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NONGEODESIC MOTION IN GENERAL RELATIVITY

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ABSTRACT

The equations of motion of a spinning body in the gravitational field of a much larger mass are found using both the Corinaldesi-Papapetrou spin supplementary condition (SSC) and the Pirani SSC. These equations of motion are compared with our previous result derived from Gupta's quantum theory of Gravitation. It is found that the spin-dependent terms differ in each of the above three results due to a different location of the center of mass of the spinning body. As expected, these terms are not affected by the choice of either Schwarzschild or isotropic coordinates. Finally, for the presently planned Stanford gyroscope experiment, we find the maximum secular displacement of the orbit of the gyro with respect to the orbit of its non-rotating housing to be of the order of (10^{-7} *cm/year*)t, *a result much smaller than Schiff's result which is proportional to time squared.*

§(1): *INTRODUCTION*

Although laboratory measurements of non-Newtonian gravitational forces [*1*] are far outside the realm of present-day capabilities, the extremely sensitive instrumentation which will be on board the satellite to measure the precession of a gyroscope [*2-10*] leads one to speculate that the differential acceleration between the gyro and its non-rotating housing might also be measured in the same experiment [*11*].

In section 2 of this paper we shall use Papapetrou's [*12*] equations of motion (first with the Corinaldesi-Papapetrou [*13*] spin supplementary condition (SSC) and then with the Pirani [*14*] SSC to find the non-geodesic equations of motion of a spinning body in the gravitational field of a much more massive body. Although the spin-dependent terms in the equations of motion *are not* affected by the use of either Schwarzschild or isotropic coordinates they *are* affected by the particular SSC used. The essential difference between the Corinaldesi-Papapetrou and Pirani SSC's is that they give rise to two different locations for the center of mass of the gyro. It is well known that the center of mass of a spinning body is not invariant under Lorentz transformations [15]. The Corinaldesi-Papapetrou SSC determines the center of mass of the gyro in the rest frame of the central attracting body while the Pirani SSC determines the center of mass of the gyro in its own rest frame [8]. In section 3 we shall show explicitly the amount by which the center of mass is shifted by changing the SSC.

In section 4 we compare a previous result [7] of ours derived from Gupta's [16] quantum theory of gravitation with the results described above using the Corinaldesi-Papapetrou and Pirani SSC's and find this previous result to correspond to a center of mass midway between the other two centers of mass.

In section 5 we find the differential acceleration of the spinning body with respect to its non-rotating housing†. We then show that the result of Schiff [11] follows from the dropping of a term that should have been retained, leading to the incorrect conclusion that the secular displacement of the spinning body with respect to its non-rotating housing should be proportional to the *time squared*.

In section 6 we find the secular precession of the orbit of the spinning body and also that of its non-rotating housing. We then conclude that (for the presently planned gyroscope experiment [4, 5]) the maximum secular displacement of the orbit of the spinning body with respect to the orbit of its non-rotating housing will be of the order of $(10^{-7}$ cm/year$)t$, a result proportional to *time to the first power*.

Finally, in section 7, we present our conclusions.

§(2): *PAPAPETROU'S EQUATIONS OF MOTION*

In this section we shall use the notation of Papapetrou [12,13] where the velocity of light is unity, Latin indices take the values of 1,2 and 3, Greek indices take the values of 1,2,3,and 4 and the line element is written as

$$ds^2 = g_{44}dt^2 + g_{ij}dx^i dx^j. \tag{1}$$

The covariant form of Papapetrou's [12] equations of motion and equations of motion of spin can be written respectively as

$$\frac{D}{Ds}\left(mu^{\alpha} + u_{\beta}\frac{DS^{\alpha\beta}}{Ds}\right) + \tfrac{1}{2}S^{\mu\nu}u^{\sigma}R^{\alpha}{}_{\nu\sigma\mu} = 0, \tag{2}$$

and

$$\frac{DS^{\alpha\beta}}{Ds} + u^{\alpha}u_{\rho}\frac{DS^{\beta\rho}}{Ds} - u^{\beta}u_{\rho}\frac{DS^{\alpha\rho}}{Ds} = 0, \tag{3}$$

† Throughout this paper we shall consider the non-rotating housing to be spherically symmetric.

where m is the mass of the spinning body,

$$S^{\alpha\beta} = - S^{\beta\alpha}, \tag{4}$$

$$u^{\alpha} = \frac{dx^{\alpha}}{ds}, \tag{5}$$

$$R^{\alpha}{}_{\nu\sigma\mu} = \Gamma^{\alpha}{}_{\nu\sigma,\mu} - \Gamma^{\alpha}{}_{\mu\sigma,\nu} + \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma} - \Gamma^{\alpha}{}_{\nu\rho}\Gamma^{\rho}{}_{\mu\sigma}, \tag{6}$$

and for any tensor $T^{\alpha\cdots}{}_{\beta}$ we have

$$\frac{DT^{\alpha}{}_{\cdot\cdot\beta}}{Ds} \equiv \frac{dT^{\alpha}{}_{\cdot\cdot\beta}}{ds} + \Gamma^{\alpha}{}_{\mu\nu}u^{\nu}T^{\mu}{}_{\cdot\cdot\beta} + \ldots - \Gamma^{\mu}{}_{\beta\nu}u^{\nu}T^{\alpha}{}_{\cdot\cdot\mu}. \tag{7}$$

Applying equation (7) to $S^{\alpha\beta}$ gives

$$\frac{DS^{\alpha\beta}}{Ds} \equiv \frac{dS^{\alpha\beta}}{ds} + \Gamma^{\alpha}{}_{\mu\nu}u^{\nu}S^{\mu\beta} + \Gamma^{\beta}{}_{\mu\nu}u^{\nu}S^{\alpha\mu}, \tag{8}$$

while using equations (6,7) in equation (2) gives

$$\frac{D}{Ds}(mu^{\alpha}) + \frac{d}{ds}\left(u_{\beta}\frac{DS^{\alpha\beta}}{Ds}\right) + \Gamma^{\alpha}{}_{\mu\nu}u^{\nu}u_{\beta}\frac{DS^{\mu\beta}}{Ds}$$

$$+ S^{\mu\nu}u^{\sigma}(\Gamma^{\alpha}{}_{\nu\sigma,\mu} + \Gamma^{\alpha}{}_{\mu\rho}\Gamma^{\rho}{}_{\nu\sigma}) = 0. \tag{9}$$

We shall consider the spinning body to be in orbit about a much more massive body. Using x^1, x^2, x^3 and $x^4 = t$ for the Cartesian Schwarzschild coordinates we have to first order in r_0, the gravitational radius of the central body,

$$g_{44} = 1 - \frac{2r_0}{r}, \tag{10}$$

$$g_{ij} = -\delta_{ij} - \frac{2r_0}{r}\frac{x^i x^j}{r^2}, \tag{11}$$

$$\Gamma^4{}_{ij} = \Gamma^i{}_{4j} = 0, \qquad \Gamma^4{}_{44} = 0, \tag{12}$$

$$\Gamma^4{}_{k4} = \Gamma^k{}_{44} = \frac{r_0}{r^3}x^k, \tag{13}$$

$$\Gamma^i{}_{j\ell} = \frac{2r_0}{r^3}\, x^i \left[\delta_{j\ell} - \frac{3x^j x^\ell}{2r^2}\right], \tag{14}$$

$$\Gamma^4{}_{k4,\ell} = \frac{r_0}{r^3}\left[\delta_{k\ell} - 3\,\frac{x^k x^\ell}{r^2}\right], \tag{15}$$

and

$$\Gamma^i{}_{j\ell,k} = \frac{2r_0}{r^3}\left[\delta_{ik} - 3\,\frac{x^i x^k}{r^2}\right]\delta_{j\ell}$$

$$- \frac{3r_0}{r^5}\left[\delta_{ik}x^j x^\ell + \delta_{jk}x^i x^\ell + \delta_{k\ell}x^i x^j - 5\,\frac{x^i x^j x^\ell x^k}{r^2}\right]. \tag{16}$$

If we use x^1, x^2, x^3 and $x^4 = t$ for the Cartesian isotropic coordinates we have to first order in r_0

$$g_{44} = 1 - \frac{2r_0}{r}, \tag{17}$$

$$g_{ij} = -\,\delta_{ij} - \frac{2r_0}{r}\,\delta_{ij}, \tag{18}$$

$$\Gamma^4{}_{ij} = \Gamma^i{}_{4j} = 0, \qquad \Gamma^4{}_{44} = 0, \tag{19}$$

$$\Gamma^4{}_{k4} = \Gamma^k{}_{44} = \frac{r_0}{r^3}\, x^k, \tag{20}$$

$$\Gamma^i{}_{j\ell} = \frac{r_0}{r^3}\,(\delta_{j\ell}x^i - \delta_{\ell i}x^j - \delta_{ij}x^\ell), \tag{21}$$

$$\Gamma^4{}_{k4,\ell} = \frac{r_0}{r^3}\left[\delta_{k\ell} - 3\,\frac{x^k x^\ell}{r^2}\right], \tag{22}$$

and

$$\Gamma^i{}_{j\ell,k} = \frac{r_0}{r^3}\,(\delta_{ik}\delta_{j\ell} - \delta_{jk}\delta_{\ell i} - \delta_{\ell k}\delta_{ji}) \qquad \text{(Contd)}$$

(Contd) $$- \frac{3r_0}{r^5} (\delta_{j\ell} x^i x^k - \delta_{\ell i} x^j x^k - \delta_{ij} x^\ell x^k). \tag{23}$$

A *The Corinaldesi-Papapetrou Spin Supplementary Condition*

The SSC used by Corinaldesi and Papapetrou is

$$S^{i4} = 0, \tag{24}$$

which holds in the rest frame of the heavy mass. Using equation (24) in equation (8) gives

$$\frac{DS^{ij}}{Ds} = \frac{dS^{ij}}{ds} + \Gamma^i{}_{k\nu} u^\nu S^{kj} + \Gamma^j{}_{k\nu} u^\nu S^{ik}, \tag{25}$$

$$\frac{DS^{i4}}{Ds} = \Gamma^4{}_{k\nu} u^\nu S^{ik}, \tag{26}$$

$$\frac{DS^{4j}}{Ds} = \Gamma^4{}_{k\nu} u^\nu S^{kj}, \tag{27}$$

and

$$\frac{DS^{44}}{Ds} = 0. \tag{28}$$

Furthermore, equation (3), after using equation (8) and (24), can be put in the form†

$$\frac{DS^{ij}}{Ds} = \Gamma^4{}_{k\nu} \frac{u^\nu}{u^4} (u^j S^{ik} - u^i S^{jk}). \tag{29}$$

Using Cartesian (Schwarzschild or isotropic) coordinates and dropping terms of order higher than r_0 in the spin-dependent terms, the ith component of equation (9), after using equations (24,26-29), can be written as

$$\frac{D}{Ds}(mu^i) + \frac{d}{ds}\left[\Gamma^4{}_{k4} \frac{u^4}{u_4} (u_j u^j S^{ik} - u_j u^i S^{jk}) + \Gamma^4{}_{k4} u^4 u_4 S^{ik}\right]$$

$$+ S^{kj} u^\ell \Gamma^i{}_{j\ell,k} = 0. \tag{30}$$

From the form of the line element, equation (1), we obtain

† See equation (2.6) of reference [18].

$$u_\alpha u^\alpha = +1. \qquad (31)$$

For the spin-dependent terms of equation (30) we can make the approximations $u_4 \approx u^4 \approx 1$, $u_i = -u^i$ and $v^i \approx u^i$ where

$$v^i = \frac{dx^i}{dt}. \qquad (32)$$

Keeping only the first order terms in v^i in the spin-dependent terms of equation (30) we obtain

$$\frac{Du^i}{Ds} = -\frac{v^\ell \Gamma^4{}_{k4,\ell} S^{ik}}{m} - \frac{v^\ell \Gamma^i{}_{j\ell,k} S^{kj}}{m}. \qquad (33)$$

Let us now put

$$S^{ij} = \epsilon_{ijk} S^k \qquad (34)$$

in equation (33) along with *either* the Schwarzschild *or* isotropic results for $\Gamma^4{}_{k4,\ell}$ and $\Gamma^i{}_{j\ell,k}$. We then obtain directly in *both* cases

$$\frac{D\vec{u}}{Ds} = \frac{3r_0}{mr^5}\{[\vec{S}\cdot(\vec{r}\times\vec{v})]\vec{r} - r^2(\vec{v}\times\vec{S}) + 2(\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S})\}. \qquad (35)$$

By using the vector identity

$$[\vec{S}\cdot(\vec{r}\times\vec{v})]\vec{r} = (\vec{S}\cdot\vec{r})(\vec{r}\times\vec{v}) - (\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S}) + r^2(\vec{v}\times\vec{S}) \qquad (36)$$

in equation (35), we finally obtain

$$\frac{D\vec{u}}{Ds} = \frac{3r_0}{mr^5}[(\vec{S}\cdot\vec{r})(\vec{r}\times\vec{v}) + (\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S})], \qquad (37)$$

which is the result of Schiff [11,17].

B *The Pirani Spin Supplementary Condition*

The SSC used by Pirani is

$$S^{\alpha\beta} u_\beta = 0. \qquad (38)$$

Differentiating the Pirani SSC we obtain

$$\frac{D}{Ds}(S^{\alpha\beta}u_\beta) = u_\beta \frac{DS^{\alpha\beta}}{Ds} + S^{\alpha\beta}\frac{Du_\beta}{Ds} = 0. \qquad (39)$$

As Du_β/Ds is of order r_0S, $u_\beta DS^{\alpha\beta}/Ds$ is of order r_0SS. Thus equation (2) reduces to

$$\frac{D}{Ds}(mu^\alpha) + S^{\mu\nu}u^\sigma\Gamma^\alpha{}_{\nu\sigma,\mu} = 0 \tag{40}$$

after dropping terms of order r_0SS and $r_0{}^2S$. Taking the ith component of equation (40) and noting that $\Gamma^i{}_{4\ell} = \Gamma^i{}_{\ell 4} = 0$ for both Cartesian Schwarzschild and Cartesian istropic coordinates, we get

$$\frac{D}{Ds}(mu^i) = -u^4\Gamma^i{}_{44,j}S^{j4} - u^\ell\Gamma^i{}_{j\ell,k}S^{kj}. \tag{41}$$

Using the Pirani SSC

$$S^{j4}u_4 + S^{j\ell}u_\ell = 0, \tag{42}$$

along with the approximations $u_4 \approx u^4 \approx 1$, $u_\ell \approx -u^\ell$ and $v^\ell \approx u^\ell$ for the spin dependent terms in equation (41) we obtain

$$\frac{Du^i}{Ds} = -\frac{v^\ell\Gamma^i{}_{44,k}S^{k\ell}}{m} - \frac{v^\ell\Gamma^i{}_{j\ell,k}S^{kj}}{m}. \tag{43}$$

Finally, using equation (34) in equation (43) along with *either* the Schwarzschild *or* isotropic results for $\Gamma^i{}_{44,k}$ and $\Gamma^i{}_{j\ell,k}$ we obtain directly in *both* cases

$$\frac{D\vec{u}}{Ds} = \frac{3r_0}{mr^5}\{2[\vec{S}\cdot(\vec{r}\times\vec{v})]\vec{r} - r^2(\vec{v}\times\vec{S}) + (\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S})\}. \tag{44}$$

Equation (44) is *not* the same as equation (35). Thus the Corinaldesi-Papapetrou and the Pirani SSC's lead to *different* nongeodesic equations of motion.

Finally, we remark that the SSC's used by Tulczyjew [18] and Dixon [19]

$$S^{\alpha\beta}p_\beta = 0, \tag{45}$$

are equivalent to the Pirani SSC when terms quadratic in the spin are neglected. This is the situation for the problem at hand, since spin squared terms are negligible in magnitude for physically acceptable solutions [15,19].

§(3): *SHIFT IN THE CENTER OF MASS*

In this section and for the remainder of the paper we shall use a notation in which the velocity of light is denoted by c. We thus have

$$r_0 = \frac{GM}{c^2}, \tag{46}$$

where M is the mass of the central body. For isotropic or harmonic coordinates we have to order $r_0{}^2$ and $r_0 v^2/c^2$

$$\frac{D\vec{u}}{Ds} = \dot{\vec{v}} + \frac{c^2 r_0 \vec{r}}{r^3} - \vec{a}_E, \tag{47}$$

where a dot denotes differentiation with respect to t and

$$\vec{a}_E = \frac{r_0}{r^3}\left[\frac{4c^2 r_0 \vec{r}}{r} - v^2\vec{r} + 4(\vec{v}\cdot\vec{r})\vec{v}\right], \tag{48}$$

which is the familiar Einstein result [7].

The nongeodesic equations of motion for a spinning body can thus be written as

$$\dot{\vec{v}}_{(CP)} + \frac{c^2 r_0 \vec{r}_{(CP)}}{r_{(CP)}{}^3} = \vec{a}_E + \vec{a}_S{}^{(CP)} \tag{49}$$

and

$$\dot{\vec{v}}_{(P)} + \frac{c^2 r_0 \vec{r}_{(P)}}{r_{(P)}{}^3} = \vec{a}_E + \vec{a}_S{}^{(P)}, \tag{50}$$

when the Corinaldesi-Papapetrou and Pirani SSC are applied respectively, and

$$\vec{a}_S{}^{(CP)} = \frac{3r_0}{mr^5}\{[\vec{S}\cdot(\vec{r}\times\vec{v})]\vec{r} - r^2(\vec{v}\times\vec{S}) + 2(\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S})\} \tag{51}$$

and

$$\vec{a}_S{}^{(P)} = \frac{3r_0}{mr^5}\{2[\vec{S}\cdot(\vec{r}\times\vec{v})]\vec{r} - r^2(\vec{v}\times\vec{S}) + (\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S})\}. \tag{52}$$

The vectors $\vec{r}_{(CP)}$ and $\vec{r}_{(P)}$ used on the left hand side of equations (49,50) are not the same vector. In equation (49) $\vec{r}_{(CP)}$ is the vector going to the center of mass of the spinning body in the rest frame of the heavy body, while in equation (50) $\vec{r}_{(P)}$ is the vector going to the center of mass of the spinning body in its own rest frame. On the right hand side of equations (49,50) the terms are of higher order and it is not necessary to distinguish between $\vec{r}_{(CP)}$ and $\vec{r}_{(P)}$, so here we shall just use $\vec{r}$.

It is easy to verify that equations (49,50) are consistent with each other if we write

$$\vec{r}_{(CP)} = \vec{r}_{(P)} + \frac{\vec{v} \times \vec{S}}{mc^2} . \tag{53}$$

This is exactly the same as the shift in the center of mass of a spinning body under a Lorentz transformation [2,15].

§(4): *EQUATIONS OF MOTION FROM QUANTUM THEORY OF GRAVITATION*

In a recent paper [7] we derived the equations of motion of a spinning body about a much more massive body. These results were based on two-body potentials for spin $\frac{1}{2}$ particles derived from Gupta's quantum theory of gravitation.

These results [7] can be put in the form

$$\dot{\vec{v}} + \frac{c^2 r_0 \vec{r}}{r^3} = \vec{a}_{\text{geodesic}} + \vec{a}_{\text{nongeodesic}}, \tag{54}$$

where

$$\vec{a}_{\text{geodesic}} = \vec{a}_Q + \vec{a}_E + \vec{a}_{S2}, \tag{55}$$

and

$$\vec{a}_{\text{nongeodesic}} = \vec{a}_S + \vec{a}_{S,S2}. \tag{56}$$

The geodesic terms are $\vec{a}_Q$ (the term due to the quadrupole moment of the heavy mass, a standard result), $\vec{a}_E$ (the Einstein term) and $\vec{a}_{S2}$ (the term due to the spin angular momentum of the heavy mass first given by Lense and Thirring [26]). The nongeodesic terms are $\vec{a}_S$ (the term due to the spin angular momentum of the small mass) and $\vec{a}_{S,S2}$ (the term due to the spin-spin interaction of both masses) [7]. The above five terms can be expressed as [7]

$$\vec{a}_Q = -\frac{3c^2 r_0 J_2}{2r^5} \left\{ \left[1 - \frac{5(\hat{S}^{(2)} \cdot \vec{r})^2}{r^2} \right] \vec{r} + 2(\hat{S}^{(2)} \cdot \vec{r}) \hat{S}^{(2)} \right\}, \tag{57}$$

$$\vec{a}_E = \frac{r_0}{r^3} \left[\frac{4c^2 r_0 \vec{r}}{r} - v^2 \vec{r} + 4(\vec{v} \cdot \vec{r}) \vec{v} \right], \tag{58}$$

$$\vec{a}_{S2} = \frac{4r_0}{Mr^5} \left\{ \frac{3}{2} \left[\vec{S}^{(2)} \cdot (\vec{r} \times \vec{v}) \right] \vec{r} - r^2 (\vec{v} \times \vec{S}^{(2)}) + \frac{3}{2} (\vec{v} \cdot \vec{r})(\vec{r} \times \vec{S}^{(2)}) \right\}, \tag{59}$$

$$\vec{a}_S = \frac{3r_G}{mr^5}\left\{\frac{3}{2}\,[\vec{S}\cdot(\vec{r}\times\vec{v})]\vec{r} - r^2(\vec{v}\times\vec{S}) + \frac{3}{2}\,(\vec{v}\cdot\vec{r})(\vec{r}\times\vec{S})\right\}, \quad (60)$$

and

$$\vec{a}_{S,S2} = -\frac{3r_G}{mMr^5}\left[(\vec{S}^{(2)}\cdot\vec{r})\vec{S} + (\vec{S}\cdot\vec{r})\vec{S}^{(2)} - \frac{5(\vec{S}\cdot\vec{r})(\vec{S}^{(2)}\cdot\vec{r})\vec{r}}{r^2} + (\vec{S}\cdot\vec{S}^{(2)})\vec{r}\right], \quad (61)$$

where $\vec{S}^{(2)}$ is the spin angular momentum of the heavy mass.

Let us note that

$$\vec{a}_S = \tfrac{1}{2}(\vec{a}_S{}^{(CP)} + \vec{a}_S{}^{(P)}). \quad (62)$$

This implies that the vector $\vec{r}$ that is used in the result derived from the quantum theory of gravitation, equation (54), lies midway between $\vec{r}_{(CP)}$ and $\vec{r}_{(P)}$, i.e.

$$\vec{r} = \tfrac{1}{2}(\vec{r}_{(CP)} + \vec{r}_{(P)}). \quad (63)$$

The quantum field theory of spin $\frac{1}{2}$ particles from which the classical result, equation (60) was derived does not have any spin supplementary conditions. This is because field theories deal with point particles and not with extended bodies. Thus in going from the quantum mechanical result to the classical result it was not clear which center of mass the vector $\vec{r}$ would go to until we compared our solution to the previous two solutions with the Corinaldesi-Papapetrou and the Pirani spin supplementary conditions.

It is not difficult to demonstrate that the result for $\vec{a}_{S,S2}$, given by equation (61), is the same for all SSC's. An application of this spin-spin interaction to black holes has already been given [21] and in this context we should note that the results are of course also applicable [22] to bodies with angular momentum larger than that allowed by use of the Kerr metric.

§(5): *DIFFERENTIAL ACCELERATION*

Let the vector $\vec{r}$ go to the center of mass of the spinning body (as in section 4) and let the vector $\vec{r}_{(h)}$ go to the center of mass of the non-rotating housing. We shall assume initially that these two centers of mass are coincident and moving with the same velocity. The equations of motion for the non-rotating housing can be written as

$$\dot{\vec{v}}_{(h)} + \frac{c^2 r_G \vec{r}_{(h)}}{r_{(h)}{}^3} = \vec{a}_{\text{geodesic}}, \quad (64)$$

while the equations of motion for the spinning body are given by equation (54).

Let us put

$$\vec{r} = \vec{r}_{(h)} + \delta\vec{r}, \tag{65}$$

where $\delta\vec{r}$ is the displacement of the center of mass of the spinning body from the center of mass of the non-rotating housing.

Using equation (65) in equation (64) we obtain

$$\dot{\vec{v}} + \frac{c^2 r_0 \vec{r}}{r^3} = \vec{a}_{\text{geodesic}} + \delta\ddot{\vec{r}} + \frac{c^2 r_0}{r^3}\left[\delta\vec{r} - 3\,\frac{\vec{r}\cdot\delta\vec{r}}{r^2}\right]\vec{r}. \tag{66}$$

Next, we equate the right hand sides of equations (54,66) to obtain

$$\delta\ddot{\vec{r}} = \vec{a}_{\text{nongeodesic}} - \frac{c^2 r_0}{r^3}\left[\delta\vec{r} - 3\,\frac{\vec{r}\cdot\delta\vec{r}}{r^2}\,\vec{r}\right], \tag{67}$$

where $\delta\ddot{\vec{r}}$ is the differential acceleration of the spinning body with respect to its non-rotating housing. Initially the differential acceleration, $\delta\ddot{\vec{r}}$, is equal only to $\vec{a}_{\text{nongeodesic}}$ as $\delta\vec{r}$ will be zero and the second term on the right of equation (67) will not contribute. We shall now show that the second term on the right of equation (67) will be of the same order of magnitude as $\vec{a}_{\text{nongeodesic}}$ after only *one* orbit.

Let us suppose that for a time, t, only $\vec{a}_{\text{nongeodesic}}$ contributes to the differential acceleration, $\delta\ddot{\vec{r}}$. Since $\vec{a}_{\text{nongeodesic}}$ is periodic with the orbital period, T, we can write

$$\delta\vec{r} = \tfrac{1}{2}t^2\vec{a}_{\text{nongeodesic secular}} + (\delta\vec{r})_{\text{nonsecular}}. \tag{68}$$

The term $\vec{a}_{\text{nongeodsic secular}}$ is the time average of $\vec{a}_{\text{nongeodesic}}$ over the orbital period, T, and $(\delta\vec{r})_{\text{nonsecular}}$ is a periodic term of period T. Thus, *if* equation (68) remained true for time $t \gg T$, then after a large number, N, of orbits with $N \gg 1$ we would have Schiff's result [*11*], viz,

$$\delta\vec{r} \approx \tfrac{1}{2}(NT)^2\vec{a}_{\text{nongeodesic secular}}, \tag{69}$$

since $(\delta\vec{r})_{\text{nonsecular}}$ could then be neglected. Let us now find $\delta\ddot{\vec{r}}$ after one orbit assuming that equation (68) remains valid. We then obtain

$$\delta\ddot{\vec{r}}_{t=T} = \vec{a}_{\text{nongeodesic}} - 2\pi^2\Big[\vec{a}_{\text{nongeodesic secular}} + \qquad \text{(Contd)}$$

$$\text{(Contd)} \qquad - 3\,\frac{\vec{r}\cdot\vec{a}_{\text{nongeodesic secular}}}{r^2}\,\vec{r}\Big]$$

$$-\frac{c^2 r_G}{r^3}\left[(\delta\vec{r})_{\text{nonsecular}} - 3\,\frac{\vec{r}\cdot(\delta\vec{r})_{\text{nonsecular}}}{r^2}\,\vec{r}\right]_{t=T}, \qquad (70)$$

where we have used a circular orbit, where

$$T^2 = \frac{4\pi^2 r^3}{c^2 r_G}\,. \qquad (71)$$

Clearly, the second term of equation (70) is of the same order of magnitude as the first term. Hence, equation (68) can only hold for $t \ll T$ and equation (69) is thus wrong. The incorrect result of equation (69) has been obtained by neglect of the second term on the right hand side of equation (67).

Instead of trying to solve equation (67) for the displacement $\delta\vec{r}$ we shall proceed in a different manner in the next section.

§(6): *SECULAR PRECESSION OF THE ORBIT*

Initially we would like the center of mass of the spinning body and the center of mass of the nonrotating housing to be coincident and moving with the same velocity. However, in the actual experiment [2-5] (where it is proposed to put a fused quartz gyroscope of 4 cm diameter, spinning at 200 revolutions per second, in a sattellite moving in a 500 mile polar orbit) there will be some error in the initial positioning of the gyro which will cause the gyro to go into a slightly different orbit than the housing. Thus, whether the gyro was spinning or not, there would be a small difference in the orbital period of the gyro and that of its housing. Hence, we should expect to see the gyro and its non-rotating housing move apart even if only Newtonian forces were present.

Let us also note that the smallest possible positioning error of the gyro is very much larger than the distance between the two centers of mass of the gyro, $|\vec{r}(CP) - \vec{r}(P)|$, since $|\vec{v}\times\vec{S}|/mc^2$ has a maximum value of only 1.7×10^{-12} cm for the above proposed experiment.

We need a theoretical result that is independent of the initial positioning error and also independent of which center of mass is used for the spinning body. The secular precession of the orbit [7] satisfies the above two criterion. The orbit of the non-rotating housing will precess with an angular velocity of $\vec{\Omega}_{\text{housing}}$ while the orbit of the spinning body will precess with an angular velocity of $\vec{\Omega}_{\text{gyro}}$ where

$$\vec{\Omega}_{\text{housing}} = \vec{\Omega}_Q + \vec{\Omega}_E + \vec{\Omega}_{S2} \qquad (72)$$

and

$$\vec{\Omega}_{gyro} = \vec{\Omega}_{housing} + \vec{\Omega}_S + \vec{\Omega}_{S,S2}, \tag{73}$$

with [7]

$$\vec{\Omega}_Q = -\frac{3c^2 r_O J_2 \frac{m}{L}}{4a^3(1 - e^2)^{3/2}} \{2(\hat{L}\cdot\hat{S}^{(2)})\hat{S}^{(2)} + [1 - 5(\hat{L}\cdot\hat{S}^{(2)})^2]\hat{L}\}, \tag{74}$$

$$\vec{\Omega}_E = \frac{3r_O \frac{L}{m}}{a^3(1 - e^2)^{3/2}} \hat{L}, \tag{75}$$

$$\vec{\Omega}_{S2} = \frac{2r_O S^{(2)} \frac{1}{M}}{a^3(1 - e^2)^{3/2}} [\hat{S}^{(2)} - 3(\hat{L}\cdot\hat{S}^{(2)})\hat{L}], \tag{76}$$

$$\vec{\Omega}_S = \frac{3r_O S \frac{1}{m}}{2a^3(1 - e^2)^{3/2}} [\hat{S} - 3(\hat{L}\cdot\hat{S})\hat{L}], \tag{77}$$

and

$$\vec{\Omega}_{S,S2} = -\frac{3r_O S S^{(2)} \frac{1}{LM}}{2a^3(1 - e^2)^{3/2}} \{(\hat{L}\cdot\hat{S})\hat{S}^{(2)} + (\hat{L}\cdot\hat{S}^{(2)})\hat{S}$$

$$+ [\hat{S}\cdot\hat{S}^{(2)} - 5(\hat{L}\cdot\hat{S})(\hat{L}\cdot\hat{S}^{(2)})]\hat{L}\}, \tag{78}$$

where $\vec{L}$ is the orbital angular momentum, a is the semi-major axis and e is the eccentricity of the orbit. The quantities $\vec{\Omega}_Q$, $\vec{\Omega}_E$, $\vec{\Omega}_{S2}$, $\vec{\Omega}_S$ and $\vec{\Omega}_{S,S2}$ are the results due to $\vec{a}_Q$, $\vec{a}_E$, $\vec{a}_{S2}$, $\vec{a}_S$ and $\vec{a}_{S,S2}$ of equations (57-61) respectively. If we denote $\vec{\Omega}_S{}^{(CP)}$ and $\vec{\Omega}_S{}^{(P)}$ as the result due to $\vec{a}_S{}^{(CP)}$ and $\vec{a}_S{}^{(P)}$ then we obtain

$$\vec{\Omega}_S{}^{(CP)} = \vec{\Omega}_S{}^{(P)} = \vec{\Omega}_S, \tag{79}$$

i.e. the precession of the orbit is independent of which center of mass is used.

Let us denote the difference of the orbital precession angular velocity of the gyro and that of the housing by $\delta\vec{\Omega}$, that is

$$\delta\vec{\Omega} = \vec{\Omega}_{gyro} - \vec{\Omega}_{housing} = \vec{\Omega}_S + \vec{\Omega}_{S,S2} \tag{80}$$

where it should be noted that $\Omega_S \gg \Omega_{S,S2}$. The orbit of the spinning body will thus be displaced a distance $\delta\vec{r}_{orbit}$ from the orbit of the housing where

$$\delta\vec{r}_{orbit} = (\delta\vec{\Omega} \times \vec{r}_{orbit})t. \tag{81}$$

Note that $\vec{r}_{orbit}$ is a vector going to an arbitrary position on the orbit not necessarily the position of the satellite.

Let us consider an elliptical orbit with a perihelion distance r_{min}, and an aphelion distance r_{max}, where

$$r_{min} = a(1 - e), \tag{82}$$

and

$$r_{max} = a(1 + e) = r_{min}\frac{(1 + e)}{(1 - e)}. \tag{83}$$

We can then rewrite equation (77) in the form

$$\vec{\Omega}_S = \frac{3r_0\frac{S}{m}}{2r_{min}{}^3}\left[\frac{1 - e}{1 + e}\right]^{3/2}[\hat{S} - 3(\hat{L}\cdot\hat{S})\hat{L}]. \tag{84}$$

If $\vec{S}$ is perpendicular to both $\vec{L}$ and the semi-major axis, we obtain

$$\max(\delta\vec{r}_{orbit}) = \frac{3r_0\frac{S}{m}}{2r_{min}{}^2}\left[\frac{1 - e}{1 + e}\right]^{\frac{1}{2}}t, \tag{85}$$

while if $\vec{S}$ is parallel to $\vec{L}$ we get

$$\max(\delta\vec{r}_{orbit}) = \frac{3r_0\frac{S}{m}}{r_{min}{}^2}\left[\frac{1 - e}{1 + e}\right]^{\frac{1}{2}}t. \tag{86}$$

Thus in both the above cases the maximum result for fixed r_{min} will be for a circular orbit where e equals zero. However, equation (86) *can not* be applied to an exact circular orbit since the orbit is displaced into itself.

Let us now consider a circular 500 mile orbit. We can always find a position on the orbit such that $\vec{r}_{orbit}$ is perpendicular to $\delta\vec{\Omega}$ so that the maximum $\delta\vec{r}_{orbit}$ will equal $(r_{orbit}\delta\Omega)t$. For the actual experiment mentioned above, $3r_0S/2ma^2 = 8.2 \times 10^{-8}$ cm/year. Hence, if we consider the special case of $\vec{S}$ perpendicular to $\vec{L}$, the maximum $\delta\vec{r}_{orbit}$ will equal $(8.2 \times 10^{-8}$ cm/year$)t$. This equation should remain valid for t such that $t\Omega_{housing} \ll 2\pi$, i.e. as long as the orbit of the satellite does not change appreciably.

§(7): *CONCLUSION*

We have examined the equations of motion of a spinning body in the gravitational field of a much larger mass. We have shown that the three versions of these equations encountered are all physically the same, their apparent differences being due only to different locations of the center of mass.

We have derived a result that is independent of the initial positioning error of the gyro with respect to its non-rotating housing and also independent of which center of mass is used for the gyro. This result shows that the maximum secular displacement of the orbit of the gyro with respect to the orbit of its non-rotating housing is of the order of $(10^{-7}$ cm/year$)t$ for the presently planned gyroscope experiment. We also obtain a maximum displacment for circular orbits for a given perihelion distance.

The magnitude of our result is considerably smaller than the result of $(4 \times 10^{-4}$ cm/year$^2)t^2$ for the displacement of the gyro with respect to its housing obtained by Schiff [*11*]. Schiff obtains his maximum result using an elliptical orbit with an eccentricity of 0.275. He gets no secular displacement for circular orbits.

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