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A Simplified Form for the Hamiltonian and Lagrangian of the Spin-Independent Gravitational Two-Body System

R. F. O'CONNELL[1]

Laboratoire de Physique Théorique et Hautes Energies,[2] Université de Paris XI, 91405 Orsay, France

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Abstract

In general, the gravitational two-body Hamiltonian, to order c^{-2}, contains GP^2, $G(\mathbf{P}\cdot\mathbf{r})^2$, and G^2 terms. We have previously shown [4–6] that a proper choice of coordinate system enables one to eliminate the $G(\mathbf{P}\cdot\mathbf{r})^2$ term. We now show that, making use of energy conservation, and coordinate transformations, we can eliminate either of the remaining two terms. In particular, we are able to write down a Hamiltonian and a Lagrangian that contain no mixed potential and kinetic terms.

The gravitational two-body equations of motion [1], to order c^{-2}, may be derived from a Hamiltonian [2], which in general contains $G\mathbf{P}^2$, $G(\mathbf{P}\cdot\mathbf{r})^2$, and G^2 terms [3]. In our work on the generalization of the two-body problem to include spin and rotation effects [4–6], we used a coordinate system in which the $G(\mathbf{P}\cdot\mathbf{r})^2$ term did not appear. We now show that it is possible to eliminate either of the remaining two terms, by making use of energy conservation and transformations in both the spatial and time coordinates.

The potential energy which we have used contains velocity-dependent terms. However, since these occur in *higher order*, we may modify their form by using the energy-conservation equation in *lowest order*, viz.,

$$P^2/2\mu = Gm_1m_2/r + E \tag{1}$$

where E is a constant.

[1] Dr. O'Connell's permanent address is Department of Physics, Louisiana State University, Baton Rouge, Louisiana 70803, U.S.A.

[2] Laboratoire associé au Centre National de la Recherche Scientifique.

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Substituting equation (1) into equation (10) of reference 5, we obtain for the complete G term in the potential

$$V_1 = -\frac{Gm_1m_2}{r}\left[B + \left(4 + \frac{3m_1}{2m_2} + \frac{3m_2}{2m_1}\right)\frac{2G\mu}{rc^2}\right] \tag{2}$$

where

$$B = 1 + \left(4 + \frac{3m_1}{2m_2} + \frac{3m_2}{2m_1}\right)\frac{2E}{Mc^2} \tag{3}$$

is another constant. Rearrangement of terms in equation (2) leads us to

$$V_1 = -\frac{Gm_1m_2}{r}\left[B + \left(1 + \frac{3M}{2\mu}\right)\frac{2G\mu}{rc^2}\right] \tag{4}$$

In addition, from equation (11) of reference 5, we have for the G^2 term in the potential

$$V_2 = \frac{Gm_1m_2}{2r}\left(1 + \frac{M}{\mu}\right)\frac{G\mu}{rc^2} \tag{5}$$

Thus, the total potential energy, to order c^{-2}, is simply

$$V \equiv V_1 + V_2 = -\frac{Gm_1m_2}{r}\left[B + \left(3 + \frac{5M}{\mu}\right)\frac{G\mu}{2c^2r}\right] \tag{6}$$

The corresponding Lagrangian is

$$\mathcal{L} = T - V \tag{7}$$

where V is given by equation (6) and [5]

$$T = \tfrac{1}{2}\,\mu v^2 + (1/8c^2)(1 - 3\mu/M)\,\mu v^4 \tag{8}$$

Now set

$$r' \equiv (r/B) \qquad \text{and} \qquad t' \equiv (t/B) \tag{9}$$

so that

$$V = -\frac{Gm_1m_2}{r'}\left[1 + \left(3 + \frac{5M}{\mu}\right)\frac{G}{2c^2r'}\frac{\mu}{B^2}\right] \tag{10}$$

However, since B appears in the highest-order term in equation (10), and recognizing from equation (3) that B is unity to lowest order (since the total energy $E \ll Mc^2$), we may replace B by unity in equation (10). Finally, dropping the prime, we have

$$V = -\frac{Gm_1m_2}{r}\left[1 + \left(3 + \frac{5M}{\mu}\right)\frac{G\mu}{2c^2r}\right] \tag{11}$$

In other words, we have eliminated the velocity-dependent terms from the potential. We note that the transformation to t' leaves V unchanged, whereas the combined transformation leaves the kinetic energy unchanged.

Actually, the v^4 term appearing in T can be rewritten, by use of equation (1), and by recognition of the fact that $P = \mu v$ to lowest order, to give

$$T = \frac{1}{2}\mu v^2 + \left(1 - \frac{3\mu}{M}\right)\left[\frac{Gm_1m_2}{r}\frac{GM}{2c^2r} + \frac{EGm_1m_2}{\mu rc^2} + \frac{E^2}{2\mu c^2}\right] \tag{12}$$

As before, we can effectively eliminate the E term by a scale transformation in the r coordinate [compare equations (3), (6), and (11)], making a corresponding scale transformation in the time coordinate to ensure the v^2 term remains unchanged. In addition, the E^2 term can be dropped as it only adds a constant term to the Lagrangian, which does not affect the equations of motion.

Thus, in essence, the total two-body Lagrangian, to order c^{-2}, may be written simply as

$$\mathcal{L} = \frac{1}{2}\mu v^2 + \frac{Gm_1m_2}{r}\left(1 + \frac{3GM}{c^2r}\right) \tag{13}$$

The corresponding Hamiltonian is

$$\mathcal{H} = \frac{P^2}{2\mu} - \frac{Gm_1m_2}{r}\left(1 + \frac{3GM}{c^2r}\right) \tag{14}$$

It follows immediately that the equation of motion is

$$\dot{\mathbf{v}} = -\frac{GM}{r^3}\left(1 + \frac{6GM}{c^2r}\right)\mathbf{r} \tag{15}$$

where the dot denotes the time derivative.

The correctness of our results may be verified by calculating the periastron precession. In contrast to the methods used in most textbooks, this may be derived simply and elegantly by use of the Runge-Lenz vector [4, 5],

$$\mathbf{A}/\mu = \mathbf{v} \times (\mathbf{v} \times \mathbf{r}) - (GM/r)\,\mathbf{r} \tag{16}$$

which is a constant of the motion for a Newtonian orbit.

For the problem at hand, we have

$$\dot{\mathbf{A}}/\mu = -(GM/r^3)(6GM/c^2r)\,\mathbf{r} \times (\mathbf{r} \times \mathbf{v}) \tag{17}$$

Averaging over an orbit, we obtain the secular result

$$\dot{\mathbf{A}}_{av} = \mathbf{\Omega}^* \times \mathbf{A}, \tag{18}$$

where

$$\mathbf{\Omega}^* = \frac{3G\omega M}{c^2a(1 - e^2)}\,\mathbf{r} \tag{19}$$

is the usual result for the periastron precession [5, 7].

It is also of interest to note that, in our simplified coordinate system, $P = \mu v$, to order c^{-2}, whereas in other coordinate systems, $P = \mu v[1 + \theta(G\mu/rc^2)]$.

In arbitrary theories of gravitation (except for the restriction that there is no preferred frame and that we have a conservative theory [8]), one may generalize [9] the two-body Lagrangian and Hamiltonian of Einstein's theory to now include the so-called Eddington parameters γ and β [8]. Proceeding as before, one may verify that the results for $\mathcal{L}$, $\mathcal{H}$, $\dot{\mathbf{v}}$, and $\mathbf{\Omega}^*$ are the same as those given in equations (13), (14), (15), and (19), except that the G^2 term is now multiplied by an additional factor $(2 + 2\gamma - \beta)/3$.

In particular, we would like to emphasize the simple form of the equation of motion, viz.,

$$\dot{\mathbf{v}} = -(GM/r^3)\,[1 + (2 + 2\gamma - \beta)\,2GM/c^2 r]\,\mathbf{r}, \tag{20}$$

This should be contrasted with the usual form [10], which contains velocity-dependent forces. The latter is used presently to experimentally determine γ and β by use of orbiting planets and spacecraft [10]. We now suggest that equation (20) is a more desirable equation to use for this purpose.

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