SHAPE OF THE LOW-ENERGY SPECTRUM FOR COULOMBIC INTERACTIONS

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There have been many calculations of transition probabilities where the potential involved is Coulombic. Such calculations are, in general, quite involved. In all cases the problem is greatly simplified by confining one's attention to threshold energies (incoming or outgoing particle having zero momentum), though this has the obvious disadvantage of giving only one point of the energy spectrum.

The purpose of this note is to show that, for reactions where the particles are described by Coulomb wave functions and the interaction Hamiltonian is independent of momentum, the shape of the momentum spectrum near threshold is parabolic or, equivalently, the shape of the energy spectrum is linear. This observation provides us with a firm basis for interpolating between threshold and points nearby.

The radial equations for a Dirac particle in a central field potential V are given by[1,2]

$$\frac{df}{dr}=\frac{\kappa-1}{r}f-(w-1-V)g, \tag{1a}$$

and

$$\frac{dg}{dr}=(w-V+1)f-\frac{\kappa+1}{r}g, \tag{1b}$$

where $w=(p^2+1)^{1/2}$. Thus, in particular, for Coulombic potentials (which may include screening and nuclear size effects) the solutions of the differential equations for f and g must be invariant under the replacement $p\rightarrow -p$, apart from a possible normalization factor which one may arbitrarily introduce.

If the result of a calculation involving Coulomb wave functions is a physically measurable quantity, it must be independent of the particular normalization chosen for the wave functions. Wave functions having different normalizations must be compensated for in the course of the calculation in such a way that they lead to the same final result. Similarly, any general conclusions regarding the final result, which are based on the properties of the wave functions, must also be independent of the particular normalization chosen.

The above considerations lead us to consider only normalization factors which are invariant under $p\rightarrow -p$ (this corresponds to the usual choice found in the literature), so that, in general, f and g are invariant under $p\rightarrow -p$. Since it is seldom obvious by inspection that f and g are invariant under $p\rightarrow -p$, we will consider a specific example, viz., the exact continuum wave functions for a Dirac particle in a Coulomb field. The solutions are[3]

$$f=i(w-1)^{1/2}N(2r)^{\gamma-1}I_{-}, \tag{2a}$$

$$g=(w+1)^{1/2}N(2r)^{\gamma-1}I_{+}, \tag{2b}$$

where

$$N=\frac{e^{\pi\nu/2}}{(\pi p)^{1/2}}\,\frac{|\Gamma(\gamma+i\nu)|}{\Gamma(2\gamma+1)}\,p^{\gamma},$$

$$\nu=\alpha ZW/p,\quad \gamma=(\kappa^2-\alpha^2Z^2)^{1/2},$$

$$e^{2i\eta}=\frac{-\kappa+i\alpha Z/p}{\gamma+i\nu},$$

and

$$I_{\pm}=e^{-ipr+i\eta}(\gamma+i\nu)_1F_1(\gamma+1+i\nu;2\gamma+1;2ipr)\pm \text{c.c.}$$

Since p always occurs as a coefficient of the imaginary number i, we see immediately that $(w\pm 1)^{1/2}I_{\pm}$, when expressed as a power series in p, is invariant under $p\rightarrow -p$. Using the relation[4]

$$\Gamma(Z)=\exp[-Z]\exp[(Z-\tfrac{1}{2})\ln Z](2\pi)^{1/2}\times\left\{1+\frac{Z^{-1}}{12}+O(Z^{-2})\right\},\quad |\arg Z|<\pi, \tag{3}$$

we find that

$$|\Gamma(\gamma+i\nu)| = \exp[-\gamma]\exp[\tfrac{1}{2}(\gamma-\tfrac{1}{2})\ln\{(\gamma^2/\nu^2)+1\}] \times\exp[-(\pi/2)\nu]\exp[\nu\tan^{-1}(\gamma/\nu)] \times\left(\frac{\alpha Z}{W}\right)^{\gamma-1/2} p^{-\gamma+1/2}(2\pi)^{1/2} \times\left|\left\{1+\frac{(\gamma+i\nu)^{-1}}{12}+O((\gamma+i\nu)^{-2})\right\}\right|. \quad (4)$$

This enables us to deduce that the normalization factor N is also invariant under $p \to -p$, and hence we see that f and g are invariant.

If the interaction Hamiltonian for a given reaction is similarly invariant, then it follows that the transition probability ω will be also. Hence, expanding ω in a Taylor series about $p=0$, we obtain

$$\omega = \alpha + \beta p^2 + O(p^4), \quad (5)$$

where α and β are independent of p. In other words, no terms odd in p appear. Thus, to order p^2, the momentum spectrum is parabolic. To the same order, $w = 1 + \frac{1}{2}p^2$, and thus the energy spectrum is linear.

Though the above argument is based on the properties of a wave function with a particular normalization, the conclusion, as pointed out above, does not depend on the normalization. Thus our result is valid regardless of the wave function employed.

We will now consider some specific applications. The threshold calculations of internal conversion coefficients performed by Spinrad[5] for the K shell and by O'Connell and Carroll[6] for all shells, as well as the K-shell threshold calculations of electron angular correlations performed by Young,[7] give results which are in fact correct to order p. This is quite useful knowledge in view of the fact that no theoretical values of internal conversion coefficients exist for electron momenta p in the range 0 to a minimum of 0.32 (the latter value corresponding to the lowest transition energy $k = 0.05$ considered by Rose[8] and by Sliv and Band[9]). In our paper[6] on internal conversion coefficients we actually calculate results to order p and show explicitly that no terms of order p appear in the final results. This constitutes a direct verification in this particular case of the general conclusions outlined above.

Deck et al.[10] have calculated the high-frequency limit of the bremsstrahlung spectrum and then interpolated from the region of validity of the Bethe-Heitler results to their result. Their interpolated curve is a straight line, but this particular shape had no theoretical basis. Our analysis now confirms that, at least for the small region of the curve close to threshold, this is a correct choice.

Although Eq. (5) is correct in general, its usefulness may be quite restricted in a particular application. For example, if $|\beta/\alpha| \approx 1$, then while $\omega = \alpha$ is certainly correct to order p, one obviously introduces a large error when using this formula for any p other than that satisfying $p^2 \ll |\alpha/\beta|$. Of course the validity of any approximation depends on the relative magnitude of the terms neglected, whether or not it is correct, and this consideration must be taken into account in each particular application.

The usefulness of this expansion in p for the calculation of internal conversion coefficients, particularly for high Z values, has been demonstrated by Olsson and Hultberg[11] for K-shell $M1$ transitions as well as by the present authors[12] for all shells and all transition energies.

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[1] M. E. Rose, Relativistic Electron Theory (John Wiley & Sons, Inc., New York, 1960), p. 159.

[2] In our units $\hbar = c = m = 1$.

[3] See reference 1, p. 194.

[4] A. Erdelyi et al., Higher Transcendental Functions (McGraw-Hill Book Company, Inc., New York, 1953), Bateman Manuscript Project; Vol. 1, Eq. (2), p. 47.

[5] B. I. Spinrad, Phys. Rev. 98, 1302 (1955).

[6] R. F. O'Connell and C. O. Carroll, "Internal Conversion Coefficients: General Formulation for all Shells and Application to Low Energy Transitions" (Phys. Rev., to be published).

[7] R. C. Young, Phys. Rev. 115, 577 (1959).

[8] M. E. Rose, Beta- and Gamma-Ray Spectroscopy, edited by K. Siegbahn (Interscience Publishers, Inc., New York, 1955).

[9] L. A. Sliv and I. M. Band, Coefficients of Internal Conversion of Gamma Radiation, Part I—K Shell and Part II—L Shell (Physico-Technical Institute, Academy of Science, Leningrad, U.S.S.R., 1956); issued in the United States as University of Illinois Reports No. 571CC K1 and No. 581CC L1 (unpublished).

[10] R. T. Deck, C. J. Mullin, and C. L. Hammer, Nuovo Cimento 32, 180 (1964).

[11] P. O. M. Olsson and S. Hultberg, Arkiv Fysik 15, 361 (1959).

[12] F. R. O'Connell and C. O. Carroll, "Internal Conversion Coefficients for Low-Energy Transitions" (to be published). It should perhaps be pointed out that we do not expand the factors which depend on the photon en-

ergy k about the atomic threshold energy k_0. Our expansion is primarily concerned with the p-dependent terms in the hypergeometric functions which appear in our results.

SU(6) AND MESON-EXCHANGE FORCES*

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It has been shown recently that if the $BB\mu$ (baryon-baryon-meson) interactions are SU(6) symmetric in the static limit, the baryon-exchange forces may lead to the bootstrapping of the B multiplet in P-wave $B\mu$ states.[1] In this paper we examine the possibility that the exchange of the 35-fold meson multiplet μ and the singlet meson leads to bound states of the types μB, $\mu\mu$, BB, and $B\bar{B}$. The approximations necessary in order to write the meson-exchange forces in an SU(6)-symmetric way are more drastic than those used in reference 1; on the other hand, the μ-exchange model leads to a large number of simple experimental predictions, because of the generality of the two-particle states that are influenced by these forces.

We consider the $\alpha\alpha\mu$ vertices, where α represents any particle multiplet that corresponds to an irreducible representation of SU(6). If the mesons are real, one may write the $\alpha\alpha\mu$ interaction in an SU(6)-symmetric manner by following the procedure discussed previously.[1,2] The P-wave pseudoscalar (P) and vector (V) mesons play the roles of spin-1 and -0 mesons, respectively. The static limit of the V and P vertices may be written in the forms Fe_0 and $i(G/M)\vec{S}\cdot\vec{q}$, where $e=(\vec{e},e_0)$ is the V-meson four-polarization vector, F and G are interaction constants, $\vec{q}$ is the three-momentum transfer, and $\vec{S}$ is a vector operator in the α spin space. The static V- and P-exchange potentials between particles of the multiplets α and β may be written in configuration space in the following manner:

$$v_V = F_\alpha F_\beta e^{-mr}/r,$$

$$v_P = (G_\alpha G_\beta/M^2)\vec{S}_\alpha\cdot\nabla\vec{S}_\beta\cdot\nabla e^{-mr}/r, \qquad (1)$$

where $r=|\vec{r}_\alpha-\vec{r}_\beta|$, and the particle labels of the interaction constants are suppressed. A positive potential represents repulsion. If only S waves are considered, the P-exchange potential may be written

$$v_P = \tfrac{1}{3}(G_\alpha G_\beta/M^2)\vec{S}_\alpha\cdot\vec{S}_\beta[(m^2e^{-mr}/r)-4\pi\delta(r)]. \qquad (2)$$

Henceforth, we neglect the singular delta-function term of v_P; our motivation is that this leads to an SU(6)-symmetric potential if the constants F amd G are related suitably.[3] Again the effective spins of the V and P particles are 0 and 1.

If the multiplet α is identical to μ, the effective spins of the real V and P are 1 and 0. The relativistic $\mu\mu\mu$ interaction should be symmetric under permutations, as well as SU(6) symmetric in the limit that the mass of the virtual meson is small. These two requirements are not strictly compatible, as may be seen from the fact that the SU(3) singlets of the real and virtual μ multiplets correspond to the ω and X^0, respectively.[1,2] Belinfante and Cutkosky have considered this problem and have shown that the two requirements may be satisfied approximately for the major part of the $\mu\mu\mu$ interaction.[4] Therefore, we ignore this difficulty in the present paper, noting only that SU(6) symmetry may be broken more strongly in the μB and $\mu\mu$ channels than in the BB and $B\bar{B}$ channels.

We now assume SU(6) symmetry and also that the ρ_0 and φ are coupled universally to the I_z and hypercharge currents.[5] If the 35 virtual mesons are taken to correspond to Hermitian fields, the interaction constants are proportional to the appropriate matrix elements of 35 orthogonal and equivalently normalized Hermitian generators J_i of SU(6). The coefficient of the Yukawa potential for the process $\alpha_k+\beta_l\rightarrow\alpha_{k'}+\beta_{l'}$ is equal to $2f^2\sum_i\langle\alpha_{k'}|J_i|\alpha_k\rangle\langle\beta_{l'}|\times J_i|\beta_l\rangle$, where f^2 is a universal constant. Because of the additive property of the $\langle J_i\rangle$, the potential corresponding to the irreducible rep-