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QUADRATIC ZEEMAN EFFECT IN THE HYDROGEN BALMER LINES FROM MAGNETIC WHITE DWARFS

G. L. SURMELIAN* AND R. F. O'CONNELL
Department of Physics and Astronomy, Louisiana State University, Baton Rouge
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ABSTRACT

Motivated by Preston's suggestion of using the quadratic Zeeman effect to detect large magnetic fields in DA white dwarfs, we use a variational method for an accurate calculation of the total Zeeman wavelength shifts of the hydrogen Balmer absorption lines (Hα–H10) for magnetic fields from 10^6 to 5×10^6 gauss. Interference effects between upper values of the principal quantum number n become significant in that range. The quadratic shifts are smaller than those given by perturbation theory, leading to an increase in the estimated magnetic field.

Subject headings: magnetic stars — white dwarf stars — Zeeman effect

I. INTRODUCTION

The origin of magnetic fields on the surface of dense stars is attributed to flux conservation in the compressed matter. Thus, fields of the order of 10^6 gauss are inferred for magnetic white dwarfs and fields of 10^{12}–10^{13} gauss for pulsars.

The suggestion to use the quadratic Zeeman shift of the hydrogen Balmer absorption lines to detect large fields in DA white dwarfs was made by Preston (1970). Comparison of quadratic Zeeman wavelength shifts obtained from perturbation theory with observational data led to the possibility of magnetic fields $\lesssim 10^5$ gauss, although the conclusions are uncertain for various reasons (Greenstein and Trimble 1972). Existing conclusions could also be altered by a more accurate theoretical calculation.

In § II of this paper we summarize earlier work on the subject based on perturbation theory and an attempt to improve upon it by Lamb and Sutherland. In § III we present a variational calculation of the quadratic Zeeman shifts, and in § IV we discuss the results.

II. BACKGROUND AND PERTURBATION THEORY

The Hamiltonian for a hydrogen atom in a magnetic field B oriented along the Z-axis is

$$H = \frac{p^2}{2\mu} - \frac{e^2}{r} + \omega_l l_z + \tfrac{1}{2}\mu\,\omega_l^2 r^2 \sin^2\theta\,, \tag{1}$$

where $\omega_l = eB/2\mu c$, and l_z is the z-component of the angular momentum. The electron spin is neglected since the spin term $m_s\omega_l$ may be dropped for electric dipole transitions and the spin-orbit term is unimportant for large magnetic fields $B > 10^4$ gauss. Since the Hamiltonian is invariant under rotations about the z-axis and under inversion, only the eigenvalues of $l_z(m)$ and parity $(\pm)$ are good quantum numbers. In low field regimes, when perturbation theory holds, we can add to these l and the principal quantum number n.

Preston's expression (Preston 1970; Trimble 1971) for the quadratic Zeeman shifts in the hydrogen Balmer absorption lines was the Van Vleck (1932) formula as used by Jenkins and Segré (1939) in their work on the alkali Lyman series. The quadratic term was taken as a first-order perturbation, and the Zeeman shift obtained was

$$\Delta\lambda_Q\,(\text{Å}) = -4.98 \times 10^{-23}\,\lambda_0^2(n^4 - n'^4)(1 + m^2)B^2\,, \tag{2}$$

where λ_0 is the zero-field wavelength in angstroms, n is the principal quantum number of the upper level, $m = 0$ for π transitions, and $m = \pm 1$ for σ transitions. In Preston's original expression, terms of order $1/n^2$ and the shift of the lower level of the Balmer lines were neglected. The term in $n' = 2$ in equation (2) takes the latter into account. In addition, Preston considered only absorptions from the $2s$ $(l = 0)$ level. However, the Balmer series include absorptions from the $2p$ $(l = 1)$ levels as well, while the Lyman series do not (Hamada 1971). Furthermore, in the Balmer lines there will be interference between states of angular momenta differing by ± 2 due to the quadratic term in the Hamiltonian, and thus l will not be a good quantum number. Lamb and Sutherland (1972) included these effects in their attempt to improve upon Preston's results. However, in their calculations n was still a good quantum number.

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Now the energy levels in rydbergs, given by perturbation theory, may be written

$$E = \frac{1}{n^2}\left\{-1 + n^2 m \frac{B}{B_0} + n^6 F(n, l, m)\left(\frac{B}{B_0}\right)^2\right\}, \tag{3}$$

where n, l, m are the quantum numbers of the eigenstates,

$$F(n, l, m) = \frac{5}{4}\frac{\{1 + \frac{1}{5}n^{-2}[1 - 3l(l + 1)]\}[l(l + 1) + m^2 - 1]}{(2l + 3)(2l - 1)},$$

and

$$B_0 \equiv \frac{\mu^2 c e^3}{\hbar^3} = 2.35 \times 10^9 \text{ gauss}.$$

A rough estimate of the critical field where perturbation theory breaks down can be found by equating the quadratic term in equation (3) to the sum of the zero-field and linear terms. For $m = 0$,

$$B_c = \frac{B_0}{n^3}\frac{1}{[F(n, l, 0)]^{1/2}}. \tag{4}$$

For $m \neq 0$,

$$B_c = B_0 \frac{-n^2 m + [n^4 m^2 + 4n^6 F(n, l, m)]^{1/2}}{2n^6 F(n, l, m)}. \tag{5}$$

In table 1 are given some values of B_c determined by equations (4) and (5). These are certainly upper limits, and perturbation theory results will not be very accurate at much smaller values of B as shown by our calculations.

III. VARIATIONAL CALCULATION

For a variational calculation of the hydrogen Balmer transitions in magnetic fields of around 10^6 gauss we used the same method as in previous work on the spectrum of hydrogen in magnetic fields (Smith *et al.* 1972; Smith *et al.* 1973). With only m and parity as good quantum numbers, a general form of the trial solution to the Hamiltonian may be written (Smith *et al.* 1972)

$$\Psi_m^{\pm}(\boldsymbol{r}) \equiv \Psi_t \equiv \sum_{il} (a_i^{(t)} r^l + b_i^{(t)} r^{l+1}) \exp(-\beta_i^{(t)} r) Y_{lm}(\theta, \phi), \tag{6}$$

where $a_i^{(t)}$, $b_i^{(t)}$, and $\beta_i^{(t)}$ are parameters. The sum on l in equation (3) over all even integers leads to the state with even parity (+), and the sum over odd l's to the state with odd parity (−). The sum over l takes also into account the mixing of the states due to the quadratic term. Using equation (3), we calculated energy levels and the dipole-length matrix elements $\boldsymbol{R}_{m'm} = \langle m'|\boldsymbol{r}|m\rangle$ (Smith *et al.* 1973) corresponding to the Balmer transitions Hα–H10.

TABLE 1

SOME CRITICAL FIELDS AS GIVEN BY EQUATIONS (4) AND (5)

n	l	m	B_c(gauss)
1	0	0	3×10^9
2	1	1	3×10^8
	0	0	4×10^8
3	1	1	1×10^8
	0	0	1×10^8
4	1	1	4×10^7
	0	0	6×10^7
5	1	1	2×10^7
	0	0	3×10^7
6	1	1	1×10^7
	0	0	2×10^7
7	1	1	9×10^6
	0	0	1×10^7
8	1	1	6×10^6
	0	0	7×10^6
9	1	1	4×10^6
	0	0	5×10^6
10	1	1	3×10^6
	0	0	4×10^6

The selection rules in the dipole approximation are parity change and $q \equiv m - m' = 0, \pm 1$. The labeling of the energy levels corresponds to the usual hydrogenic energy levels in the absence of a magnetic field. Thus the lower levels are $2s$ and $2p_0$, $2p_1$, $2p_{-1}$ with $\pi(q = 0)$ and $\sigma(q = \pm 1)$ transitions.

Of course, one may choose, for example, Coulomb radial functions instead of the Slater-type orbitals used in equation (6). However, the simplicity in form of the latter permits the matrix elements to be expressed in a simple analytic form. Moreover, we note that Slater-type orbitals and Coulomb radial functions may be written simply as linear combination of each other.

In principle, completeness demands that the extent of the Hilbert space encompassed by our expansion in equation (6) should include continuum states. This can be achieved by choosing noninteger values for the β_i^{-1}. The various nonlinear parameters β_i are chosen input data whereas the linear parameters a_i are evaluated numerically in the course of diagonalization of the Hamiltonian. As before (Smith *et al.* 1972), we truncated our trial wave function when we achieved convergence. For $B = 5 \times 10^6$ gauss, we used a partial wave expansion with values of l up to $11 + |m|$, in addition to 17 Slater orbitals.

The orthogonalization of our wave functions was built into our computer code. The accuracy of the wave functions used in calculating the matrix elements was checked, as before (Smith *et al.* 1973), by demanding that the transition probabilities calculated in both the dipole-length and dipole-momentum approximations agree to the accuracy of our calculation.

The individual transition energies were weighted with the square of the dipole-length matrix elements (an alternate weighting with the oscillator strengths made no appreciable difference). The mean transition energy from all four lower levels is thus given by

$$\bar{E}_{(2\to n;q)} = \left\{ \sum_{m'=0,1-1} \sum_{l} [E(nl;q) - E(2p_{m'})] \, |\langle nl;q|\mathbf{r}|2p_{m'}\rangle|^2 + \sum_{l} [E(nl;q) - E(2s)] |\langle nl;q|\mathbf{r}|2s\rangle|^2 \right\} \times \left[\sum_{m'=0,1-1} \sum_{l} |\langle nl;q|\mathbf{r}|2p_{m'}\rangle|^2 + \sum_{l} |\langle nl;q|\mathbf{r}|2s\rangle|^2 \right]^{-1} \equiv A/B \, ; \tag{7}$$

and the mean of the π and σ transitions is

$$\bar{E}_{(2\to n)} = \sum_{q=0,1-1} A \Big/ \sum_{q=0,1-1} B \, . \tag{8}$$

The quadratic Zeeman energy displacement is written

$$\Delta E_{Q(2\to n;q)} = \bar{E}_{(2\to n;q)} - (E_0 + \Delta E_L) \tag{9}$$

and

$$\langle \Delta E_Q \rangle_{(2\to n)} = \bar{E}_{(2\to n)} - E_0 \, , \tag{10}$$

where $E_0 = (\frac{1}{4} - 1/n^2)$ is the transition energy (in rydbergs) in the absence of a magnetic field and $\Delta E_L = q\omega_L$ is the linear Zeeman energy displacement. The corresponding quadratic Zeeman wavelength shifts are obtained from

$$\Delta\lambda_{Q(2\to n;q)} = \lambda_{(2\to n;q)} - (\lambda_0 + \Delta\lambda_L) \tag{11}$$

and

$$\langle \Delta\lambda_Q \rangle_{(2\to n)} = \lambda_{(2\to n)} - \lambda_0 \, , \tag{12}$$

where

$$\lambda_0 = \lambda_{\mathrm{Ry}}/E_0 \tag{13}$$

is the zero-field wavelength;

$$\lambda_{(2\to n;q)} = \lambda_{\mathrm{Ry}}/\bar{E}_{(2\to n;q)} \tag{14}$$

and

$$\lambda_{(2\to n)} = \lambda_{\mathrm{Ry}}/\bar{E}_{(2\to n)} \tag{15}$$

are the wavelengths of the transitions in a magnetic field; and

$$\Delta\lambda_L = -\lambda_0 \frac{\Delta E_L}{E_0 + \Delta E_L} \tag{16}$$

is the linear Zeeman shift (the energies are in rydbergs, and λ_{Ry} is the rydberg wavelength).

IV. RESULTS AND DISCUSSION

Tables 2, 3, 4, and 5 give the values λ_0, λ, $\Delta\lambda_L$, $\Delta\lambda_{Q(2\to n;q)}$, and $\langle \Delta\lambda_Q \rangle_{(2\to n)}$ for the Balmer transitions Hα–H10, in fields of 10^6, 2×10^6, 3.2×10^6, and 5×10^6 gauss. Also shown are values for $\Delta\lambda_{Q(2\to n;q)}$ obtained by Lamb and Sutherland (1971) for $B = 10^6$ gauss and values for $\langle \Delta\lambda_Q \rangle_{(2\to n)}$ obtained from Preston's expression (eq. [3]) for $B = 10^6$ and $B = 3.2 \times 10^6$ gauss.

TABLE 2

QUADRATIC ZEEMAN SHIFTS (Å) FOR A MAGNETIC FIELD OF 10^6 GAUSS

					$-\Delta\lambda_{Q(2\to n;q)}$		$-\langle\Delta\lambda_Q\rangle_{(2\to n)}$		
n	λ_0	λ	q	$\Delta\lambda_L$	Present Work	Lamb & Sutherland	Present Work	Preston	ΔV_r(km s^{-1})
Hα........	6561.12	6560.98	0 +1 −1	0.0 −20.02 +20.14	0.090 0.175 0.177	0.091 0.178 0.178	0.147	0.20	− 6.7
Hβ........	4860.09	4859.72	0 +1 −1	0.0 −10.99 +11.04	0.244 0.437 0.441	0.258 0.451 0.451	0.374	0.42	− 23.1
Hγ........	4339.36	4338.60	0 +1 −1	0.0 − 8.76 + 8.80	0.474 0.919 0.897	0.573 0.983 0.983	0.763	0.86	− 52.8
Hδ........	4100.70	4099.21	0 +1 −1	0.0 − 7.83 + 7.86	0.950 1.75 1.76	1.12 1.91 1.91	1.49	1.61	−108.8
Hε........	3969.07	3966.47	0 +1 −1	0.0 − 7.33 + 7.36	1.65 3.07 3.09	2.00 3.41 3.41	2.61	2.80	−197.0
H8........	3888.07	3883.79	0 +1 −1	0.0 − 7.04 + 7.06	2.71 5.04 5.07	3.33 5.67 5.67	4.28	4.61	−330.5
H9........	3834.42	3827.73	0 +1 −1	0.0 − 6.84 + 6.87	4.29 7.84 7.88	5.24 8.93 8.93	6.69	7.20	−523.9
H10.......	3796.95	3786.83	0 +1 −1	0.0 − 6.71 + 6.74	6.55 11.74 11.73	7.89 13.4 13.4	10.11	10.7	−799.2

We notice that as the field becomes stronger there are interference effects between the upper n levels. For these cases we have given the values of $\lambda(2\to n; q)$. For example (see table 4), at $B = 3.2 \times 10^6$ gauss, $\sigma = -1$ transitions to $n = 10$ overlap $\sigma = +1$ transitions to $n = 9$. Thus the usual procedure to obtain $\lambda_{(2\to n)}$, the weighted mean of σ and π transitions, is not valid since one has to define, somewhat arbitrarily, regions of the spectrum corresponding to transitions to $n = 9$, 10, etc. These interference effects will become more important with increasing field strength and affect lower values of n, and the σ and π transitions in the spectrum have to be observed separately.

TABLE 3

QUADRATIC ZEEMAN SHIFTS (Å) FOR $B = 2 \times 10^6$ GAUSS

n	λ_0	q	$\lambda - \lambda_{(2\to n;q)}$	$-\Delta\lambda_L$	$-\Delta\lambda_{Q(2\to n;q)}$	$-\langle\Delta\lambda_Q\rangle_{(2\to n)}$	ΔV_r(km s^{-1})
Hα........	6561.12	0 +1 −1	6560.53	0.0 +39.91 −40.40	0.360 0.694 0.711	0.588	− 26
Hβ........	4860.09	0 +1 −1	4858.60	0.0 +21.93 −22.13	0.974 1.74 1.77	1.49	− 92
Hγ........	4339.36	0 +1 −1	4336.23	0.0 +17.49 −17.64	2.01 3.66 3.72	3.13	− 216
Hδ........	4100.70	0 +1 −1	4094.76	0.0 +15.63 −15.75	3.78 6.96 7.06	5.94	− 434
Hε........	3969.07	0 +1 −1	3958.68	0.0 +14.64 −14.75	6.55 12.18 12.36	10.4	− 785
H8........	3888.07	0 +1 −1	3871.17	0.0 +14.05 −14.15	10.60 19.76 20.05	16.9	−1304
H9........	3834.42	0 +1 −1	3811.42	0.0 +13.67 −13.76	16.70 26.71 27.11	23.00	−1807
H10	3796.95	0 +1 −1	3774.73 3756.37 3782.90	0.0 +13.40 −13.50	22.22 27.18 27.55	...	...

TABLE 4

QUADRATIC ZEEMAN SHIFTS (Å) FOR $B = 3.2 \times 10^6$ GAUSS

n	λ_0	q	$\lambda - \lambda_{(2\to n;q)}$	$-\Delta\lambda_L$	$-\Delta\lambda_{Q(2\to n;q)}$	$\langle\Delta\lambda_Q\rangle_{(2\to n)}$ Present Work	$\langle\Delta\lambda_Q\rangle_{(2\to n)}$ Preston	ΔV_r(km s^{-1})
Hα........	6561.12	0		0.0	0.922			
		+1	6559.62	+63.62	1.76	1.50	2.0	− 68
		−1		−64.88	1.83			
Hβ........	4860.09	0		0.0	2.49			
		+1	4856.28	+35.00	4.41	3.81	4.2	− 235
		−1		−35.51	4.54			
Hγ........	4339.36	0		0.0	5.14			
		+1	4331.36	+27.99	9.31	8.01	8.6	− 553
		−1		−28.29	9.56			
Hδ........	4100.70	0		0.0	9.59			
		+1	4085.56	+24.34	17.64	15.14	16.1	−1107
		−1		−25.25	18.07			
Hε........	3969.07	0		0.0	16.4			
		+1	3942.98	+23.37	30.41	26.10	28.0	−1971
		−1		−23.65	30.96			
H8........	3888.07	0		0.0	26.05			
		+1	3851.06	+22.43	40.87	37.01	46.1	−2904
		−1		−22.69	41.17			
H9........	3834.42	0	3801.69	0.0	32.73			
		+1	3771.06	+21.82	41.55	...	72.0	...
		0	3813.99	−22.07	42.50			
H10.......	3796.95	0	3754.62	0.0	42.32			
		+1	3729.07	+21.39	46.48	...	107	...
		−1	3771.05	−21.64	47.53			

We notice also that quadratic shifts obtained in the present work are smaller than those given by earlier calculations. The line velocities corresponding to the wavelength shifts are given by (Trimble 1971)

$$\Delta V_r = c\langle\Delta\lambda_Q\rangle/\lambda_0\,, \tag{17}$$

where ΔV_r and c are in km s^{-1} and $\langle\Delta\lambda_Q\rangle$ is given by equation (12). The mean radial velocity for $B = 10^6$ gauss, thus derived from the lines Hγ–H8 (equally weighted) is −172 km s^{-1} compared with −186 km s^{-1} found by using equation (2), while Preston's original figure was −188 km s^{-1}.

TABLE 5

QUADRATIC ZEEMAN SHIFTS (Å) FOR $B = 5 \times 10^6$ GAUSS

n	λ_0(Å)	q	$\lambda - \lambda_{(2\to n;q)}$	$-\Delta\lambda_L$	$-\Delta\lambda_{Q(2\to n;q)}$	$-\langle\Delta\lambda_Q\rangle_{(2\to n)}$	ΔV_r(km s^{-1})
Hα........	6561.12	0		0.0	2.25		
		+1	6557.46	+ 98.87	4.24	3.67	− 167
		−1		−101.95	4.51		
Hβ........	4860.09	0		0.0	6.05		
		+1	4850.82	+ 54.46	10.63	9.27	− 572
		−1		− 55.71	11.13		
Hγ........	4339.37	0		0.0	12.47		
		+1	4319.97	+ 43.47	22.35	19.40	−1341
		−1		− 44.36	23.27		
Hδ........	4100.70	0		0.0	22.96		
		+1	4064.72	+ 38.84	41.47	35.97	−2631
		−1		− 39.59	42.91		
Hε........	3969.07	0		0.0	38.45		
		+1	3919.19	+ 36.40	57.06	49.89	−3795
		−1		− 37.08	59.38		
H8........	3888.07	0	3842.05	0.0	46.02		
		+1	3794.85	+ 34.94	58.28	...	...
		−1	3863.23	−35.57	60.42		
H9........	3834.42	0	3771.43	0.0	61.99		
		+1	3731.47	+ 33.98	68.97	...	...
		−1	3797.56	− 34.60	71.45		
H10.......	3796.95	0	3711.92	0.0	85.02		
		+1	3664.16	+ 33.32	99.46	...	...
		−1	3727.87	− 33.92	102.99		

Preston (1970) had set the upper limit of the averaged surface field of magnetic white dwarfs at 5×10^5 gauss. Trimble and Greenstein (1972) reported that probable instrumental errors had been neglected in their earlier (1967) observations, used by Preston, which would lower the limits of the magnetic fields to $(1–2) \times 10^5$ gauss. In addition, Borra (1973) has pointed out that the *interpretation* of the wavelength shifts is not simple. At any rate, comparison of the smaller quadratic shifts and line velocities calculated in the present work with accurate observational data would lead to fields stronger than previous estimates.

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R. F. O'CONNELL: Louisiana State University, Department of Physics and Astronomy, Baton Rouge, LA 70803
G. L. SURMELIAN: Haigazian College, Physics Department, P.O. Box 1748, Beirut, Lebanon