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Effect of the gyro's quadrupole moment on the relativity gyroscope experiment

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We show that the quadrupole moment of the gyro affects the drift rate of the presently planned Stanford gyroscope experiment by (0.0254″/yr)sin2α, where α is the angle between the gyro's spin axis and the orbit plane. Thus, if the gyro's spin axis is off by more than one degree from being either in the orbit plane or perpendicular to the orbit plane the drift rate will exceed 0.001″/yr, which is the expected accuracy of the experiment, and, hence, this predictable drift rate will have to be taken into account to preserve the expected accuracy of the experiment. We also calculate the precession of the orbit of the gyro due to the gyro's quadrupole moment and find it two orders of magnitude smaller than the precession of the orbit due to the gyro's spin.

I. INTRODUCTION

A new test of general relativity, proposed by Schiff,[1,2] is the measurement of the precession of the spin of a gyroscope in orbit about the earth. A team of physicists and engineers in the Stanford physics and aeronautics departments expect to carry out this experiment[3,4] in the near future by launching a satellite containing two pairs of superconducting gyros in a polar orbit 500 miles[4] above the earth. The spin of one pair will be parallel to the earth's axis and the spin of the other pair will be perpendicular to the plane of the orbit. Although the effect of general relativity on the precession of the spin of a gyroscope has been thoroughly discussed in the literature,[1-10] the Newtonian effect of the gyro's quadrupole moment on the precession of the spin has been less thoroughly discussed.

In Sec. II we shall calculate the quadrupole moment of the gyro assuming that it is a perfect sphere at rest and becomes distorted when it is spinning. In Sec. III we find the precession of the spin axis of the gyro due to the gyro's quadrupole moment. In Sec. IV we find the precession of the orbit due to the gyro's quadrupole moment and compare it to the precession of the orbit due to the gyro's spin. Finally, we give our conclusions in Sec. V.

II. CALCULATION OF $\Delta r_0/r_0$ AND $\Delta I/I$

Let us consider the gyro at rest to be a perfect sphere of radius r_0, with mass m and mass density ρ. We shall denote the coordinates of the point occupied by a particle in the unstrained state of the body by $\bar{x}$, $\bar{y}$, and $\bar{z}$, and the coordinates of the point occupied by the same particle in the strained state by $x'=\bar{x}+u$, $y'=\bar{y}+v$, and $z'=\bar{z}+w$. If the gyro rotates with an angular velocity ω about the $\bar{z}$ axis the complete expressions for the displacements are given by[11]

$$\frac{u}{\bar{x}}=\frac{v}{\bar{y}}=\frac{\rho\omega^2}{3}\left[\frac{1}{5(\lambda+2\mu)}\left(\frac{5\lambda+6\mu}{3\lambda+2\mu}r_0^2-\bar{r}^2\right)+\frac{1}{\mu(19\lambda+14\mu)}\left[(4\lambda+3\mu)r_0^2-\tfrac{1}{2}(5\lambda+4\mu)\bar{r}^2+(\lambda+\mu)(\bar{x}^2+\bar{y}^2-2\bar{z}^2)\right]\right] \tag{1}$$

and

$$\frac{w}{\bar{z}}=\frac{\rho\omega^2}{3}\left[\frac{1}{5(\lambda+2\mu)}\left(\frac{5\lambda+6\mu}{3\lambda+2\mu}r_0^2-\bar{r}^2\right)+\frac{1}{\mu(19\lambda+14\mu)}\left[-(8\lambda+6\mu)r_0^2+(5\lambda+4\mu)\bar{r}^2+(\lambda+\mu)(\bar{x}^2+\bar{y}^2-2\bar{z}^2)\right]\right], \tag{2}$$

where $\bar{r}=(\bar{x}^2+\bar{y}^2+\bar{z}^2)^{1/2}$ and

$$\lambda=\frac{E\sigma}{(1+\sigma)(1-2\sigma)},\quad \mu=\frac{E}{2(1+\sigma)},\quad k=\frac{E}{3(1-2\sigma)}. \tag{3}$$

In the above E is Young's modulus, σ is Poisson's ratio, and k is the compressibility.[11]

Using Eqs. (1) and (2) we obtain

$$\frac{\Delta r_0}{r_0}\equiv\frac{u(\bar{x}=r_0)-w(\bar{z}=r_0)}{r_0}=\frac{\rho\omega^2 r_0^2}{14}\frac{(35\lambda+28\mu)}{\mu(19\lambda+14\mu)}. \tag{4}$$

The moment-of-inertia tensor for the spinning gryo is given by

$$I_{ij} = \int dv'(r'^2\delta_{ij} - x_i' x_j')\,\rho(\vec{r}') , \tag{5}$$

where $r' = (x'^2 + y'^2 + z'^2)^{1/2}$ and the integration is over the distorted sphere. The quadrupole moment of the gyro is just a constant times ΔI, where

$$\Delta I \equiv I_{33} - \tfrac{1}{2}(I_{11} + I_{22}) = \tfrac{1}{2}\int dv'\,(r'^2 - 3z'^2)\,\rho(\vec{r}') . \tag{6}$$

In the above integration we can replace $\rho(\vec{r}')dv'$ by $\rho\,d\bar{v}$ since the mass of volume element $d\bar{v}$ goes into the volume element dv'. We thus get, after neglecting u^2, v^2, w^2 terms,

$$\Delta I = \rho \int d\bar{v}\,(\bar{x}u + \bar{y}v - 2\bar{z}w) , \tag{7}$$

where the integration is now over the sphere. We also have

$$I = \tfrac{2}{5} m r_0^2 = \tfrac{8}{15}\pi\rho r_0^5 . \tag{8}$$

Using Eqs. (1) and (2) in (7), and then Eqs. (3) and (4), we obtain

$$\frac{\Delta I}{I} = \frac{\rho\omega^2 r_0^2}{14}\,\frac{(35\lambda + 26\mu)}{\mu(19\lambda + 14\mu)} = \frac{\Delta r_0}{r_0}\left[1 - \frac{1}{14}\left(\frac{1 - 2\sigma}{1 + \sigma/2}\right)\right] . \tag{9}$$

In the above integration only the integrals of the type

$$\begin{aligned} &\int r_0^2\,\bar{x}^2\,d\bar{v} = \tfrac{4}{15}\pi r_0^7 , \\ &\int \bar{r}^2\,\bar{x}^2\,d\bar{v} = \tfrac{4}{21}\pi r_0^7 , \\ &\int \bar{x}^2\bar{y}^2\,d\bar{v} = \tfrac{4}{105}\pi r_0^7 , \\ &\int \bar{x}^4\,d\bar{v} = \tfrac{4}{35}\pi r_0^7 \end{aligned} \tag{10}$$

were needed. Contrary to what is stated in Refs. 3 and 4, it should be noted that $\Delta I/I$ does not equal $\Delta r_0/r_0$. This is because the mass density, $\rho(r')$, for the distorted sphere is not uniform. If we had an oblate spheroid of uniform density then it is easy to show that $\Delta I/I$ would be equal to $\Delta r_0/r_0$.

For the relativity gyroscope experiment[4] the gyro is to be a sphere of fused quartz 4 cm in diameter spinning 200 rev/sec. At a temperature of 2 °K fused quartz has the following properties[12,13]:

$$\rho = 2.2\ \mathrm{g/cm^3} , \tag{11}$$

$$E = 7\times 10^{11}\ \mathrm{dyn/cm^2} , \tag{12}$$

$$\sigma = 0.16 . \tag{13}$$

The above values can vary slightly by a few percent from one sample of fused quartz to another. Using the above values in Eqs. (4) and (9) we obtain

$$\Delta r_0/r_0 = 6.38\times 10^{-6} \tag{14}$$

and

$$\Delta I/I = 6.09\times 10^{-6} . \tag{15}$$

In other words, we see that for fused quartz $\Delta I/I$ is 4.5% smaller than $\Delta r_0/r_0$. In fact, for all real solids, it is clear from Eq. (9) that $\Delta I/I$ is less than $\Delta r_0/r_0$. The difference becomes smaller the larger the value of σ. For an ideal incompressible solid, for which σ is 0.5 and thus the compressibility k of Eq. (3) is infinite, the correction vanishes, as it should because the density of the medium is constant and the result has to coincide with the equality between $\Delta I/I$ and $\Delta r_0/r_0$ in an unstrained oblate spheroid.

III. PRECESSION OF THE SPIN

Although the precession of the spin of an oblate spheroid in a Newtonian gravitational field has been studied in various astronomical[14] and space contexts,[15] we shall derive here the precession of the spin of a gyro in the gravitational field of the earth due to the gyro's guadrupole moment, using a method that extends naturally for calculations in Sec. IV.

Let $\vec{R}$ be a vector (with $R^2 = X^2 + Y^2 + Z^2$) from the center of the earth of mass M to the center of the gyro and let r' be a vector from the center of the gyro to some point in the gyro. Also let $\vec{n}^{(2)} = \hat{Z}$ be a unit vector in the direction of the earth's spin and $\vec{n}^{(1)} = \hat{z}'$ be a unit vector in the direction of the gyro's spin. The potential energy, $V(\vec{R})$, of the gyro in orbit about the earth is given by

$$V(\vec{R}) = -\int dv'\,\rho(\vec{r}')\,\Phi(\vec{R} + \vec{r}') , \tag{16}$$

where[6]

$$\Phi(\vec{R}) = \frac{GM}{R}\left(1 - \frac{1}{R^2}J_2P_2(\cos\Theta) - \frac{1}{R^3}J_3P_3(\cos\Theta) - \cdots\right), \tag{17}$$

and Θ is the angle between $\vec{n}^{(2)}$ and $\vec{R}$. Expanding $\Phi(\vec{R} + \vec{r}')$ we get

$$\Phi(\vec{R} + \vec{r}') = \Phi(\vec{R}) + x_i' \left.\frac{\partial\Phi}{\partial x_i'}\right|_0 + \tfrac{1}{2}x_i' x_j' \left.\frac{\partial^2\Phi}{\partial x_i'\partial_j'}\right|_0 + \cdots , \tag{18}$$

where the derivatives are evaluated at $\vec{r}' = 0$. We also note that

$$m = \int dv'\,\rho(\vec{r}') , \tag{19}$$

$$\vec{d} \equiv \int dv'\,\vec{r}'\,\rho(\vec{r}') = 0 , \tag{20}$$

$$D_{ij} \equiv \int dv'\,(r'^2\delta_{ij} - 3x_i' x_j')\,\rho(\vec{r}')\ , \tag{21}$$

$$\frac{\partial^2\Phi}{\partial x_i' \partial x_i'} = 0\ , \tag{22}$$

$$D_{ii} = 0\ . \tag{23}$$

Usings Eqs. (18)–(22) in Eq. (16) we obtain

$$V(\vec{R}) = -\,m\Phi(\vec{R}) + \tfrac{1}{6} D_{ij} \left.\frac{\partial^2\Phi}{\partial x_i' \partial x_j'}\right|_0\ . \tag{24}$$

For the gyro spinning along the z' axis we have $D_{12} = D_{23} = D_{31} = 0$ and $D_{11} = D_{22}$. Then using Eq. (23) we have

$$2\Delta I = D \equiv D_{33} = -2D_{11} = -2D_{22}\ . \tag{25}$$

Thus using Eqs. (22) and (25) in Eq. (24) we have

$$V(\vec{R}) = -\,m\Phi(\vec{R}) + \tfrac{1}{4} D \left.\frac{\partial^2\Phi}{\partial z' \partial z'}\right|_0\ . \tag{26}$$

Let x, y, z be a set of axes with the origin at the center of the gyro and parallel to the X, Y, Z axes. Then we have

$$x_i' = a_{ij} x_j\ , \tag{27}$$

where a_{ij} can be written in terms of the Euler angles ϕ, θ, ψ and

$$\frac{\partial^2\Phi}{\partial X_k \partial X_l} = \left.\frac{\partial^2\Phi}{\partial x_k x_l}\right|_0\ . \tag{28}$$

In the primed system we have

$$n_i^{(1)\prime} = (0, 0, 1)\ , \tag{29}$$

while in the unprimed system we have

$$\begin{aligned} n_i^{(1)} &= a_{ji} n_j^{(1)\prime} \\ &= a_{3i} \\ &= (\sin\theta\sin\phi, -\sin\theta\cos\phi, \cos\theta)\ . \end{aligned} \tag{30}$$

We thus obtain

$$\begin{aligned} \left.\frac{\partial^2\Phi}{\partial x_3' \partial x_3'}\right|_0 &= n_i^{(1)\prime} n_j^{(2)\prime} \left.\frac{\partial^2\Phi}{\partial x_i' \partial x_j'}\right|_0 \\ &= n_i^{(1)} n_j^{(1)} \frac{\partial^2\Phi}{\partial X_i \partial X_j}\ . \end{aligned} \tag{31}$$

The Lagrangian for the gyro in orbit about the earth is

$$\begin{aligned} \mathcal{L} &= \tfrac{1}{2} m v^2 + \tfrac{1}{2} I\omega^2 + m\Phi(\vec{R}) \\ &\quad - \tfrac{1}{4} D n_i^{(1)} n_j^{(1)} \frac{\partial^2\Phi}{\partial X_i \partial X_j}\ . \end{aligned} \tag{32}$$

Then using Eqs. (30) and (32) we get

$$\frac{\partial\mathcal{L}}{\partial\phi} = \tfrac{1}{2} D\epsilon_{3jk} n_k^{(1)} n_l^{(1)} \frac{\partial^2\Phi}{\partial X_j \partial X_l}\ , \tag{33}$$

$$p_\phi = \frac{\partial\mathcal{L}}{\partial\dot\phi} = I\omega_3\ , \tag{34}$$

where we have used the space set of axes for $\vec{\omega}$. The Euler-Lagrange equations are

$$I\dot\omega_i = \tfrac{1}{2} D\epsilon_{ijk} n_k^{(1)} n_l^{(1)} \frac{\partial^2\Phi}{\partial X_j \partial X_l}\ . \tag{35}$$

Since $\dot{\vec{n}}^{(1)}$ is perpendicular to $\vec{n}^{(1)}$ we may write Eq. (35) as

$$\dot n_i^{(1)} = \epsilon_{ijk}\Omega_j n_k^{(1)}\ , \quad \Omega_j = \frac{\Delta I}{I\omega} n_l^{(1)} \frac{\partial^2\Phi}{\partial X_j \partial X_l} \tag{36}$$

or in vector form as

$$\dot{\vec{n}}^{(1)} = \vec{\Omega}\times\vec{n}^{(1)}\ , \quad \vec{\Omega} = \frac{\Delta I}{I\omega}(\vec{n}^{(1)}\cdot\vec{\nabla})\vec{\nabla}\Phi\ . \tag{37}$$

We shall now consider the result due to

$$\vec{\Omega}_q = \frac{\Delta I}{I\omega}(\vec{n}^{(1)}\cdot\vec{\nabla})\vec{\nabla}\Phi^{(0)}\ , \tag{38}$$

where

$$\Phi^{(0)}(\vec{R}) = GM/R\ . \tag{39}$$

We then have

$$\vec{\Omega}_q = \frac{GM\Delta I}{I\omega}\left(\frac{3(\vec{n}^{(1)}\cdot\vec{R})\vec{R}}{R^5} - \frac{\vec{n}^{(1)}}{R^3}\right)\ . \tag{40}$$

It should be noted that $\vec{\Omega}_q$ has the same form (though not the same direction) as the Lense-Thirring term,[6] $\vec{\Omega}_{LT}$, and hence the average value over one period will also be similar. We thus get

$$\vec{\Omega}_{q\,\mathrm{av}} = \frac{GM\Delta I}{2I\omega a^3(1-e^2)^{3/2}}\,[\vec{n}^{(1)} - 3(\vec{n}\cdot\vec{n}^{(1)})\vec{n}]\ , \tag{41}$$

where a, e, and $\vec{n}$ are the semimajor axis, eccentricity, and unit vector in the direction of the orbital angular momentum, respectively.

The secular drift rate of the gyro due to its quadrupole moment is given by

$$|\dot{\vec{n}}^{(1)}_{\mathrm{av}}| = \frac{3}{4}\,\frac{\Delta I}{I}\,\frac{GM}{\omega a^3(1-e^2)^{3/2}}\sin 2\alpha\ , \tag{42}$$

where α is the angle between $\vec{n}^{(1)}$ and the plane of the orbit. Note that $|\sin 2\beta| = \sin 2\alpha$, where β is the angle between $\vec{n}^{(1)}$ and $\vec{n}$. The result of Eq. (42) is a factor of 2 smaller than that given in Refs. 3 and 4 but is consistent with the mean secular torque of Ref. 15.

For an orbit 500 miles above the surface of the earth we get

$$\frac{3}{4}\,\frac{\Delta I}{I}\left(\frac{GM}{\omega a^3}\right) = 0.0254\,''/\mathrm{yr}\ . \tag{43}$$

It should be emphasized that Eq. (42) is very general in the sense that the ΔI refers to a quadrupole moment regardless of what caused it. Thus it could be used to analyze the effect of a quadrupole moment due to nonsphericity in the manufactured gyroscope. Here we have concentrated on the effect due to the *intrinsic* quadrupole moment due solely to the gyro's rotation. Of course, both these effects will have to be taken into account to preserve the expected accuracy of the experiment.

IV. DIFFERENTIAL PRECESSION OF THE ORBIT

The potential energy for the gyro in orbit about the earth is

$$V(\vec{R}) = -m\Phi(\vec{R}) + \tfrac{1}{4} D n_i^{(1)} n_j^{(1)} \frac{\partial^2 \Phi}{\partial X_i\, \partial X_j} , \qquad (44)$$

which can be written as

$$V(\vec{R}) = -GmM/R + V_Q(\vec{R}) + V_q(\vec{R}) + \cdots , \qquad (45)$$

where

$$V_Q(\vec{R}) = \frac{GJ_2 mM}{2R^3}\left(\frac{3(\vec{n}^{(2)}\cdot\vec{R})^2}{R^2} - 1\right) , \qquad (46)$$

$$V_q(\vec{R}) = \frac{Gj_2 mM}{2R^3}\left(\frac{3(\vec{n}^{(1)}\cdot\vec{R})^2}{R^2} - 1\right) , \qquad (47)$$

and we have used

$$mj_2 = \Delta I = \tfrac{1}{2} D . \qquad (48)$$

Since $V_q(\vec{R})$ has the same form as $V_Q(\vec{R})$ the term $\vec{\Omega}^{(q)}$ for the precession of the orbit due to the gyro's quadrupole moment will be of the same form as $\vec{\Omega}^{(Q)}$, which is the term for the precession of the orbit due to the earth's quadrupole moment.[6] We thus have

$$\vec{\Omega}^{(q)} = \frac{-3GmMj_2/L}{4a^3(1-e^2)^{3/2}} \times \{2(\vec{n}\cdot\vec{n}^{(1)})\vec{n}^{(1)} + [1 - 5(\vec{n}\cdot\vec{n}^{(1)})^2]\vec{n}\} , \qquad (49)$$

where L is the orbital angular momentum and

$$\left(\frac{GM}{a^3}\right)^{1/2} = \frac{L/m}{a^2(1-e^2)^{1/2}} . \qquad (50)$$

We also note that the terms[6] for the precession of the orbit due to the gyro's spin are given by

$$\vec{\Omega}^{(1)} = \frac{3GS^{(1)}M/m}{2c^2a^3(1-e^2)^{3/2}} [\vec{n}^{(1)} - 3(\vec{n}\cdot\vec{n}^{(1)})\vec{n}] , \qquad (51)$$

$$\vec{\Omega}^{(1,2)} = \frac{-3GS^{(1)}S^{(2)}/L}{2c^2a^3(1-e^2)^{3/2}} \times \{(\vec{n}\cdot\vec{n}^{(1)})\vec{n}^{(2)} + (\vec{n}\cdot\vec{n}^{(2)})\vec{n}^{(1)} + [\vec{n}^{(1)}\cdot\vec{n}^{(2)} - 5(\vec{n}\cdot\vec{n}^{(1)})(\vec{n}\cdot\vec{n}^{(2)})]\vec{n}\} , \qquad (52)$$

where $S^{(1)}$ is the spin angular momentum of the gyro and is equal to $I\omega$ and $S^{(2)}$ is the spin angular momentum of the earth.

Let us now consider an idealized experiment[16,17] of the same gyro at the center of a nonrotating spherically symmetric housing. Then the differential precession of the orbit of the gyro with respect to that of the housing would be given by the term $\delta\vec{\Omega}$, where

$$\delta\vec{\Omega} = \vec{\Omega}^{(1)} + \vec{\Omega}^{(1,2)} + \vec{\Omega}^{(q)} \qquad (53)$$

and

$$\delta\vec{R}_{\text{orbit}} = (\delta\vec{\Omega}\times\vec{R}_{\text{orbit}})\, t . \qquad (54)$$

The vector $\delta\vec{R}_{\text{orbit}}$ gives the displacement of the orbit of the gyro from the orbit of the housing at the position $\vec{R}_{\text{orbit}}$. For a circular orbit 500 miles above the surface of earth we have

$$\frac{3GS^{(1)}M}{2c^2a^2m} = 8.18\times10^{-8} \text{ cm/yr} , \qquad (55)$$

$$\frac{3GS^{(1)}S^{(2)}}{2c^2a^2L} = 1.50\times10^{-9} \text{ cm/yr} , \qquad (56)$$

$$\frac{3GmMj_2}{4a^2L} = 3.33\times10^{-10} \text{ cm/yr} . \qquad (57)$$

Thus the effect of the gyro's spin on the precession of the orbit is two orders of magnitude greater than that due to the gyro's quadrupole moment.

V. CONCLUSION

We have analyzed the effect of the quadrupole moment of the gyro on the relativity gyroscope experiment and found the effect to be important if the spin axis of the gyro is off by more than one degree from its planned orientation. Even if the gyros whose spins are to be perpendicular to the plane of the orbit are in perfect alignment initially, they will not remain that way because the orbit will precess due to the earth's quadrupole moment. Initially, let $\vec{n}^{(1)}$ be in the $\vec{n}$ direction where $\dot{\vec{n}}_{\text{av}} = \vec{\Omega}^{(Q)}\times\vec{n}$ and $\vec{\Omega}^{(Q)}$ can be obtained from Eq. (49) by changing j_2 and $\vec{n}^{(1)}$ into J_2 and $\vec{n}^{(2)}$, respectively. We then have, after use of Eq. (50),

$$|\dot{\vec{n}}_{\text{av}}| = \frac{3J_2}{4(1-e^2)^2}\left(\frac{GM}{a^7}\right)^{1/2} |\sin 2\epsilon| , \qquad (58)$$

where $\epsilon = \gamma - \frac{1}{2}\pi$ and γ is the angle between $\vec{n}$ and $\vec{n}^{(2)}$. For a circular orbit 500 miles above the surface of the earth we have

$$|\dot{\vec{n}}_{\text{av}}| = (3.287°/\text{day})|\sin 2\epsilon| . \qquad (59)$$

Thus, if the orbit is off by more than 0.024° from being a polar orbit, $\vec{n}$ will precess by more than 1° in a year. Furthermore in actual practice the gyros will be aimed at stars[10] whose locations may not be exactly in the directions perpendicular to the plane of the orbit or parallel to the plane of the orbit. However, even if the drift rate due to the quadrupole moment of the gyro is greater than 0.001″/yr the relativistic terms can still be measured to the required accuracy.[4]

We have also shown that $\Delta I/I$ is about 4.5% smaller than $\Delta r_0/r_0$ and not equal to $\Delta r_0/r_0$ as has been previously assumed.[3,4] We also have an expression for the drift rate of the gyro which is a factor of 2 smaller than that given previously.[3,4]

Finally, we have shown that the precession of the orbit of the gyro caused by the spin of the gyro is two orders of magnitude greater than that caused by the quadrupole moment of the gyro.

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