Lagrangian-Hamiltonian formalism for the gravitational two-body problem with spin and parametrized post-Newtonian parameters $\gamma$ and $\beta$

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We generalize the Lagrangian and the Hamiltonian of our previous work on the gravitational two-body problem with spin by including the parametrized post-Newtonian parameters $\gamma$ and $\beta$. By this procedure we are able to obtain the precession of the orbit as well as the precession of the spin. Equations of motion corresponding to an arbitrary spin supplementary condition are also given. Finally we show how the masses of the binary pulsar PSR 1913 + 16 and its companion are related to the orbit and spin precessions. Combining this with a result derivable from the second-order Doppler effect and the gravitational red-shift, we obtain a relation constraining the values that $\gamma$ and $\beta$ can take.

I. INTRODUCTION

We have recently given a complete post-Newtonian solution (including a Lagrangian and a Hamiltonian) to the gravitational two-body problem with spin within the framework of general relativity. In previous work we used the one-graviton-exchange interaction of two spin-$1$ particles derived from Gupta's quantum theory of gravitation to obtain a classical potential-energy term, which we then used to construct a Hamiltonian and a Lagrangian. This enabled us to obtain the precession of the spin of either body, the equations of motion, and the precession of the orbit.

Our results were soon verified by several purely classical calculations, including a simple heuristic derivation of our own. Börner, Ehlers, and Rudolph, using the formalism given by Misner, Thorne and Wheeler (MTW), found the spin precession in two-body systems where the parametrized post-Newtonian (PPN) parameter $\gamma$ was included. Cho and Dass, using Schwinger's source theory, gave the Hamiltonian and the spin precession for the two-body problem in general relativity. Also, D'Eath, using a method of matched asymptotic expansions of the Kerr geometry, derived the spin precession and the equations of motion for two black holes in general relativity. The papers of Refs. 6, 8, and 10 did not include the results for the spin-dependent terms in the precession of the orbit.

In this paper we wish to generalize our previous results by giving a Lagrangian and a Hamiltonian with PPN parameters $\gamma$ and $\beta$. We believe that it is highly desirable to give a Lagrangian and a Hamiltonian regardless of what type of derivation one might be using, for then it is possible to obtain the precession of the spin, the equations of motion, and the precession of the orbit (including spin-dependent terms). We shall restrict ourselves to conservative theories with no preferred frames; then, the number of PPN parameters is reduced to only two ($\gamma$ and $\beta$).

In Sec. II we shall give the $\gamma, \beta$ form of the Lagrangian and the Hamiltonian without spin. We shall then make the same coordinate transformation as was made in Sec. II of Ref. 1. This will introduce a new dimensionless parameter $\alpha$ into the Hamiltonian and the Lagrangian which should not be confused with the Eddington-$\alpha$, parameter $\alpha$, which is set equal to unity throughout this paper to ensure the correct Newtonian limit. In Sec. III we modify the spin-dependent terms in the Lagrangian and the Hamiltonian of Ref. 1 to include $\gamma$ (there is no $\beta$ term) and give the total Lagrangian and Hamiltonian. As the quadrupole-moment results are strictly Newtonian, they would not be modified by including the PPN parameters $\gamma$ and $\beta$. Therefore in this paper we will not include any quadrupole-moment effects but instead refer the reader to Refs. 1 and 3. In Sec. IV we give both the instantaneous and the averaged precession of the spin. In Sec. V the equations of motion are presented. A shift in the centers of mass of the bodies is then made so that two dimensionless parameters $\lambda_1$ and $\lambda_2$ are introduced into the equation of motion, corresponding to an arbitrary spin supplementary condition. After the large-mass approximation is made we can see which spin supplementary condition is associated with certain values of these parameters. In Sec. VI we find the precession of the orbit and note...
that it is independent of the parameters \( \lambda_1, \lambda_2, \) and \( \alpha. \) It should be emphasized that the values of \( \gamma \) and \( \beta \) depend on which theory of gravitation one is using, while \( \lambda_1, \lambda_2, \) and \( \alpha \) can take arbitrary values in any theory of gravitation. In Sec. VII we show how the masses of the binary pulsar PSR 1913+16 and its companion may be evaluated from a knowledge of the orbital and spin precessions. A mass relation derivable from the second-order Doppler effect and the gravitational redshift is combined with the above to give an equation constraining the values that \( \gamma \) and \( \beta \) can have. In Sec. VIII we present our conclusions.

II. LAGRANGIAN AND HAMILTONIAN WITHOUT SPIN

The \( n \)-body post-Newtonian Lagrangian and equations of motion containing the PPN parameter \( \gamma \) (but not \( \beta \)) are given by Moyer, and the \( n \)-body equations of motion containing the PPN parameters \( \gamma \) and \( \beta \) are given by Anderson. From the above results it is easy to show that the two-body Lagrangian with PPN parameters \( \gamma \) and \( \beta \) is

\[
\mathcal{L}(\vec{r}_1, \vec{v}_1, \vec{r}_2, \vec{v}_2) = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{k m_1 m_2}{r^{3}} + \frac{\alpha m_1 m_2}{r^{2}} - \frac{G m_1 m_2}{r} \left[ 1 + \frac{1 + 2\gamma}{2c^2} (v_1^2 + v_2^2) - \frac{3 + 4\gamma}{2c^2} \left( \frac{1}{r} \cdot \vec{v}_1 \right) \cdot \vec{r}_1 \right] - \frac{1 - 2\beta}{2c^2} \frac{G^3 m_1 m_2 (m_1 + m_2)}{r^3},
\]

where \( c \) and \( G \) are the speed of light and the gravitational constant, respectively, \( \vec{r} = \vec{r}_1 - \vec{r}_2 \), and thus \( \vec{v} = \vec{v}_1 - \vec{v}_2 \). For general relativity we have \( \gamma = \beta = 1 \), while for Brans-Dicke theory \( \gamma = (1 + \omega)/(2 + \omega) \) and \( \beta = 1 \). By the usual standard procedure we obtain the Hamiltonian

\[
3c(\vec{r}, \vec{P}, \vec{r}_2, \vec{P}_2) = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2} - \frac{P_1^4}{8m_1^3 c^2} - \frac{P_2^4}{8m_2^3 c^2} - \frac{G m_2 m_2}{r} \left[ 1 + \frac{1 + 2\gamma}{2c^2} \left( \frac{P_1^2}{m_1^3} + \frac{P_2^2}{m_2^3} \right) - \frac{3 + 4\gamma}{2c^2} \frac{\vec{P}_1 \cdot \vec{r}_1 (\vec{P}_2 \cdot \vec{r}_2)}{m_1 m_2} \right] + \frac{2\beta - 1}{2c^2} G^2 m_1 m_2 (m_1 + m_2)
\]

Going to center-of-mass coordinates where \( \vec{P} = \vec{P}_1 = - \vec{P}_2 \) we obtain

\[
3c(\vec{r}, \vec{P}) = \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) P^2 - \frac{1}{8} \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) P^4 c^2 - \frac{G m_1 m_2}{r} \left[ 1 + \frac{1 + 2\gamma}{2c^2} \left( \frac{1}{m_1^3} + \frac{1}{m_2^3} \right) P^2 + \frac{3 + 4\gamma}{2c^2} \frac{P_2^2}{m_1 m_2} + \frac{(\vec{P} \cdot \vec{r})^2}{2c^2 r^2 m_1 m_2} \right] + \frac{2\beta - 1}{2c^2} G^2 m_1 m_2 (m_1 + m_2)
\]

The Lagrangian corresponding to Eq. (3) is

\[
\mathcal{L}(\vec{v}, \vec{\dot{v}}) = \frac{1}{2} \mu \dot{v}^2 + \frac{3\mu}{M} \frac{GM^2}{c^2} \left[ 1 + \frac{1 + 2\gamma}{2} \frac{\dot{v}^2}{c^2} + \frac{1}{2} \frac{\mu}{M} \frac{c^2}{c^2 r^2} \right] + (1 - 2\beta) \frac{G^2 \mu M^2}{2c^2 r^2},
\]

where the reduced mass and the total mass are given by \( \mu = m_1 m_2/(m_1 + m_2) \) and \( M = m_1 + m_2 \), respectively.

Let us change notation slightly and denote \( \vec{r} \) and \( \vec{P} \) of Eq. (3) as \( \vec{r}_{EHI, \gamma \beta}, \vec{P}_{EHI, \gamma \beta} \), and \( \vec{v}_{EHI, \gamma \beta} \). We do this because the coordinates of Eqs. (1)–(4) are Einstein–Infeld–Hoffmann (EIH) coordinates when \( \gamma = \beta = 1 \). We shall now make the coordinate transformation

\[
\vec{r}_{EHI, \gamma \beta} = \vec{r} \left( 1 - \frac{\alpha}{c^2 r} \frac{GM}{c^2 r} \right),
\]

which implies that

\[
\vec{P}_{EHI, \gamma \beta} = \vec{P} + \alpha \frac{GM}{c^2 r} \left( \vec{P} - \frac{\vec{P} \cdot \vec{r}}{r^2} \vec{r} \right),
\]

\[
\vec{v}_{EHI, \gamma \beta} = \vec{v} - \alpha \frac{GM}{c^2 r} \left( \vec{v} - \frac{\vec{v} \cdot \vec{r}}{r^2} \vec{r} \right).
\]
where $\alpha$ is an arbitrary dimensionless parameter. The Hamiltonian of Eq. (3) in the new coordinates is

$$\mathcal{H} = 3\mathcal{H}_0 + V_1(\alpha) + V_2(\alpha),$$

where

$$\mathcal{H}_0 = \frac{1}{2} \left( \frac{1}{m_1} + \frac{1}{m_2} \right) P^2 - \frac{1}{8} \left( \frac{1}{m_1^2} + \frac{1}{m_2^2} \right) \frac{P^4}{c^2},$$

$$V_1(\alpha) = -\frac{Gm_1m_2}{r} \left\{ 1 + \left( \frac{1}{2} + \frac{1}{\gamma} - \alpha \right) \frac{M}{\mu} + \left( \frac{1}{2} + \alpha \frac{M}{\mu} \right) \frac{(\vec{P} \cdot \vec{r})^2}{m_1m_2c^2} \right\},$$

$$V_2(\alpha) = (2\beta - 1 - 2\alpha) \frac{G^2\mu M^2}{2c^2r^2}.$$  

The Lagrangian corresponding to Eq. (8) can be written as

$$\mathcal{L}(\alpha) = \frac{1}{2} \mu v^2 + \frac{1}{8} \left( 1 - \frac{3\mu}{M} \right) \frac{\mu v^2}{c^2} + \frac{G\mu M}{r} \left\{ 1 + \left( \frac{1}{2} + \alpha + \frac{1}{2} \frac{M}{\mu} \right) \frac{v^2}{c^2} + \left( \alpha + \frac{1}{2} \frac{\mu}{M} \right) \frac{(\vec{v} \cdot \vec{r})^2}{c^2r^2} \right\} + (1 - 2\beta + 2\alpha) \frac{G^2\mu M^2}{2c^2r^2}.$$  

The Lagrangian of Eq. (12) can be written in a very simple form if we set $\alpha = -\mu/2M$ to obtain

$$\mathcal{L}(\alpha = -\mu/2M) = \frac{1}{2} \mu v^2 + \frac{G\mu M}{r} \left[ 1 + \left( \frac{1}{2} + \gamma - \alpha + \frac{1}{2} \frac{M}{\mu} \right) \frac{v^2}{c^2} + \left( \alpha + \frac{1}{2} \frac{\mu}{M} \right) \frac{(\vec{v} \cdot \vec{r})^2}{c^2r^2} \right] + (1 - 2\beta + 2\alpha) \frac{G^2\mu M^2}{2c^2r^2},$$

where

$$k_1 = 1 - 3\mu/M,$$

$$k_2 = \frac{3}{2} + \frac{1}{\gamma} + \frac{3}{2} \frac{\mu}{M},$$

$$k_3 = 2\beta - 1 - \mu/M.$$

It should be noted that if we make the large-mass approximation $m_1 \gg m_2$, and set $\alpha = \beta = 1$, then $k_1 = k_2 = k_3 = 1$. The same will hold for $k_4$, $k_5$, and $k_6$ of Eqs. (33)–(35).

III. TOTAL LAGRANGIAN AND HAMILTONIAN

The total Hamiltonian for the two-body problem with spin and PPN parameters $\gamma$ and $\beta$ is given by

$$\mathcal{H}(\alpha) = M^2 + \mathcal{H}(\alpha) + V_{S_1} + V_{S_2} + V_{S_1,s} + V_{S_2,s},$$

where $\mathcal{H}(\alpha)$ is given by Eq. (8) and

$$V_{S_1} = \frac{G}{c^2r} \left[ \gamma + 1 + \left( \frac{1}{2} + \frac{m_2}{m_1} \right) \frac{\mathcal{S}^{(1)} \cdot (\vec{r} \times \vec{P})}{r} \right],$$

$$V_{S_2} = \frac{G}{c^2r} \left[ \gamma + 1 + \left( \frac{1}{2} + \frac{m_1}{m_2} \right) \frac{\mathcal{S}^{(2)} \cdot (\vec{r} \times \vec{P})}{r} \right],$$

$$V_{S_1,s} = \frac{G}{2c^2r^3} \left[ \frac{\mathcal{S}^{(1)} \cdot (\vec{r} \times \vec{P})}{r^2} - \mathcal{S}^{(1)} \cdot \mathcal{S}^{(2)} \right],$$

$$V_{S_2,s} = \frac{G}{2c^2r^3} \left[ \frac{\mathcal{S}^{(2)} \cdot (\vec{r} \times \vec{P})}{r^2} - \mathcal{S}^{(2)} \cdot \mathcal{S}^{(1)} \right],$$

where $\mathcal{S}^{(1)}$, $\mathcal{S}^{(2)}$, $\vec{u}^{(1)}$, $\vec{u}^{(2)}$ are the spin, moment of inertia, and angular velocity of bodies 1 and 2, respectively. To lowest order, $\mathcal{S}^{(1)} = f^{(1)}\mathcal{O}^{(1)}$, $\mathcal{S}^{(2)} = f^{(2)}\mathcal{O}^{(2)}$, and $\vec{P} = \mu\vec{v}$.

The potential-energy terms $V_{S_1}$, $V_{S_2}$, and $V_{S_1,s}$, $V_{S_2,s}$ of Eqs. (18)–(20) have been obtained by modifying the results of Ref. 1 to include $\gamma$. We check that the factors of $(\gamma + 1)$ and $(\gamma + \frac{1}{2})$ in Eqs. (18)–(20) are correct by making the large-mass approximation $(m_2 \gg m_1)$ and comparing our results with known results for the one-body problem. The $(\gamma + \frac{1}{2})$ term in $V_{S_2}$ leads to the correct de Sitter (or geodetic) spin precession as given by Schif.15 The $(\gamma + 1)$ term of $V_{S_1,s}$ leads to the correct Lense-Thirring spin precession as given by O’Connell.16 When $m_2 \gg m_1$, the $V_{S_2}$ and $V_{S_1,s}$ terms are both proportional to the off-diagonal terms in the metric tensor and hence must both be proportional to the same factor $(\gamma + 1)$. Then by symmetry the other terms in $V_{S_1}$ and $V_{S_2}$, in the general case when $m_1$ and $m_2$ are arbitrary, are determined.

The above argument for the determination of the factors in $V_{S_1}$ and $V_{S_2}$ or $V_{S_1,s}$ and $V_{S_2,s}$ cannot tell us whether or not there is an additional factor proportional to $(1 - \gamma)\mu/M$ which would be zero in general relativity or zero in the large-mass approximation.3 However, such a term does not exist since the spin precession in the two-body problem, given in Sec. III and derived from Eqs. (18)–(20), agrees with the results of Börner, Ehlers, and Rudolph.6

We note that $M^2$ of Eq. (17) contains the rest energy and rotational energy of both bodies. Thus we have
\[ Mc^2 = (m_{gi}c^2 + m_{ga}c^2) + \left( \frac{1}{2} I(1) \omega^{(1)} + \frac{1}{2} I(2) \omega^{(2)} + \cdots \right), \]
\[ (21) \]
and the total Lagrangian corresponding to Eq. (17) is
\[ L_\ell(\alpha) = -(m_{gi}c^2 + m_{ga}c^2) \]
\[ + \left( \frac{1}{2} I(1) \omega^{(1)} \omega^{(1)} + \frac{1}{2} I(2) \omega^{(2)} \omega^{(2)} + \cdots \right) \]
\[ + \mathcal{L}(\alpha) - V_{s1} - V_{s2} - V_{s1} - s_2, \]
\[ (22) \]
where \( \mu \mathbf{v} \) replaces \( \mathbf{P} \) in the potential-energy terms.

**IV. PRECESSION OF THE SPIN**

From Eq. (22) we obtain Lagrange's equations\(^{1,17}\) for the Euler angles\(^{17}\), \( \phi^{(1)}, \phi^{(2)}, \) and \( \phi^{(3)} \) in which \( \omega^{(i)} \) can be expressed. For body 1 we find (similar results can be given for body 2 by interchanging indices 1 and 2)
\[ \frac{\dot{\mathbf{n}}^{(i)}}{n} = \mathbf{\Omega}^{(i)} \times \mathbf{n}^{(i)}, \]
\[ (23) \]
where\(^{1,6}\) (dS means de Sitter and LT means Lense-Thirring)
\[ \Omega^{(i)} = \Omega^{(i)}_{\text{dS}} + \Omega^{(i)}_{\text{LT}}, \]
\[ (24) \]
and
\[ \Omega^{(i)}_{\text{dS}} = \frac{G}{2c^3u(1 - e^2)} \left[ \gamma + 1 + (\gamma + 2) \frac{m_1}{m_2} \frac{\mu \mathbf{v} \times \mathbf{v}}{c^2r^3} \right] \]
\[ = \frac{G}{2c^3u(1 - e^2)} \left[ (2y + 1)m_2 + \mu \mathbf{v} \times \mathbf{v} \right], \]
\[ (25) \]
and \( e \) is the eccentricity, \( a \) is the semimajor axis, \( \omega \) is the average orbital angular velocity, \( \mathbf{L} = \mathbf{r} \times \mathbf{P} \) is the orbital angular momentum, and \( \mathbf{n} \) is a unit vector in the \( \mathbf{L} \) direction.

**V. EQUATIONS OF MOTION**

Using Eq. (22), we obtain Lagrange's equations\(^{1}\) for the coordinate \( \mathbf{r} \) and obtain the equations of motion
\[ \dot{\mathbf{v}} + \frac{GM\mathbf{r}}{r^3} = \mathbf{B}(\alpha), \]
\[ (30) \]
where
\[ \mathbf{B}(\alpha) = \mathbf{B}^{(E)}(\alpha) + \mathbf{B}^{(1)} + \mathbf{B}^{(2)} + \mathbf{B}^{(1,2)} \]
\[ (31) \]
and
\[ \mathbf{B}^{(E)}(\alpha) = -\frac{1}{2} \frac{\mu}{M} \times \frac{GM\mathbf{r}}{c^3 \alpha^3} \left[ 4k_4 \frac{GM\mathbf{r}}{r} - k_6 \mathbf{v} \times \mathbf{v} + 4k_6 (\mathbf{v} \times \mathbf{v}) \right], \]
\[ (32) \]
with
\[ k_4 = \frac{1}{2} k_2 + \frac{1}{2} k_3 = \frac{1}{2} \gamma + \frac{1}{2} \beta + \frac{1}{2} \mu / M, \]
\[ (33) \]
\[ k_5 = \frac{3}{2} k_2 - \frac{1}{2} k_3 = \gamma - \frac{1}{2} \mu / M, \]
\[ (34) \]
\[ k_6 = \frac{1}{2} k_2 + \frac{1}{2} k_3 = \frac{1}{2} \gamma + \frac{1}{2} - \frac{1}{2} \mu / M, \]
\[ (35) \]
and
\[ \mathbf{B}^{(1)} = \frac{G}{c^3 \alpha^3} \left[ 2y + 2 + (2y + 1) \frac{m_2}{m_1} \right] \left[ \frac{3}{2} \left( \mathbf{S}^{(1)} \times \mathbf{v} \right) \mathbf{v} + r^2 \mathbf{S}^{(1)} \times \mathbf{v} + \frac{3}{2} (\mathbf{v} \times \mathbf{v}) \times \mathbf{v} - \frac{3}{2} (\mathbf{v} \times \mathbf{v}) \times \mathbf{v} \right], \]
\[ (36) \]
\[ \mathbf{B}^{(2)} = \frac{G}{c^3 \alpha^3} \left[ 2y + 2 + (2y + 1) \frac{m_1}{m_2} \right] \left[ \frac{3}{2} \left( \mathbf{S}^{(2)} \times \mathbf{v} \right) \mathbf{v} + r^2 \mathbf{S}^{(2)} \times \mathbf{v} + \frac{3}{2} (\mathbf{v} \times \mathbf{v}) \times \mathbf{v} - \frac{3}{2} (\mathbf{v} \times \mathbf{v}) \times \mathbf{v} \right], \]
\[ (37) \]
\[ E^{(1, 2)} = \frac{3(\gamma + 1)G}{2c^2} \left[ (S^{(2)} \cdot \vec{r})S^{(1)} + (S^{(1)} \cdot \vec{r})S^{(2)} - 5(S^{(1)} \cdot \vec{r})(S^{(2)} \cdot \vec{r})/r^2 + (S^{(1)} \cdot S^{(2)})\vec{r} \right]. \]  

(38)

The terms \( E^{(1)}(\alpha), E^{(2)}, \) and \( E^{(1, 2)} \) are a consequence of the Lagrangian terms \( L(\alpha), V_{\text{eq}}, V_{\text{eq}}^2, \) and \( V_{\text{eq}}, V_{\text{eq}}^2 \), respectively, of Eq. (22).

A. Shift in the center of mass

The vectors \( \vec{r}_1 \) and \( \vec{r}_2 \) go to the centers of mass of bodies 1 and 2, respectively. The centers of mass of bodies 1 and 2 can be shifted by making the coordinate transformation

\[ \vec{r}'_{(1)} = \vec{r} + \lambda_1 \frac{\vec{v}_1 \times \vec{S}^{(1)}}{m_1 c^2}, \]

(39)

\[ \vec{r}'_{(2)} = \vec{r} + \lambda_2 \frac{\vec{v}_2 \times \vec{S}^{(2)}}{m_2 c^2}, \]

(40)

which corresponds to a different spin supplementary condition than is inherent in our Lagrangian–Hamiltonian formalism.\(^{1, 9}\) The dimensionless parameters \( \lambda_1 \) and \( \lambda_2 \) are arbitrary and determine an arbitrary spin supplementary condition. The new vectors \( \vec{r}'_{(1)} \) and \( \vec{r}'_{(2)} \) now go to the new centers of mass of bodies 1 and 2, respectively. Let us set

\[ \vec{r}'_{(1, 2)} = \vec{r}'_{(1)} - \vec{r}'_{(2)}, \]

(41)

so that we have

\[ \vec{r}'_{(1, 2)} = \vec{r} + \lambda_1 \frac{\vec{v}_1 \times \vec{S}^{(1)}}{m_1 c^2} - \lambda_2 \frac{\vec{v}_2 \times \vec{S}^{(2)}}{m_2 c^2}, \]

(42)

which can be put in the form

\[ \vec{r}'_{(1, 2)} = \vec{r} + \lambda_1 \frac{m_1 \vec{v} \times \vec{S}^{(1)}}{m_1 M c^2} + \lambda_2 \frac{m_2 \vec{v} \times \vec{S}^{(2)}}{m_2 M c^2}. \]

(43)

Using Eq. (43) in Eq. (30) we obtain

\[ \frac{\dot{\vec{V}}_{(1, 2)}}{\vec{r}'_{(1, 2)}} = \vec{B}(\alpha_1, \lambda_1, \lambda_2), \]

(44)

where

\[ \vec{B}(\alpha, \lambda_1, \lambda_2) = \vec{E}^{(2)}(\alpha) + \vec{E}^{(1)}(\lambda_1) + \vec{E}^{(2)}(\lambda_2) + \vec{E}^{(1, 2)} \]

(45)

and

\[ \vec{E}^{(1)}(\lambda_1) = \vec{E}^{(1)} + \frac{3Gm_1 m_2}{c^2 \gamma^2 m_1} \left[ (\vec{v} \cdot \vec{r}) \vec{S}^{(1)} - [\vec{S}^{(1)} \cdot (\vec{r} \times \vec{v})] \vec{r}^{\perp} \right], \]

(46)

\[ \vec{E}^{(2)}(\lambda_2) = \vec{E}^{(2)} + \frac{3Gm_1 m_2}{c^2 \gamma^2 m_2} \left[ (\vec{v} \cdot \vec{r}) \vec{S}^{(2)} - [\vec{S}^{(2)} \cdot (\vec{r} \times \vec{v})] \vec{r}^{\perp} \right]. \]

(47)

The term \( \vec{B}(\alpha, \lambda_1, \lambda_2) \) is a higher-order term, and thus, to the approximation we are using \( \vec{r}_{(1, 2)} \) may be replaced by \( \vec{r} \) in this term. Further, we have

\[ \vec{E}^{(1)}(\lambda_1) = \frac{(2\gamma + 2Gc^2)}{c^2 \gamma^2 m_1} \left[ \frac{1}{2} [\vec{S}^{(1)} \cdot (\vec{r} \times \vec{v})] \vec{r}^{\perp} + \frac{1}{2} r^2 (\vec{S}^{(1)} \times \vec{r}) \cdot \frac{1}{2} (\vec{S}^{(1)} \times \vec{r}) \right] \]

\[ + \frac{3Gm_1}{c^2 \gamma^2 m_1} \left[ \frac{1}{2} (1 + 2\gamma - 2\lambda_1) [\vec{S}^{(1)} \cdot (\vec{r} \times \vec{v})] \vec{r}^{\perp} + \frac{1}{2} (1 + 2\gamma) r^2 (\vec{S}^{(1)} \times \vec{r}) \cdot \frac{1}{2} (1 + 2\gamma + 2\lambda_1) (\vec{v} \cdot \vec{r}) (\vec{S}^{(1)} \times \vec{r}) \right], \]

(48)

\[ \vec{E}^{(2)}(\lambda_2) = \frac{(2\gamma + 2Gc^2)}{c^2 \gamma^2 m_2} \left[ \frac{1}{2} [\vec{S}^{(2)} \cdot (\vec{r} \times \vec{v})] \vec{r}^{\perp} + \frac{1}{2} r^2 (\vec{S}^{(2)} \times \vec{r}) \cdot \frac{1}{2} (\vec{S}^{(2)} \times \vec{r}) \right] \]

\[ + \frac{3Gm_2}{c^2 \gamma^2 m_2} \left[ \frac{1}{2} (1 + 2\gamma - 2\lambda_2) [\vec{S}^{(2)} \cdot (\vec{r} \times \vec{v})] \vec{r}^{\perp} + \frac{1}{2} (1 + 2\gamma) r^2 (\vec{S}^{(2)} \times \vec{r}) \cdot \frac{1}{2} (1 + 2\gamma + 2\lambda_2) (\vec{v} \cdot \vec{r}) (\vec{S}^{(2)} \times \vec{r}) \right]. \]

(49)

One may verify that the spin-dependent terms in D’Eath’s Eq. (6.7) corresponds to the particular case of Eqs. (48) and (49) with \( \gamma = 1 \) and \( \lambda_1 = \lambda_2 = \frac{1}{2} \). Equations (48) and (49) with \( \gamma = 1 \) and \( \lambda_1 = \lambda_2 = 0 \) have been given by us in Eqs. (52) and (53) of Ref. 1.
B. Large-mass approximation

We shall now make the large-mass approximation \( m_2 \gg m_1 \) and apply it to \( \tilde{B}^{(1)}(\lambda_2) \) and \( \tilde{B}^{(2)}(\lambda_2) \). We thus have

\[
\tilde{B}^{(1)}(\lambda_2) = \frac{3Gm_2}{c^3r^2m_1} \left[ \frac{1}{3}(1 + 2\gamma - 2\lambda_2) \left[ \vec{S}^{(1)} \cdot (\vec{T} \times \vec{r}) \right] \vec{T} + \frac{1}{3}(1 + 2\gamma)(\vec{S}^{(1)} \times \vec{r}) - \frac{1}{3}(1 + 2\gamma + 2\lambda_2)(\vec{r} \cdot \vec{T}) \left[ \vec{S}^{(1)} \cdot \vec{r} \right] \right],
\]

\[
\tilde{B}^{(2)}(\lambda_2) = \frac{(2\gamma + 2)G}{c^3r^2m_1} \left[ \frac{1}{3} \left[ \vec{S}^{(2)} \cdot (\vec{T} \times \vec{r}) \right] \vec{T} + r^2(\vec{S}^{(2)} \times \vec{r}) - \frac{1}{3}(\vec{r} \cdot \vec{T}) \left[ \vec{S}^{(2)} \cdot \vec{r} \right] \right].
\]

The values \( \lambda_2 = 0, \pm \frac{1}{2} \) are of particular interest. We have

\[
\tilde{B}^{(1)}(0) = \frac{(1 + 2\gamma)Gm_2}{c^3r^2m_1} \left[ \frac{1}{3} \left[ \vec{S}^{(1)} \cdot (\vec{T} \times \vec{r}) \right] \vec{T} + r^2(\vec{S}^{(1)} \times \vec{r}) - \frac{1}{3}(\vec{r} \cdot \vec{T}) \left[ \vec{S}^{(1)} \cdot \vec{r} \right] \right],
\]

\[
\tilde{B}^{(1)}(\pm \frac{1}{2}) = \frac{3Gm_2}{c^3r^2m_1} \left[ \gamma(\vec{S}^{(1)} \cdot (\vec{T} \times \vec{r}) \vec{T} + \frac{1}{3}(1 + 2\gamma)r^2(\vec{S}^{(1)} \times \vec{r}) - (1 + \gamma)(\vec{r} \cdot \vec{T}) \left[ \vec{S}^{(1)} \cdot \vec{r} \right] \right],
\]

\[
\tilde{B}^{(1)}(-\frac{1}{2}) = \frac{3Gm_2}{c^3r^2m_1} \left[ (1 + \gamma)(\vec{S}^{(1)} \cdot (\vec{T} \times \vec{r}) \vec{T} + \frac{1}{3}(1 + 2\gamma)r^2(\vec{S}^{(1)} \times \vec{r}) - (1 + \gamma)(\vec{r} \cdot \vec{T}) \left[ \vec{S}^{(1)} \cdot \vec{r} \right] \right],
\]

where it should be noted\(^1\) that

\[
\tilde{B}^{(1)}(\mp \frac{1}{2}) = \frac{1}{2} \left[ \tilde{B}^{(1)}(\frac{1}{2}) \mp \tilde{B}^{(1)}(-\frac{1}{2}) \right].
\]

The term \( \tilde{B}^{(1)}(\frac{1}{2}) \) corresponds to the Corinaldesi–Papapetrou\(^{19,20}\) spin supplementary condition, the term \( \tilde{B}^{(1)}(-\frac{1}{2}) \) corresponds to the Pirani\(^{19,22}\) spin supplementary condition, and the term \( \tilde{B}^{(1)}(0) \) corresponds to the spin supplementary condition of Pryce\(^{23}\) and Newton and Wigner.\(^{24}\) For general relativity (with \( \gamma = 1 \)) the terms \( \tilde{B}^{(2)}(0) \), \( \tilde{B}^{(1)}(\frac{1}{2}) \), and \( \tilde{B}^{(1)}(-\frac{1}{2}) \) are discussed in Refs. 1 and 19. Wald\(^{25}\) also gives the general-relativity result for \( \tilde{B}^{(1)}(-\frac{1}{2}) \) and Schiff\(^{26}\) gives the result with \( \gamma \) for the term \( \tilde{B}^{(2)}(\frac{1}{2}) \). To get our result into the form of Schiff\(^{26}\) we must use the vector identity

\[
(\vec{S}^{(1)} \cdot \vec{T}) \vec{T} \times \vec{r} = [\vec{S}^{(1)} \cdot (\vec{T} \times \vec{r})] \vec{T} + r^2(\vec{S}^{(1)} \times \vec{r}) - (\vec{r} \cdot \vec{T}) \vec{S}^{(1)} \times \vec{r}
\]

to obtain

\[
\tilde{B}^{(1)}(\pm \frac{1}{2}) = \frac{3Gm_2}{c^3r^2m_1} \left[ \frac{1}{3}(1 + 2\gamma)(\vec{S}^{(1)} \cdot \vec{T}) \vec{T} + \frac{1}{3}(2 + \gamma)(\vec{r} \cdot \vec{T}) \vec{S}^{(1)} \times \vec{r} + \frac{1}{2}(\gamma - 1)(\vec{r} \cdot \vec{T}) \vec{S}^{(1)} \times \vec{r} \right].
\]

VI. PRECESSION OF THE ORBIT

The secular results for the precession of the orbit can be written as

\[
\dot{E}_x = 0, \quad \dot{L}_x = \vec{G} \times \vec{L}, \quad \dot{\vec{A}}_x = \vec{G} \times \vec{A},
\]

where \( E, \vec{L} \), and \( \vec{A} \) are the Newtonian energy, orbital angular momentum, and Runge-Lenz vector, respectively.\(^{1,17}\) and

\[
\vec{G} = \vec{G}^{(E)} + \vec{G}^{(1)} + \vec{G}^{(2)} + \vec{G}^{(1,2)};
\]

with

\[
\vec{G}^{(E)} = \frac{(2 + 2\gamma - \beta)GmM}{c^3a(1 - e^2)},
\]

\[
\vec{G}^{(1)} = \frac{GS^{(1)}}{2c^3a^2(1 - e^2)^{3/2}} \left[ \vec{n} \times \vec{a} - 3(\vec{a} \cdot \vec{n})\vec{n} \right],
\]

\[
\vec{G}^{(2)} = \frac{GS^{(2)}}{2c^3a^2(1 - e^2)^{3/2}} \left[ \vec{n} \times \vec{a} - 3(\vec{a} \cdot \vec{n})\vec{n} \right],
\]

\[
\vec{G}^{(1,2)} = \frac{-2(\gamma + 1)GS^{(1)S^{(2)}}}{4c^3a^2(1 - e^2)^{3/2}} \left[ \vec{n} \times \vec{a} \right] - \frac{5(\vec{a} \cdot \vec{n})}{4c^3a^2(1 - e^2)^{3/2}} \left[ (\vec{a} \cdot \vec{n})\vec{n} + (\vec{a} \times \vec{n}) \vec{a} + 5(\vec{a} \cdot \vec{n})\vec{n} + 5(\vec{a} \times \vec{n}) \vec{a} \right].
\]
The terms \( \tilde{\Omega}^{(E)}, \tilde{\Omega}^{(a)}, \tilde{\Omega}^{(b)}, \) and \( \tilde{\Omega}^{(a,b)} \) are the results corresponding to \( \tilde{B}^{(a,b)}(\alpha), \tilde{B}^{(a)}(\lambda_1), \tilde{B}^{(b)}(\lambda_2), \) and \( \tilde{B}^{(a,b)} \), respectively. It should be noted that the precession of the orbit is independent of the parameters \( \alpha, \lambda_1, \) and \( \lambda_2 \). The factor \( (2 - 2\gamma - \beta) \) in \( \tilde{\Omega}^{(E)} \) comes from evaluating the expression

\[
4k_0 - 2k_1 + k_2 = \left( \frac{1}{2} k_1 + 3k_2 - \frac{1}{2} k_0 \right) = (2 + 2\gamma - \beta).
\]

(64)

It is interesting to note that in the above evaluation all the \( \mu/M \) terms cancel out. However, by the time one has reached Eq. (4) it is clear that any \( \mu/M \) terms would have had to be independent of \( \gamma \) or \( \beta \).

The result for \( \tilde{\Omega}^{(E)} \) in general relativity was

\[
\frac{3}{2 + 2\gamma - \beta} \Omega^{(E)} = \frac{6}{4\gamma + 3} \Omega^{(1)}_{ds} + \Omega^{(2)}_{ds} + \frac{2}{4\gamma + 3} \left[ 9(\Omega^{(1)}_{ds})^2 + (\Omega^{(2)}_{ds})^2 \right]^{1/2}.
\]

(65)

In the equal-mass case we have

\[
\frac{3}{2 + 2\gamma - \beta} \tilde{\Omega}^{(E)} = \frac{24}{4\gamma + 3} \tilde{\Omega}^{(a,b)}_{ds}.
\]

(66)

while in the large-mass approximation \( (m_2 > m_1) \) we have

\[
\frac{3}{2 + 2\gamma - \beta} \tilde{\Omega}^{(E)} = \frac{6}{2\gamma + 1} \tilde{\Omega}^{(1)}_{LT}.
\]

(67)

There are two other precession relations:

\[
\tilde{\Omega}^{(a)} = \frac{2\gamma + 2 + (2\gamma + 1)m_2/m_1}{\gamma + 1} \tilde{\Omega}^{(1)}_{LT}.
\]

(68)

and

\[
\tilde{\Omega}^{(a,b)} = \frac{2\gamma + 2 + (2\gamma + 1)m_2/m_1}{\gamma + 1} \tilde{\Omega}^{(1)}_{LT}.
\]

(69)

In the equal-mass case we have

\[
\tilde{\Omega}^{(a)} = \frac{8\gamma + 6}{\gamma + 1} \tilde{\Omega}^{(1)}_{LT}, \quad \tilde{\Omega}^{(a,b)} = \frac{8\gamma + 6}{\gamma + 1} \tilde{\Omega}^{(1)}_{LT}.
\]

(70)

while in the large-mass approximation we have

\[
\tilde{\Omega}^{(a)} = 4\tilde{\Omega}^{(1)}_{LT}.
\]

(71)

The total angular momentum \( \tilde{J} = \tilde{J}^{(1)} + \tilde{J}^{(a)} + \tilde{J}^{(a,b)} \) satisfies the relation

\[
\tilde{J}_{av} = \tilde{\Omega}^{(a)} \times \tilde{J}^{(1)} + \tilde{\Omega}^{(a,b)} \times \tilde{J}^{(a)} + \tilde{\Omega}^{(a)} \times \tilde{J}^{(a,b)} = 0,
\]

(72)

i.e., the total angular momentum is conserved.

VII. MASS RELATIONS FOR PSR 1913 + 16

By using the relation

\[
\tilde{\omega}^2 a^3 = GM,
\]

(73)

we can rewrite \( \tilde{\Omega}^{(a)} \) and \( \tilde{\Omega}^{(a,b)} \) of Eqs. (60) and (28) first given by Robertson\(^{27}\) and the result for Brans-Dicke theory was given by Estabrook.\(^{14}\) The result for \( \tilde{\Omega}^{(E)} \) with \( \gamma \) and \( \beta \) and other PPN parameters has been given by Will.\(^{28}\) The result for \( \tilde{\Omega}^{(E)} \) with \( \gamma \) and \( \beta \) in the large-mass approximation \( (m_2 > m_1) \) is widely quoted.\(^{29}\)

The results for \( \tilde{\Omega}^{(a)}, \tilde{\Omega}^{(a,b)} \), and \( \tilde{\Omega}^{(a,b)} \) in general relativity were first given in Ref. 1. The quadrupole-moment results \( \tilde{\Omega}^{(a)} \) and \( \tilde{\Omega}^{(a,b)} \) and the corresponding terms \( \tilde{B}^{(a)} \) and \( \tilde{B}^{(a,b)} \) in the equations of motion are given in Refs. 1 and 3 and are omitted in this paper as they are independent of the PPN parameters \( \gamma \) and \( \beta \).

There is an interesting relation involving the de Sitter terms \( \tilde{\Omega}^{(1)}_{de} \) and \( \tilde{\Omega}^{(a,b)}_{de} \) and the Einstein term \( \tilde{\Omega}^{(E)} \):

\[
\tilde{\Omega}^{(E)} = \frac{3G^2/3G^3}{c^2(1 - e^2)} f_E \tilde{\Omega}^{(1)}_{de} + \frac{3G^2/5G^3}{2c^2(1 - e^2)} f^{(1)}_{de} \tilde{\Omega}^{(a,b)}_{de},
\]

(74)

where

\[
f_E = \frac{2 + 2\gamma - \beta}{3} f_{de} = \frac{2 + 2\gamma - \beta}{3} \frac{M^2}{e^3}.
\]

(77)

The observed elements of the orbit of PSR 1913 + 16 have been given by Hulse and Taylor.\(^{30}\) They are the period \( T = 2\pi/\tilde{\omega} = 27.908 \) sec and the eccentricity \( e = 0.615 \). Using these values in Eqs. (74) and (75) we obtain

\[
\tilde{\Omega}^{(E)} = \frac{3.34}{\gamma r} g_{\tilde{\Omega}} \tilde{\Omega}^{(1)}_{de}, \quad \tilde{\Omega}^{(a,b)}_{de} = \frac{0.974}{\gamma r} g_{\tilde{\Omega}}^{(1)} \tilde{\Omega}^{(a,b)}_{de},
\]

(78)

(79)

where

\[
g_{\tilde{\Omega}}(m_1, m_2, \gamma, \beta) = \frac{f_E(m_1, m_2, \gamma, \beta)}{f_E(m_0, m_0, 1, 1)},
\]

(80)

\[
g_{\tilde{\Omega}}^{(1)}(m_1, m_2, \gamma) = \frac{f^{(1)}(m_1, m_2, \gamma)}{f^{(1)}(m_0, m_0, 1, 1)}.
\]

(81)

and \( m_0 \) is the mass of the sun.

Let body 1 be the pulsar and body 2 be its companion. We shall assume the companion to be a compact object. Then \( \tilde{\Omega}^{(a,b)} \) will be negligible and \( \tilde{\Omega}^{(E)} \) will be the dominant contribution to the orbit precession. Also, \( \tilde{\Omega}^{(1)}_{de} \) is the dominant contribution to the spin precession of the pulsar even if the companion is not assumed to be compact. Then it is clear that a measurement of \( \tilde{\Omega}^{(E)} \) will
enable one to determine \( f_{\text{E}} \) while a measurement\(^{35} \)
of \( \tilde{\Omega}_{\text{AS}} \) will enable one to determine \( f_{\text{AS}} \). We can, thus, obtain the values of the individual masses (for given \( \gamma \) and \( \beta \)) as

\[
m_1 = M \left\{ \left( \gamma + 1 \right)^2 - \frac{3 f_{\text{E}}^{(1)}}{M \gamma} \right\}^{1/2} - \gamma
\]

\[m_2 = M - m_1,
\]

where

\[
M = \left( \frac{3 f_{\text{E}}}{2 + 2 \gamma - \beta} \right)^{3/2}.
\]

By an analysis of pulse arrival-time data\(^{30-34} \) involving the second-order Doppler effect and gravitational red-shift a quantity\(^{32,33} \)

\[\beta^{(1)} = \frac{e (2G/c^2)^{2/3}}{(1 - e^2)^{1/3}} f_{\text{B}}^{(1)},\]

with

\[f_{\text{B}}^{(1)} = \frac{m_2 (m_2 + M)}{M^{4/3}},\]

can be determined. A measurement of \( \beta^{(1)} \) will enable one to determine \( f_{\text{B}}^{(1)} \). The above result is independent of \( \gamma \) or \( \beta \). There is another effect,\(^{31} \) the relativistic time delay, which depends on \( \gamma \), which we shall not consider because it is of the order of the third-order Doppler effect.\(^{31} \)

Using the masses given in Eqs. (82) and (83) in Eq. (86) we obtain the relation

\[(1 + \gamma)(3 + 2 \gamma) - \frac{(2 + 2 \gamma - \beta) (f_{\text{AS}}^{(1)} + \frac{1}{2} f_{\text{B}}^{(1)})}{f_{\text{E}}} = (3 + 2 \gamma) \left\{ (\gamma + 1)^2 - \frac{(2 + 2 \gamma - \beta) f_{\text{AS}}^{(1)}}{f_{\text{E}}} \right\}^{1/2},\]

which gives a constraint on the values that \( \gamma \) and \( \beta \) can have.

Because of the massive dense bodies forming PSR 1913 + 16, results involving the PPN parameters, such as \( \tilde{\Omega}_{\text{AS}}^{(1)} \) and \( \tilde{\Omega}_{\text{AS}}^{(1)} \), should be corrected for the Nordtvedt\(^{13,55} \) effect (which does not exist in general relativity) or else these results could have an error\(^{6} \) of about 1%. The 1% error is obtained by noting that \( \tilde{\Omega}_{\text{AS}}^{(1)} \) and \( \tilde{\Omega}_{\text{AS}}^{(1)} \) should be corrected by some factors proportional to

\[
\left[ \text{const} \times (1 - \gamma) + \text{const} \times (1 - \beta) \right] \frac{U}{mc^3},
\]

where \( U \) is the gravitational binding energy. Assuming \( |(1 - \gamma)| \leq 0.1, \ (1 - \beta) \leq 0.1, \) and\(^{35} \) \( |U/mc^2| \leq 0.1, \) we obtain the 1% figure.\(^{37} \)

The modification of the Newtonian potential-energy term to include the Nordtvedt effect has been given,\(^{13} \) and we can write the corresponding Lagrangian as

\[
\mathcal{L} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + \frac{1}{2} f^{(1)} (\omega^{(11)} + \frac{1}{2} f^{(1)} \omega^{(22)}) + \frac{G (m_1 + (4 \beta - \gamma - 3) U_1/c^2) \omega^{(12)} [m_2 + (4 \beta - \gamma - 3) U_2/c^2]}{r}
\]

-(rest-energy terms) + (post-Newtonian terms),

where

\[
U_1 = -\frac{G}{2} \int \rho_1 \left( \frac{\rho_1 (\mathbf{T}) \rho_1 (\mathbf{T}) dV dV'}{|\mathbf{T} - \mathbf{T}'|} \right),
\]

\[
U_2 = -\frac{G}{2} \int \rho_2 (\mathbf{T}) \rho_2 (\mathbf{T}) dV dV' \left| \frac{|\mathbf{T} - \mathbf{T}'|}{|\mathbf{T} - \mathbf{T}'|} \right|
\]

and \( \rho_1 \) and \( \rho_2 \) are the mass densities of bodies 1 and 2 respectively. However, corrections to the post-Newtonian terms (which are essential in calculating \( \tilde{\Omega}_{\text{AS}}^{(1)} \) and \( \tilde{\Omega}_{\text{AS}}^{(1)} \), with the Nordtvedt effect included) have not been given. These corrections would be post-post-Newtonian effects and thus apparently beyond the normal scope of PPN (parametrized post-Newtonian) theory.

Eardley\(^{38} \) has given a fully relativistic Nordtvedt-effect correction to the Newtonian term for Brans-Dicke theory. Assuming \( |(1 - \gamma)| \leq 0.1 \) and a sensitivity\(^{38} \) factor of 0.78 implies a correction to the Newtonian term of about 5%.\(^{39} \)

**VIII. CONCLUSION**

We have given the Lagrangian and the Hamiltonian for the gravitational two-body problem with spin and PPN parameters \( \gamma \) and \( \beta \). We have emphasized the importance of Lagrangian-Hamiltonian formalism in that this allows one to obtain the spin precession, the equations of motion, and the orbit precession. In addition to the parameter \( \alpha \), which we introduced into the Hamiltonian without spin, we also introduced the parameters \( \lambda_1 \) and \( \lambda_2 \) into the spin-dependent terms in the equations of motion, which gave us equations of motion corresponding to an arbitrary spin supplementary condition. However, it was noted that the orbit precession is independent of the parameters \( \alpha, \lambda_1, \) and \( \lambda_2 \). The spin-dependent terms with \( \gamma \) in the equations of motion and in the precession of the orbit are new results.

Finally, we showed how the masses of the binary pulsar PSR 1913 + 16 are related to the orbit and the spin precession. Combining this with a result derivable from the second-order Doppler effect and the gravitational red-shift we obtained a relation constraining the values that \( \gamma \) and \( \beta \) can take.

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14 LAGRANGIAN-HAMILTONIAN FORMALISM FOR THE...

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11See p. 1093 of Ref. 7.
16R. F. O'Connell, Phys. Rev. Lett. 20, 69 (1968). Note that $\gamma = \frac{1}{\kappa}$ in Brans-Dicke theory; see also Eq. 40.37 on p. 1119 of Ref. 7 and note that $1 + \gamma = \frac{1}{2} (\Delta_1 + \Delta_2)$ since $\alpha_1 = 0$ in our case (see p. 1093 of Ref. 7).
29See, for example, p. 1110, Eq. (40.18) of Ref. 7.
37We obtain a result much less than 1% if we use the recent experimental results for $\gamma$ and $\beta$ of J. G. Williams et al., Phys. Rev. Lett. 36, 551 (1976); I. I. Shapiro, C. C. Counselman III, and R. W. King, ibid. 36, 555 (1976).
39We obtain a result much less than 5% if we use the recent experimental results for $\gamma$ in the papers of Ref. 37.