orbit contributions to the Lagrangian, which can give rise to repulsive gravitational interactions.[15,16] Furthermore, there is another kind of contact contribution, viz.

$$\Delta\mathcal{L}_E^{(2)} = -\frac{7\pi}{3} m c^2 r_s \lambda_c \sum_{a>b} \sum_b \delta(\vec{r}_a - \vec{r}_b). \quad (12)$$

This term is analogous to the so-called Darwin term, which contributes to the ground-state fine-structure energy of the hydrogen atom. The physical origin of the Darwin term is generally considered to be associated with the *Zitterbewegung* of the electron. It is thus very suggestive that $\Delta\mathcal{L}_E^{(2)}$ is due to "gravitational *Zitterbewegung*." Because it gives rise to a repulsive interaction, regardless of spin orientations, we conclude that it is more likely to impede gravitational collapse than the spin contact terms. At the least, its contribution at densities[13] $\approx m/r_s \lambda_c^2 \approx m/L_P^2 \lambda_c \approx 10^{54}$ g cm^{-3} [$L_P = (\hbar c k)^{1/2}$ is the Planck length] is as important as mass contributions.

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[1]É. Cartan, C. R. Acad. Sci. 174, 593 (1972), and Ann. Ec. Norm. Sup. 40, 325 (1923), and 41, 1 (1924), and 42, 17 (1925).

[2]D. W. Sciama, *Recent Developments in General Relativity* (Pergamon, New York, 1962).

[3]D. W. Sciama, Rev. Mod. Phys. 36, 463, 1103 (1964).

[4]T. W. B. Kibble, J. Math. Phys. (N.Y.) 2, 212 (1961).

[5]S. A. Goudsmit and G. E. Uhlenbeck, Physica (The Hague) 5, 266 (1925).

[6]F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, Rev. Mod. Phys. 48, 393 (1976).

[7]R. F. O'Connell, Gen. Relat. Grav. 6, 99 (1975) [see Eqs. (2e) and (3)].

[8]B. M. Barker, S. N. Gupta, and R. D. Haracz, Phys. Rev. 149, 1027 (1966).

[9]B. M. Barker and R. F. O'Connell, Phys. Rev. D 2, 1428 (1970), and 12, 329 (1975), and 14, 861 (1976), and Astrophys. J. 199, L25 (1975).

[10]A key element which emerges from the work of Ref. 9 is that one can write down a Hamiltonian H which describes both microscopic and macroscopic two-body interactions—from the gravitational interaction of elementary spinning particles with arbitrary spin values to the gravitational interaction of massive rotating celestial bodies. The reliability of these results is attested to by the fact that the spin-independent part of H reproduces the Einstein-Infeld-Hoffmann results whereas the spin-dependent parts have been verified by a variety of purely classical calculations.

[11]Whereas this is a common choice for the spin density, one could envisage a different form resulting from the transition from a continuum field to a particle picture. Fields whose spin density is significantly different from this form may give rise to different spin-spin contact interactions in the ECSK theory. This point merits further investigation.

[12]J. Frenkel, Z. Phys. 37, 243 (1926); F. Pirani, Acta Phys. Pol. 15, 389 (1956).

[13]A. Trautman, Nature (London) Phys. Sci. 242, 7 (1973).

[14]We should emphasize that the ECSK spin contact terms under discussion are these which do not occur in general relativity. Thus, the quantum version of the ECSK theory should also have terms of the same kind as those which occur in the quantum version of Einstein's theory, in addition to those terms peculiar to the usual classical version.

[15]R. F. O'Connell, Phys. Lett. 32A, 402 (1970).

[16]In fact, it turns out that the gravitational spin-spin contact and spin-spin noncontact terms may be obtained from the corresponding terms in electrodynamics by simply letting $e_1e_2 \rightarrow -Gm_1m_2$. However, such a replacement does not hold for the spin-orbit terms (see Ref. 9).