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General-Relativistic Effects in Binary Systems.

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1. – Introduction.

The discovery in 1974, by HULSE and TAYLOR [1], of a pulsar in a binary system has given impetus to the study of two-body effects in general relativity. The gravitational two-body equations of motion for arbitrary masses without spin were first derived in 1938 by EINSTEIN, INFELD and HOFFMAN (EIH) [2]. Immediately afterwards, ROBERTSON [3] applied these results to calculate the precession of the periastron.

The extension of the two-body problem to include spin effects (**) was not considered until 1970, when, in a paper [4] which was mainly concerned with spin effects in the framework of the large-mass approximation, we pointed out that our basic starting equation «... has not been subject to the large-mass approximation and thus we could have proceeded in the same manner without this approximation». Stimulated by the discovery of PSR 1913 + 16 [1], we have now carried out this extension [5, 6]. Here, we consider some new aspects of this work and, in addition, the detailed application of our results to binary stellar systems.

In sect. 2, we discuss our previous derivation. In particular, we emphasize that our approach is based on well-accepted rigorous techniques from quantum field theory. We point out that our method has the advantage of giving us *both* a quantum Hamiltonian and a classical Hamiltonian for the gravitational interaction of two bodies with arbitrary masses and spins. Next, we present a summary of our main results. In sect. 3, we present a heuristic classical der-

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(**) We use «spin» in the generic sense of meaning «internal spin» in the case of an elementary particle and «rotation» in the case of a macroscopic body.

ivation of our relativistic results, for the purpose of giving a better physical insight into their nature. In sect. 4, we write down our general results (including quadrupole moment as well as spin effects) in terms of measurable quantities, for ready application to binary systems. In particular, we carry out a detailed analysis of the pulsar PSR 1913 + 16 and its companion.

2. – The gravitational two-body problem with arbitrary masses and spins.

With the exception of the work of EIH [2] and related papers, most research in gravitation, involving equations of motion, has been concerned with the motion of a test body in the field of a large mass (the « large-mass approximation »). The classic work of PAPAPETROU [7, 8] and CORINALDESI [8] dealt with the case of a test particle with spin. However, it is by no means clear how one might extend this work in a rigorous way, within the confines of classical treatments, to the case of two arbitrary masses with spins.

When faced with such a problem one naturally thinks of how the corresponding problem is treated in electrodynamics. In 1920, DARWIN [9] derived classically (up to lowest-order in the relativistic terms) the Lagrangian for two charged spinless particles of arbitrary mass (the corresponding gravitational result is that of EIH). Spin effects were introduced, essentially for the one-body problem, by THOMAS [10], who considered the spin-orbit interaction, and by FERMI [11], who treated the spin-spin interaction—which, of course, led to the fine-structure and hyperfine splittings, respectively, in hydrogenic atoms.

The inclusion of electromagnetic spin interactions between bodies of arbitrary mass was developed by BREIT [12]. The next stage was the development of a rigorous derivation of the Breit interaction using the techniques of quantum field theory [13]. In the latter approach, one visualizes the lowest-order interaction as arising from the exchange of a single photon.

As a result, we feel that the most satisfactory way to treat the gravitational two-body problem is to adopt from the outset the most sophisticated approach available, *viz.* the method of quantum field theory, which treats the gravitational interaction as being due to the exchange of a zero-mass, spin-two graviton [14]. In contrast to the more conventional approaches [7, 8], the basic idea is to introduce spin effects at the *microscopic* level. A further advantage of our method is that we obtain both a quantum Hamiltonian (which can be used, for example, to treat the gravitational interaction of neutrons in dense matter, as in gravitational collapse [15]) and a classical Hamiltonian (which can be used to analyze various macroscopic binary systems).

The initial results obtained pertain to the interaction of two elementary particles [4, 14]. Next, a classical limit is taken [4, 5], from which one derives the basic Hamiltonian. At this stage, we simply apply the basic techniques of analytical mechanics to obtain such results as the precession of the orbit and the spin precessions. Here, we will simply write down our principal results.

Let m_1, $\boldsymbol{r}_1$, $\boldsymbol{v}_1$, $\boldsymbol{P}_1$, $\boldsymbol{S}^{(1)}$, $\boldsymbol{n}^{(1)}$, $I^{(1)}$ and $\boldsymbol{\omega}^{(1)}$ denote the mass, position, velocity, momentum, spin, unit vector in the spin direction, moment of inertia and angular velocity, respectively, of body 1. The same symbols, with $1 \to 2$, denote the corresponding quantities for body 2. In the center-of-mass system we have $\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2$ and $\boldsymbol{P} = \boldsymbol{P}_1 = -\boldsymbol{P}_2$. The reduced mass and the total mass are given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}, \qquad M = m_1 + m_2, \tag{1}$$

respectively. Then, the total Hamiltonian $\mathscr{H}_T$ for arbitrary masses, spins and quadrupole moments may be written as

$$\mathscr{H}_T = Mc^2 + \mathscr{H} + V_{S1} + V_{S2} + V_{S1,S2} + V_{Q1} + V_{Q2}, \tag{2}$$

where

$$\mathscr{H} = \mathscr{H}_0 + V_1 + V_2, \tag{3}$$

$$\mathscr{H}_0 = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\boldsymbol{P}^2 - \frac{1}{8}\left(\frac{1}{m_1^3} + \frac{1}{m_2^3}\right)\frac{\boldsymbol{P}^4}{c^2}, \tag{4}$$

$$V_1 = -\frac{Gm_1m_2}{r}\left\{1 + \left[4 + \frac{3m_1}{2m_2} + \frac{3m_2}{2m_1}\right]\frac{\boldsymbol{P}^2}{m_1m_2c^2}\right\}, \tag{5}$$

$$V_2 = \frac{G^2\mu M(\mu + M)}{2c^2r^2}, \tag{6}$$

$$V_{S1} = \frac{G}{c^2r^3}\left(2 + \frac{3m_2}{2m_1}\right)\boldsymbol{S}^{(1)}\cdot(\boldsymbol{r}\times\boldsymbol{P}), \tag{7}$$

$$V_{S2} = \frac{G}{c^2r^3}\left(2 + \frac{3m_1}{2m_2}\right)\boldsymbol{S}^{(2)}\cdot(\boldsymbol{r}\times\boldsymbol{P}), \tag{8}$$

$$V_{S1,S2} = \frac{G}{c^2r^3}\left(\frac{3(\boldsymbol{S}^{(1)}\cdot\boldsymbol{r})(\boldsymbol{S}^{(2)}\cdot\boldsymbol{r})}{r^2} - \boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}\right), \tag{9}$$

$$V_{Q1} = \frac{GJ_2^{(1)}m_1m_2}{2r^3}\left(\frac{3(\boldsymbol{n}^{(1)}\cdot\boldsymbol{r})^2}{r^2} - 1\right), \tag{10}$$

$$V_{Q2} = \frac{GJ_2^{(2)}m_1m_2}{2r^3}\left(\frac{3(\boldsymbol{n}^{(2)}\cdot\boldsymbol{r})^2}{r^2} - 1\right). \tag{11}$$

The quantities $J_2^{(1)}$ and $J_2^{(2)}$ for bodies 1 and 2, respectively, are given by (see eqs. (38) and (40) of [4] and eqs. (6) and (48) of [16])

$$m_1J_2^{(1)} = \Delta I^{(1)} = \tfrac{1}{2}\int \mathrm{d}V'\,[r'^2 - 3(\boldsymbol{n}^{(1)}\cdot\boldsymbol{r}')^2]\,\varrho_1(\boldsymbol{r}'), \tag{12}$$

$$m_2J_2^{(2)} = \Delta I^{(2)} = \tfrac{1}{2}\int \mathrm{d}V'\,[r'^2 - 3(\boldsymbol{n}^{(2)}\cdot\boldsymbol{r}')^2]\,\varrho_2(\boldsymbol{r}'), \tag{13}$$

where $\varrho_1(\boldsymbol{r}')$ and $\varrho_2(\boldsymbol{r}')$ are the mass densities of body 1 and body 2, respectively.

We note that m_1c^2 and m_2c^2 contain the rotational energy of body 1 and body 2 as well as the rest energy. We thus have [4]

$$Mc^2 = (m_{01}c^2 + m_{02}c^2) + (\tfrac{1}{2}I^{(1)}\omega^{(1)^2} + \tfrac{1}{2}I^{(2)}\omega^{(2)^2} + \dots)\,. \tag{14}$$

The total Lagrangian corresponding to eq. (2) is given by

$$\mathscr{L}_t = -(m_{01}c^2 + m_{02}c^2) + \left(\frac{1}{2}I^{(1)}\omega^{(1)^2} + \frac{1}{2}I^{(2)}\omega^{(2)^2} + \dots\right) + \frac{1}{2}\mu v^2 + \frac{1}{8}\left(1 - 3\frac{\mu}{M}\right)\frac{\mu v^4}{c^2} - (V_1 + V_2 + V_{s1} + V_{s2} + V_{s1,s2} + V_{Q1} + V_{Q2})\,, \tag{15}$$

where $\mu\boldsymbol{v}$ replaces $\boldsymbol{P}$ in the potential-energy terms.

It should be noted that $\mathscr{H}$, as given in eqs. (3)-(6), is related to the EIH Hamiltonian by the co-ordinate transformation [5]

$$\boldsymbol{r}_{\text{EIH}} = \boldsymbol{r}\left(1 + \frac{G\mu}{2c^2r}\right), \tag{16}$$

which implies that

$$\boldsymbol{P}_{\text{EIH}} = \boldsymbol{P} - \frac{G\mu}{2c^2r}\left[\boldsymbol{P} - \frac{(\boldsymbol{P}\cdot\boldsymbol{r})\boldsymbol{r}}{r^2}\right]. \tag{17}$$

In addition, the form of the spin-orbit terms depends on the choice of a specific spin supplementary condition [5], which in essence corresponds to a particular choice for the centers of mass of the spinning bodies. As emphasized previously [5, 17], the orbit and spin precessions are independent both of the choice of the co-ordinate system and of the spin supplementary condition.

If we use the above Lagrangian, it follows [5] that the secular results for the spin precession of body 1 are given by

$$\dot{\boldsymbol{n}}^{(1)}_{\text{av}} = \boldsymbol{\Omega}^{(1)}_{\text{av}} \times \boldsymbol{n}^{(1)}\,, \tag{18}$$

where

$$\boldsymbol{\Omega}^{(1)}_{\text{av}} = \boldsymbol{\Omega}^{(1)}_{DS\,\text{av}} + \boldsymbol{\Omega}^{(1)}_{LT\,\text{av}} + \boldsymbol{\Omega}^{(1)}_{Q1\,\text{av}}\,, \tag{19}$$

and

$$\boldsymbol{\Omega}^{(1)}_{DS\,\text{av}} = A^{(1)}_{DS}\boldsymbol{n}\,, \tag{20}$$

$$\boldsymbol{\Omega}^{(1)}_{LT\,\text{av}} = A^{(1)}_{LT}[\boldsymbol{n}^{(2)} - 3(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})\boldsymbol{n}]\,, \tag{21}$$

$$\boldsymbol{\Omega}^{(1)}_{Q1\,\text{av}} = A^{(1)}_{Q1}[\boldsymbol{n}^{(1)} - 3(\boldsymbol{n}\cdot\boldsymbol{n}^{(1)})\boldsymbol{n}]\,, \tag{22}$$

with

$$A^{(1)}_{DS} = \frac{3G\bar{\omega}(m_2 + \mu/3)}{2c^2 a(1-e^2)}, \tag{23}$$

$$A^{(1)}_{LT} = \frac{GS^{(2)}}{2c^2 a^3(1-e^2)^{\frac{3}{2}}}, \tag{24}$$

$$A^{(1)}_{Q1} = \frac{Gm_2 \Delta I^{(1)}}{2I^{(1)}\omega^{(1)} a^3(1-e^2)^{\frac{3}{2}}}. \tag{25}$$

Here e is the eccentricity, a is the semi-major axis, $\bar{\omega}$ is the average orbital angular velocity, $\boldsymbol{L} = \boldsymbol{r}\times\boldsymbol{P}$ is the orbital angular momentum, and $\boldsymbol{n}$ is a unit vector in the $\boldsymbol{L}$-direction. Also, we have the relation

$$\frac{L/\mu}{a^2(1-e^2)^{\frac{1}{2}}} = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}} = \frac{2\pi}{T} = \bar{\omega}, \tag{26}$$

where T is the orbital period.

In particular, it is notable that, in going from the large-mass approximation ($m_2 \gg m_1$) to the case of arbitrary masses, the result for $\boldsymbol{\Omega}^{(1)}_{DS\,\mathrm{av}}$ is obtained by the replacement $m_2 \to m_2 + \mu/3$.

Now we turn to the precession of the orbit. Introducing the Runge-Lenz vector

$$\boldsymbol{A}/\mu = \boldsymbol{v}\times(\boldsymbol{r}\times\boldsymbol{v}) - GM\boldsymbol{r}/r, \tag{27}$$

and noting that

$$\boldsymbol{L}/\mu = \boldsymbol{r}\times\boldsymbol{v}, \tag{28}$$

we find that the secular results for the precession of the orbit are

$$\dot{\boldsymbol{L}}_{\mathrm{av}} = \boldsymbol{\Omega}^* \times \boldsymbol{L}, \tag{29}$$

$$\dot{\boldsymbol{A}}_{\mathrm{av}} = \boldsymbol{\Omega}^* \times \boldsymbol{A}, \tag{30}$$

where

$$\boldsymbol{\Omega}^* = \boldsymbol{\Omega}^{*(E)} + \boldsymbol{\Omega}^{*(1)} + \boldsymbol{\Omega}^{*(2)} + \boldsymbol{\Omega}^{*(1,2)} + \boldsymbol{\Omega}^{*(Q1)} + \boldsymbol{\Omega}^{*(Q2)}, \tag{31}$$

and

$$\boldsymbol{\Omega}^{*(E)} = A^{*(E)}\boldsymbol{n}, \tag{32}$$

$$\boldsymbol{\Omega}^{*(1)} = A^{*(1)}[\boldsymbol{n}^{(1)} - 3(\boldsymbol{n}\cdot\boldsymbol{n}^{(1)})\boldsymbol{n}], \tag{33}$$

$$\boldsymbol{\Omega}^{*(2)} = A^{*(2)}[\boldsymbol{n}^{(2)} - 3(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})\boldsymbol{n}], \tag{34}$$

$$\boldsymbol{\Omega}^{*(1,2)} = A^{*(1,2)}\{(\boldsymbol{n}\cdot\boldsymbol{n}^{(1)})\boldsymbol{n}^{(2)} + (\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})\boldsymbol{n}^{(1)} + [\boldsymbol{n}^{(1)}\cdot\boldsymbol{n}^{(2)} - 5(\boldsymbol{n}\cdot\boldsymbol{n}^{(1)})(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})]\boldsymbol{n}\}, \tag{35}$$

$$\boldsymbol{\Omega}^{*(Q1)} = A^{*(Q1)}\{2(\boldsymbol{n}\cdot\boldsymbol{n}^{(1)})\boldsymbol{n}^{(1)} + [1 - 5(\boldsymbol{n}\cdot\boldsymbol{n}^{(1)})^2]\boldsymbol{n}\}, \tag{36}$$

$$\boldsymbol{\Omega}^{*(Q2)} = A^{*(Q2)}\{2(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})\boldsymbol{n}^{(2)} + [1 - 5(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})^2]\boldsymbol{n}\}, \tag{37}$$

with

$$A^{*(E)} = \frac{3G\bar{\omega}M}{c^2 a(1-e^2)}, \tag{38}$$

$$A^{*(1)} = \frac{GS^{(1)}(4+3m_2/m_1)}{2c^2a^3(1-e^2)^{\frac{3}{2}}}, \tag{39}$$

$$A^{*(2)} = \frac{GS^{(2)}(4+3m_1/m_2)}{2c^2a^3(1-e^2)^{\frac{3}{2}}}, \tag{40}$$

$$A^{*(1,2)} = \frac{-3GS^{(1)}S^{(2)}/\mu\bar{\omega}}{2c^2a^3(1-e^2)^2}, \tag{41}$$

$$A^{*(Q1)} = \frac{-3GMJ_2^{(1)}/\bar{\omega}}{4a^3(1-e^2)^2}, \tag{42}$$

$$A^{*(Q2)} = \frac{-3GMJ_2^{(2)}/\bar{\omega}}{4a^3(1-e^2)^2}. \tag{43}$$

3. – Heuristic classical derivation of the spin-dependent terms in the gravitational interaction of two arbitrary masses.

The motion of a spinning test mass, in the gravitational field of a large mass, was analyzed in detail by PAPAPETROU [7, 8] and CORINALDESI [8]. In the case of a stationary gravitational field, due to a spinning body of mass m_2, one may readily write down the equations of motion of the test body of mass m_1 ($m_1 \ll m_2$). Next, we make a co-ordinate transformation, corresponding in essence to a change from the use of the Corinaldesi-Papapetrou spin supplementary condition [8] to the use of the spin supplementary condition of Pryce [18] and Newton-Wigner [19]. This condition has the advantage that, in the transition to quantum-mechanical operator language, the operators corresponding to the different components of position commute with each other [18, 19]; it corresponds to what we have used in our previous work [17]. As already mentioned, the choice of spin supplementary conditions does not affect the observable results, *viz.* the precession of the orbit and the spin precessions.

The Hamiltonian, from which these equations of motion may be derived, is then obtained. We find that the result for the spin-dependent part of the Hamiltonian, which we will denote by $\tilde{\mathscr{H}}_S$ (here, and elsewhere, a tilde will denote the large-mass approximation), is

$$\tilde{\mathscr{H}}_S = \tilde{V}_{S1} + \tilde{V}_{S2} + \tilde{V}_{S1,S2}, \tag{44}$$

where

$$\hat{V}_{S1} = \frac{3G}{2c^2 r^3}\frac{m_2}{m_1}\boldsymbol{S}^{(1)}\cdot(\boldsymbol{r}\times\boldsymbol{P}), \tag{45}$$

$$\hat{V}_{S2} = \frac{2G}{c^2 r^3}\boldsymbol{S}^{(2)}\cdot(\boldsymbol{r}\times\boldsymbol{P}), \tag{46}$$

$$\hat{V}_{S1,S2} = \frac{G}{c^2 r^3}\left(\frac{3(\boldsymbol{S}^{(1)}\cdot\boldsymbol{r})(\boldsymbol{S}^{(2)}\cdot\boldsymbol{r})}{r^2} - \boldsymbol{S}^{(1)}\cdot\boldsymbol{S}^{(2)}\right). \tag{47}$$

One may readily verify that the contribution of $\hat{V}_{S2}$ to the precession of the perihelion of the test body m_1 agrees with the result obtained by LENSE and THIRRING [20, 21]. Furthermore, the contribution of $\hat{V}_{S1}$ and $\hat{V}_{S1,S2}$ to the precession of the spin of m_1 agrees with the results obtained by SCHIFF [22] for the precession of a «gyroscope». Thus we can proceed with *confidence in the correctness of eqs.* (45)-(47). Our task now is to generalize these results to the case where the large-mass approximation no longer holds.

The results, in the case where m_1 and m_2 are arbitrary, clearly must satisfy two criteria:

a) for $m_2 \gg m_1$, they must reduce to eqs. (45)-(47);

b) for $m_1 \gg m_2$, they must reduce to eqs. (45) to (47), after making the replacement $1 \leftrightarrow 2$ everywhere.

These criteria are met if we choose V_{S1}, V_{S2} and $V_{S1,S2}$ as given in eqs. (7)-(9). Thus we have heuristically derived the spin-dependent terms in the gravitational interaction of two arbitrary masses. However, we note that the choice of the spin-dependent terms, based on criteria *a*) and *b*), is *not unique*, because one could also include terms which are proportional to, say, μ/M, which vanishes both for $m_1 \gg m_2$ and $m_2 \gg m_1$. As it turns out, our rigorous derivation [5] shows that there are no extra terms, such as μ/M, which vanish both for $m_1 \gg m_2$ and $m_2 \gg m_1$.

4. – Application to binary systems, particularly the binary pulsar PSR 1913+16.

In the observation of stellar binary systems, among the quantities which are immediately inferred from the observations are the average angular velocity $\bar{\omega} = 2\pi/T$ (T is the orbital period), the eccentricity e, and, in the case where one of the bodies (say m_1) is a pulsar, the angular velocity of rotation $\omega^{(1)}$. Hence, it is desirable to rewrite our previous equations in terms of as few unknown quantities as possible. The semi-major axis a may be eliminated by means of the relation

$$\bar{\omega}^2 a^3 = GM. \tag{48}$$

Hence, we obtain

(49) $$A^{(1)}_{DS} = \frac{3G^{\frac{2}{3}}\bar{\omega}^{\frac{5}{3}}}{2c^2(1-e^2)}\,\frac{m_2+\mu/3}{M^{\frac{1}{3}}},$$

(50) $$A^{(1)}_{LT} = \frac{\bar{\omega}^2}{2c^2(1-e^2)^{\frac{3}{2}}}\,\frac{S^{(2)}}{M},$$

(51) $$A^{(1)}_{Q1} = \frac{\bar{\omega}^2}{2\omega^{(1)}(1-e^2)^{\frac{3}{2}}}\,\frac{\Delta I^{(1)}}{I^{(1)}}\,\frac{m_2}{M},$$

and

(52) $$A^{\theta(E)} = \frac{3G^{\frac{2}{3}}\bar{\omega}^{\frac{5}{3}}}{c^2(1-e^2)}\,M^{\frac{2}{3}},$$

(53) $$A^{\theta(1)} = \frac{\bar{\omega}^2}{2c^2(1-e^2)^{\frac{3}{2}}}\,\frac{S^{(1)}(4+3m_2/m_1)}{M},$$

(54) $$A^{\theta(2)} = \frac{\bar{\omega}^2}{2c^2(1-e^2)^{\frac{3}{2}}}\,\frac{S^{(2)}(4+3m_1/m_2)}{M},$$

(55) $$A^{\theta(1,2)} = \frac{-3\bar{\omega}^{\frac{7}{3}}}{2G^{\frac{1}{3}}c^2(1-e^2)^2}\,\frac{S^{(1)}S^{(2)}}{\mu M^{\frac{5}{3}}},$$

(56) $$A^{\theta(Q1)} = \frac{-3\bar{\omega}^{\frac{7}{3}}}{4G^{\frac{2}{3}}(1-e^2)^2}\,\frac{\Delta I^{(1)}}{m_1 M^{\frac{2}{3}}},$$

(57) $$A^{\theta(Q2)} = \frac{-3\bar{\omega}^{\frac{7}{3}}}{4G^{\frac{2}{3}}(1-e^2)^2}\,\frac{\Delta I^{(2)}}{m_2 M^{\frac{2}{3}}}.$$

Now we will apply these results to the case of the binary pulsar PSR 1913+16, which we designate as m_1. The relevant observed quantities are [1]

(58) $$e = 0.615,$$

(59) $$\bar{\omega} = 2.25\cdot 10^{-4}\ \text{rad/s}$$

and

(60) $$\omega^{(1)} = 1.06\cdot 10^{2}\ \text{rad/s}.$$

It is also convenient to define

(61) $$I_0 \equiv 10^{44}\ \text{g cm}^2,$$

(62) $$\Delta I_0 \equiv 10^{38}\ \text{g cm}^2$$

and

(63) $$\omega_0 \equiv 10^{2}\ \text{rad/s}.$$

Thus

(64) $$\Delta I_0/I_0 = 10^{-6}$$

and

$$S_0 = I_0 \omega_0 = 10^{46}\ \text{g cm}^2\,\text{s}^{-1}. \tag{65}$$

For neutron stars, we note that, as pointed out in [23], estimates of $\Delta I/I$ range from about 10^{-5} to 10^{-7}, whereas estimates of I range from about $7\cdot 10^{43}$ to $7\cdot 10^{44}$ g cm^2 [24]. Furthermore, $S_0 \approx 10^{-3} S_\odot$, which is related to the fact that, in the evolution of main-sequence stars to the neutron star stage, the angular momentum is approximately conserved, so that pulsars have initial angular velocities $\approx 10^4$ rad s^{-1}, which, however, very rapidly drop to $\approx 10^2$ rad s^{-1} because of intense radiation losses.

It follows that the various A and A^* values can be written simply as follows:

$$A^{(1)}_{DS} = (0.974\ \text{deg/y}) \left\{ \frac{2^{\frac{1}{3}}}{\frac{7}{6}} \, \frac{m_2 + \mu/3}{M^{\frac{1}{3}} m_\odot^{\frac{2}{3}}} \right\}, \tag{66}$$

$$A^{(1)}_{LT} = (0.261\cdot 10^{-6}\ \text{deg/y}) \frac{S^{(2)}}{S_0} \left\{ \frac{2m_\odot}{M} \right\}, \tag{67}$$

$$A^{(1)}_{Q1} = (0.439\cdot 10^{-8}\ \text{deg/y}) \frac{\Delta I^{(2)}/I^{(2)}}{\Delta I_0/I_0} \left\{ \frac{2m_2}{M} \right\}, \tag{68}$$

and

$$A^{*(S)} = (3.34\ \text{deg/y}) \left\{ \left(\frac{M}{2m_\odot} \right)^{\frac{2}{3}} \right\}, \tag{69}$$

$$A^{*(1)} = (0.183\cdot 10^{-5}\ \text{deg/y}) \frac{S^{(1)}}{S_0} \left\{ \frac{8m_1 m_\odot - 6m_2 m_\odot}{7m_1 M} \right\}, \tag{70}$$

$$A^{*(2)} = (0.183\cdot 10^{-5}\ \text{deg/y}) \frac{S^{(2)}}{S_0} \left\{ \frac{8m_2 m_\odot - 6m_1 m_\odot}{7m_2 M} \right\}, \tag{71}$$

$$A^{*(1,2)} = -(0.147\cdot 10^{-11}\ \text{deg/y}) \frac{S^{(1)} S^{(2)}}{S_0 S_0} \left\{ \frac{2^{\frac{2}{3}} m_\odot^{\frac{8}{3}}}{\mu M^{\frac{5}{3}}} \right\}, \tag{72}$$

$$A^{*(Q1)} = -(0.132\cdot 10^{-11}\ \text{deg/y}) \frac{\Delta I^{(1)}}{\Delta I_0} \left\{ \frac{2^{\frac{2}{3}} m_\odot^{\frac{5}{3}}}{m_1 M^{\frac{2}{3}}} \right\}, \tag{73}$$

$$A^{*(Q2)} = -(0.132\cdot 10^{-11}\ \text{deg/y}) \frac{\Delta I^{(2)}}{\Delta I_0} \left\{ \frac{2^{\frac{2}{3}} m_\odot^{\frac{5}{3}}}{m_2 M^{\frac{2}{3}}} \right\}. \tag{74}$$

We have written the eqs. (66)-(74) so that, when $m_1 = m_2 = m_\odot$, all the quantities in the curly brackets reduce to unity. For reference, we note that $2^{\frac{2}{3}} = 1.587$ and $2^{\frac{1}{3}}/\frac{7}{6} = 1.080$.

With regard to the precession of the spin of the pulsar PSR 1913 + 16, it is clear that, regardless of whether or not the pulsar's companion (body 2) is compact, $A^{(1)}_{LT}$ and $A^{(1)}_{Q1}$ are negligible compared to the $A^{(1)}_{DS}$ (which we recall arises from the spin-orbit contribution).

Turning our attention now to the precession of the orbit, we can write $\boldsymbol{\Omega}^*$ in the form that the astronomers or experimentalists use as [4]

$$\boldsymbol{\Omega}^* = \frac{\mathrm{d}\Omega}{\mathrm{d}t}\boldsymbol{n}_0 - \frac{\mathrm{d}\omega}{\mathrm{d}t}\boldsymbol{n} - \frac{\mathrm{d}i}{\mathrm{d}t}\frac{\boldsymbol{n}_0\times\boldsymbol{n}}{|\boldsymbol{n}_0\times\boldsymbol{n}|}, \tag{75}$$

where Ω, ω and i denote the longitude of the ascending node, the argument of the periastron and the inclination of the orbit, respectively, in the reference system of the plane of the sky (the tangent plane to the celestial sphere at the center of mass of the binary system). In addition, $\boldsymbol{n}_0$ is a unit vector normal to the plane of the sky directed from the center of mass of the binary system towards the Earth. The angle between $\boldsymbol{n}_0$ and $\boldsymbol{n}$ is the inclination i.

Actually, the dominant A^* term, *viz.* $A^{*(E)}$, causes a change in ω only. Contributions to a change in Ω and i arise from all the other terms, but it is clear that the only term of potentially significant magnitude (in the case where the companion is not a neutron star) is $A^{*(Q2)}$.

If we use eq. (37), it is clear that we may write [4]

$$\boldsymbol{\Omega}^{*(Q2)} = A^{(Q2)}[c_1\boldsymbol{n}_0 + c_2\boldsymbol{n} + c_3(\boldsymbol{n}_0\times\boldsymbol{n})/\sin i], \tag{76}$$

where

$$c_1 = 2(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})[\boldsymbol{n}^{(2)}\cdot\boldsymbol{n}_0 - (\boldsymbol{n}^{(2)}\cdot\boldsymbol{n})\cos i]/\sin^2 i, \tag{77}$$

$$c_2 = 2(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})[\boldsymbol{n}^{(2)}\cdot\boldsymbol{n} - (\boldsymbol{n}^{(2)}\cdot\boldsymbol{n}_0)\cos i]/\sin^2 i + [1 - 5(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})^2], \tag{78}$$

$$c_3 = 2(\boldsymbol{n}\cdot\boldsymbol{n}^{(2)})[\boldsymbol{n}^{(2)}\cdot(\boldsymbol{n}_0\times\boldsymbol{n})]/\sin i. \tag{79}$$

Thus, in general, the quadrupole moment of the companion star changes Ω, ω and i. However, in the particular case where $\boldsymbol{n}_2$ is perpendicular to or parallel to $\boldsymbol{n}$, then $\boldsymbol{\Omega}^{*(Q2)}$ does not contribute to a change in Ω or i. In addition, there will be no change in i if $\boldsymbol{n}^{(2)}$ is in the plane containing $\boldsymbol{n}_0$ and $\boldsymbol{n}$.

Now, if the companion star is a neutron star, then $\Delta I^{(2)} \approx \Delta I_0$ and $A^{*(Q2)}$ is negligible.

However, for certain models of white dwarfs and normal stars, one can obtain relatively large values of $\Delta I^{(2)}$. In fact, for normal stars, values $\approx 10^{14}\Delta I_0$ are possible, with corresponding values of $A^{*(Q2)}$ as large as $\approx 10^2\Omega^{*(E)}$. The observational determination of the periastron precession will put severe constraints on acceptable models [1] and, in fact, it already seems clear that the pulsar's companion is certainly no larger than a white dwarf.

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