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Gravitational analogue of magnetic force

THE gravitational analogue of the magnetic force has been discussed[1,2] within the framework of special relativity. The difference of opinion between these authors stems basically from the fact that a rigorous calculation of this force must be carried out using the general theory of relativity, and we give here such a calculation.

Consider two massive charged particles with masses, charges, coordinates and velocities, denoted by m_i, e_i, r_i and v_i, respectively ($i = 1, 2$). Then, to order c^{-2}, the Lagrangian for the system is[3]

$$L = L_0 - V \equiv L_0 - (V_{EIH} - V_D - V_M) \tag{1}$$

where L_0–V_{EIH} is the Einstein–Infeld–Hoffmann Lagrangian for uncharged particles[4], L_0–V_D is the Darwin Lagrangian of classical electrodynamics for a system of two charges (see ref. 4, p. 168) and V_M consists of mixed mass–charge terms. Explicitly, we have

$$L_0 = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \frac{1}{8c^2}(m_1v_1^4 + m_2v_2^4) \tag{2}$$

$$-V_{EIH} = \frac{Gm_1m_2}{r} - \frac{Gm_1m_2}{2rc^2}[\mathbf{v}_1\cdot\mathbf{v}_2 - (\mathbf{v}_1\cdot\mathbf{n})(\mathbf{v}_2\cdot\mathbf{n}) - 3(\mathbf{v}_1-\mathbf{v}_2)^2] - \frac{G^2m_1m_2(m_1-m_2)}{2r^2c^2} \tag{3}$$

$$-V_D = -\frac{e_1e_2}{r} + \frac{e_1e_2}{2rc^2}[\mathbf{v}_1\cdot\mathbf{v}_2 - (\mathbf{v}_1\cdot\mathbf{n})(\mathbf{v}_2\cdot\mathbf{n})] \tag{4}$$

and

$$-V_M = \frac{G}{2r^2c^2}[2e_1e_2(m_1-m_2) - (e_1^2m_2 + e_2^2m_1)] \tag{5}$$

where $\mathbf{n}$ is a unit vector along $\mathbf{r}_1 - \mathbf{r}_2$.

The existence of a gravitational analogue of the magnetic force (the Gv_iv_j terms) is now apparent. It is instructive, however, to consider, as in refs 1 and 2, the two particles to be in equilibrium in their rest frames, from the balance of Newtonian and Coulomb forces. Then, we consider a frame of reference in which the particles, while maintaining the same distance apart, are moving with uniform velocity $\mathbf{v}_1 = \mathbf{v}_2 \equiv \mathbf{v}$ with $\mathbf{v}_1\cdot\mathbf{n} = \mathbf{v}_2\cdot\mathbf{n} = 0$. Thus

$$L_0 = \tfrac{1}{2}(m_1+m_2)v^2 + \frac{v^4}{8c^2}(m_1+m_2) \tag{6}$$

$$-V_{EIH} = \frac{Gm_1m_2}{r}\left(1 - \frac{v^2}{2c^2}\right) - \frac{G^2m_1m_2(m_1+m_2)}{2r^2c^2} \tag{7}$$

and

$$-V_D = -\frac{e_1e_2}{r}\left(1 - \frac{v^2}{2c^2}\right) \tag{8}$$

The corresponding 'generalised force' on particle 1 from particle 2 is

$$F = \left|\left(\frac{\partial L}{\partial \mathbf{r}_1}\right)\right| = -\frac{Gm_1m_2}{r^2}\left(1 - \frac{v^2}{2c^2}\right) + \frac{G^2m_1m_2(m_1-m_2)}{r^2c^2} + \frac{e_1e_2}{r^2}\left(1 - \frac{v^2}{2c^2}\right) - \frac{G}{r^2c^2}\left(2e_1e_2(m_1-m_2) - (e_1^2m_2 + e_2^2m_1)\right) \tag{9}$$

Thus, the condition for equilibrium in the moving frame is $e_i = \pm G^{1/2}m_i$. In other words, as previously noted[5,6] we have the same condition for equilibrium in the general relativistic case as in the Newtonian case. In addition, we notice that the gravitational analogue of the magnetic force, in this particular example, is simply obtained from the magnetic force $-e_1e_2v^2/2r^2c^2$ by the replacement $e_1e_2 \to -Gm_1m_2$ and, to lowest order in (v/c), this agrees with the result given in ref. 1. On examination of equations (3) and (4), however, it is clear that this is not true in general.

In fact, it is now clear that arguments based on the use of special relativity only[1,2] are inadequate in two major respects: (1) They fail to uncover the existence of various higher-order terms (in addition to the Gv^2 terms, the so-called gyron force[1] terms), such as the purely gravitational G^2 terms (which, at least for bound-state problems—perihelion precession of Mercury—are comparable in magnitude to the Gv^2 terms), nor the mixed Ge_ie_j terms. The existence of these nonlinear terms is intimately related to a well-known facet of general relativity theory—that the electromagnetic field and the gravitational field itself both act as sources of the gravitational field. (2) They provide no unambiguous way of deciding on the correct law of transformation of forces, whereas such a transformation emerges naturally in the general relativistic approach.

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1 Salisbury, W. W., and Menzel, D. H., *Nature*, 252, 664 (1974).
2 Lorrain, P., *Nature*, 257, 161 (1975).
3 Bazanski, S., *Acta phys. pol.*, 15, 363 (1956); *ibid.*, 16, 423 (1957).
4 Landau, L. D., and Lifshitz, E. M., *The Classical Theory of Fields*, 330 (Pergamon, Oxford, 1971).
5 Papapetrou, A., *Proc. R. Ir. Acad.*, 51, 191 (1947).
6 Majumdar, S. D., *Phys. Rev.*, 72, 390 (1947).

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