Generalized Conservation Laws for Free Fields with Mass.

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The purpose of this letter is twofold. First, we wish to extend our method of obtaining generalized conservation laws for massless free fields (1) (which contain, as particular cases, the generalized conservation laws of Lipkin (2) and Morgan (3)) to the case of free fields with mass. Second, we wish to show how the usual conservation laws for free fields with mass may be derived, without the use of Lagrangians or Noether's theorem, by a method similar to that used by Good and Hammer (4) for massless free fields.

Dirac (5), Fierz (6), Pauli (7) and Bhabha (8) have derived relativistic wave equations for particles of any integral or odd half-integral spin. This discussion will be based on the Bhabha equation which may be written in the form (9)

$$\left(i\alpha_\mu \frac{\partial}{\partial x_\mu} + \varkappa\right)\psi(x) = 0\,, \tag{1}$$

where α_μ are four-matrices determining the spin properties and $\varkappa$ is a constant related to the mass of the particle. The α_μ may always be chosen to have the properties

$$\alpha_i^\dagger = -\alpha_i\,, \qquad \alpha_4^\dagger = \alpha_4\,. \tag{2}$$

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(1) R. F. O'Connell and D. R. Tompkins: *Journ. Math. Phys.*, in press.

(2) D. M. Lipkin: *Journ. Math. Phys.*, **5**, 696 (1964).

(3) T. A. Morgan: *Journ. Math. Phys.*, **5**, 1659 (1964).

(4) R. H. Good jr.: *Phys. Rev.*, **105**, 1914 (1957); C. L. Hammer and R. H. Good jr.: *Phys. Rev.*, **108**, 882 (1957).

(5) P. A. M. Dirac: *Proc. Roy. Soc.*, A **155**, 447 (1936).

(6) M. Fierz: *Helv. Phys. Acta*, **12**, 3 (1939).

(7) M. Fierz and W. Pauli: *Proc. Roy. Soc. (London)*, A **173**, 211 (1939).

(8) H. J. Bhabha: *Rev. Mod. Phys.*, **17**, 200 (1945); *Proc. Ind. Acad. Sci.*, A **21**, 241 (1945).

(9) In our units $\hbar = c = 1$. Furthermore, Greek indices run from 1 to 4 and Latin indices from 1 to 3.

Furthermore, a matrix D always exists such that

$$D\alpha_i = -\alpha_i D\,, \qquad D\alpha_4 = \alpha_4 D \tag{3}$$

and $D^+ = D$. These properties enable one to write down the adjoint equation

$$-i\frac{\partial\psi^\dagger}{\partial x_\mu}D\alpha_\mu + \varkappa\psi^\dagger D = 0\,. \tag{4}$$

From eqs. (1) and (4) BHABHA obtained the charge-current conservation laws:

$$\frac{\partial}{\partial x_\mu}(\psi^\dagger D\alpha\psi) = 0\,. \tag{5}$$

Consider now the infinitesimal transformation of co-ordinates

$$x_\mu \to x'''_\mu = x_\mu + \delta x_\mu \tag{6}$$

and the corresponding transformation of the wave-function

$$\psi(x) \to \psi'''(x''')\,. \tag{7}$$

Introduce a linear operator $\mathcal{O}$ defined by

$$\psi'''(x) = \mathcal{O}\psi(x)\,. \tag{8}$$

This enables us to find the general conservation law

$$\frac{\partial}{\partial x_\mu}(\psi^\dagger D\alpha\mathcal{O}\psi) = 0\,. \tag{9}$$

This equation, which was derived without the use of Lagrangians or Noether's theorem is similar in form to the equation derived by GOOD and HAMMER [4] for massless free fields. Corresponding to the various co-ordinate transformations of displacement, rotation etc., we obtain appropriate expressions for $\mathcal{O}$. The various operators $\mathcal{O}$ when substituted into eq. (9) yield the usual conservation laws of energy-momentum, angular momentum, and so on. Let us consider some examples. The identity transformation $x'''_\nu = x_\nu$ gives $\mathcal{O} = 1$ so that eq. (9) reduces to eq. (5). The infinitesimal space and time displacements

$$x'''_\nu = x_\nu + \varepsilon_\nu\,, \tag{10}$$

where ε_ν is an infinitesimal constant, imply that

$$\psi'''(x''') = \psi(x) \tag{11}$$

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which results in

$$\mathcal{O} = 1 - \varepsilon_\nu \frac{\partial}{\partial x_\nu} . \tag{12}$$

This gives

$$\frac{\partial}{\partial x_\mu} T_{\mu\nu} = 0 , \tag{13}$$

where

$$T_{\mu\nu} = i\psi^\dagger D\alpha_\mu \frac{\partial}{\partial x_\nu} \psi . \tag{14}$$

This is the energy-momentum conservation law which BHABHA derived using a Lagrangian. We next consider the infinitesimal 4-space rotation [10]

$$x'''_\mu = x_\mu + \tfrac{1}{2} \varepsilon_{\varrho\sigma} I^{\mu\nu}_{\varrho\sigma} x_\nu , \tag{15}$$

where $\varepsilon_{\varrho\sigma}$ are the infinitesimal parameters and $I_{\varrho\sigma}$ the infinitesimal generators. The corresponding wave-function transformation

$$\psi'''(x''') = \psi(x) + \tfrac{1}{2} \varepsilon_{\varrho\sigma} I_{\varrho\sigma} \psi(x) \tag{16}$$

results in

$$\mathcal{O} = 1 + \frac{1}{2} \varepsilon_{\mu\nu} \left[\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} \right) + I_{\mu\nu} \right] . \tag{17}$$

This gives the conservation law

$$\frac{\partial}{\partial x_\sigma} M_{\mu\nu,\sigma} = 0 ,$$

where

$$M_{\mu\nu,\sigma} = x_\mu T_{\nu\sigma} - x_\nu T_{\mu\sigma} + i\psi^\dagger D\alpha_\sigma I_{\mu\nu} \psi . \tag{18}$$

This is the angular-momentum density which BHABHA obtained using a Lagrangian.

In general, we see that we may obtain all of the usual conservation laws from eq. (9). We now wish to generalize these conservation laws in the same manner as employed for massless fields [1].

Consider any operator V so selected that $V\psi(x)$ is a (generalized) solution of eq. (1). Thus we can write

$$\left(i\alpha_\mu \frac{\partial}{\partial x_\mu} + \varkappa \right) \psi'(x) = 0 \tag{19}$$

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[10] See, for example, P. ROMAN: *Theory of Elementary Particles* (Amsterdam, 1961), pp. 41 and 95.

and

$$\left(i\alpha_\mu \frac{\partial}{\partial x_\mu} + \varkappa\right) \psi''(x) = 0\,, \tag{20}$$

where

$$\psi' = V'\psi \qquad \text{and} \qquad \psi'' = V''\psi\,. \tag{21}$$

Using a procedure analogous to that used in deriving eq. (9) we obtain

$$\frac{\partial}{\partial x_\mu}(\psi'^{\dagger} D\alpha_\mu \mathcal{O}\psi'') = 0\,, \tag{22}$$

which constitute our generalized conservation laws for free fields with mass.

As a particular example of our general results we consider the generalized conservation laws associated with the Dirac field. In this case eq. (22) reduces to

$$\frac{\partial}{\partial x_\mu}(\psi'^{\dagger}\gamma_4\gamma_\mu \mathcal{O}\psi'') = 0\,, \tag{23}$$

where the γ_μ are the usual Dirac matrices. Taking $\psi' = \psi'' = \psi$ we obtain the usual Dirac field conservation laws. An example of a generalized conservation law for the Dirac field is

$$\frac{\partial}{\partial x_\mu}\{(\partial_{\alpha_1}\partial_{\alpha_2} \dots \partial_{\alpha_n}\psi^{\dagger})\gamma_4\gamma_\mu \mathcal{O}(\partial_{\beta_1} \dots \partial_{\beta_m}\psi)\} = 0 \tag{24}$$

corresponding to the particular choice of

$$V_1 = \partial_{\alpha_1} \dots \partial_{\alpha_n}\,, \qquad V_2 = \partial_{\beta_1}\partial_{\beta_2} \dots \partial_{\beta_m}\,. \tag{25}$$