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POLARIZED RADIATION FROM MAGNETIC WHITE DWARFS. II. SOLUTION OF KEMP'S MODEL AT ALL TEMPERATURES

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ABSTRACT

Our exact solution of Kemp's model for polarized radiation from magnetic white dwarfs at low temperatures is extended to all temperatures. The implications of the results are discussed.

I. INTRODUCTION

Recently we (Chanmugam, O'Connell, and Rajagopal 1972, hereafter called Paper I) presented a solution of Kemp's harmonic-oscillator model (Kemp 1970*a*, *b*) correct to all orders in the magnetic field B, for polarized radiation from low-temperature magnetic white dwarfs.

However, because of the low-temperature restriction, the results of Paper I are not applicable in the infrared ($\hbar\omega \ll kT$) region to a magnetic white dwarf such as *Grw*+ 70°8247. In this paper we extend our results to all temperatures and discuss the implications of our results. We also discuss the applicability and limitations of Kemp's model.

II. FORMULATION

In Paper I we restricted our discussion to low temperatures. What this meant was that only the lowest excited states of the oscillator were considered and transitions from higher excited states were ignored. Here, we take into consideration the population of excited states. We follow the notation of Paper I.

Let us begin by computing the matrix elements for transition between any two states designated $|n'_+, n'_-, n'_z\rangle$ and $|n_+, n_-, n_z\rangle$. The expression for the interaction Hamiltonian which causes transitions giving rise to circularly polarized radiation is

$$H_I = \frac{ieEe^{-i\omega t}}{2\omega}\left(\frac{\hbar}{m\omega_c}\right)^{1/2}[\omega_c(A_\mp - A^\dagger{}_\pm) \mp \Omega(A_\mp + A^\dagger{}_\pm)]\,, \tag{1}$$

where $\Omega = (eB/2mc)$, $\omega_c = (\omega_0{}^2 + \Omega^2)^{1/2}$, and ω_0 is the natural frequency of the oscillator. The $\pm$ signs correspond to left and right circularly polarized radiation, respectively. For left circularly polarized radiation, setting

$$X = \tfrac{1}{2}ieEe^{-i\omega t}\left(\frac{\hbar}{m}\right)^{1/2},$$

we have

$$\begin{aligned}\langle n_+n_-n_z|H_I|n'_+n'_-n'_z\rangle &= \frac{X}{\omega\omega_c{}^{1/2}}\,\delta(n_z, n'_z)\\ &\times [(\omega_c - \Omega)(n_- + 1)^{1/2}\delta(n'_- - 1, n_-)\delta(n_+, n'_+)\delta(\omega - \omega_c + \Omega)\\ &- (\omega_c + \Omega)n_+{}^{1/2}\delta(n'_+ + 1, n_+)\delta(n_-, n'_-)\delta(\omega + \omega_c + \Omega)]\,.\end{aligned} \tag{2}$$

The δ functions in the frequency ω have been introduced to remind us that $\hbar\omega = E(n'_+, n'_-, n'_z) - E(n_+, n_-, n_z)$. If $n_+ = n'_+$, $n_- = n'_- - 1$, $n_z = n'_z$, then $\omega = (\omega_c - \Omega)$, while if $n_+ = n'_+ + 1$, $n_- = n'_-$, $n_z = n'_z$, then $\omega = -(\omega_c + \Omega)$. Since we are considering spontaneous (downward) transitions, the term containing $\delta(\omega + \omega_c + \Omega)$ in equation (2), which corresponds to an upward transition, is dropped. Hence

$$|\langle n_+ n_- n_z | H_I | n'_+ n'_- n'_z \rangle|^2 = \frac{X^2}{\omega_c} \delta(n_z, n'_z)\delta(n_+, n'_+)\delta(n_-, n'_- - 1)\delta(\omega - \omega_c + \Omega)(n_- + 1) . \quad (3)$$

The populations of the various levels (which are nondegenerate) of the oscillator are given by Maxwell-Boltzmann statistics, so that the probability that level $|n'_+, n'_-, n'_z\rangle$ is occupied is given by

$$P(n'_+, n'_-, n'_z) = \frac{1}{Z} \exp \{-\hbar\beta[(\omega_c + \Omega)(n'_+ + \tfrac{1}{2}) + (\omega_c - \Omega)(n'_- + \tfrac{1}{2}) + \omega_0(n'_z + \tfrac{1}{2})]\} , \quad (4)$$

where

$$Z = \sum_{n_+, n_-, n_z} \exp \{-\beta\hbar[(\omega_c + \Omega)(n_+ + \tfrac{1}{2}) + (\omega_c - \Omega)(n_- + \tfrac{1}{2}) + \omega_0(n_z + \tfrac{1}{2})]\}$$

and $\beta = 1/kT$. Then the intensity of left circularly polarized radiation is given by

$$I_L(\omega) = \frac{\hbar\omega X^2}{\omega_c Z} \sum_{n_+, n_-, n_z} \exp \{-\beta\hbar[(\omega_c + \Omega)(n_+ + \tfrac{1}{2}) + (\omega_c - \Omega)(n_- + \tfrac{3}{2}) + \omega_0(n_z + \tfrac{1}{2})]\}(n_- + 1)\delta(\omega - \omega_c + \Omega) . \quad (5)$$

Cancelling the terms in the summation over n_+, n_z with those in Z, we find that

$$I_L(\omega) = \frac{\hbar\omega X^2 S}{\omega + \Omega} , \quad (6)$$

where

$$S \equiv \frac{\sum_n (n + 1) \exp [-\beta\hbar\omega(n + 1)]}{\sum_n \exp [-\beta\hbar\omega n]} = \frac{1}{e^{\beta\hbar\omega} - 1} . \quad (7)$$

Similarly we find that the intensity of right circularly polarized light (here, for $\Omega \leq \omega$, the natural frequency of the oscillator is different so that $\omega_c = \omega - \Omega$) is

$$I_R(\omega) = \frac{\hbar\omega X^2 S}{\omega - \Omega} . \quad (8)$$

Hence for $2\Omega \leq \omega$ the fractional circular polarization q is given by

$$q = \Omega/\omega . \quad (9)$$

If $2\Omega > \omega$, then one has only left circularly polarized radiation so that

$$q = 1 . \quad (10)$$

These results are modified if one includes the effects of the field on the distribution of oscillators. In the quantum-mechanical case $\omega_0 = (\omega^2 \pm 2\omega\Omega)^{1/2}$, in agreement with the classical case (Kemp 1970*a*). We hence find in the quantum-mechanical case that

$$q(\omega) = (S^2 - 1)^{1/2} - S\,, \qquad \text{where } S \equiv \frac{\omega}{2\Omega} \geq 1\,,$$
$$q(\omega) = 1 \qquad \text{if } 0 < S < 1\,, \tag{11}$$

in agreement with Kemp's classical results.

III. LINEAR POLARIZATION

In this case if we look in a plane perpendicular to the z-axis and choose the x-axis along the line of sight, we observe two σ-components and one π-component. The interaction Hamiltonian for linear polarization is (from Paper I)

$$H_I = \frac{ieEe^{-i\omega t}}{\omega}\left[\left(\frac{\hbar}{4m\omega_c}\right)^{1/2}\{\omega_c(A_+ - A_- + A^\dagger{}_+ - A^\dagger{}_-) + \Omega(A_+ + A_- + A^\dagger{}_+ + A^\dagger{}_-)\} - i\left(\frac{\hbar\omega_0}{2m}\right)^{1/2}(a_z - a^\dagger{}_z)\right].$$

Considering only downward transitions, we find that

$$\langle n_+n_-n_z|H_I|n'_+n'_-n'_z\rangle = -iee^{i\omega t}\left[\left(\frac{\hbar}{4m\omega_c}\right)^{1/2}\{\delta(n_z, n'_z)\delta(n_-, n'_-)\delta(n'_+ - 1, n_+) \times (n_+ + 1)^{1/2}\delta(\omega - \omega_c - \Omega) - \delta(n_+, n'_+)\delta(n_z, n'_z) \times \delta(n'_z - 1, n_-)(n_- + 1)^{1/2}\delta(\omega - \omega_c + \Omega)\} - i\left(\frac{\hbar\omega_0}{2m}\right)^{1/2}\frac{1}{\omega}\,\delta(n_+, n'_+)\delta(n_-, n'_-) \times \delta(n'_z - 1, n_z)(n_z + 1)^{1/2}\delta(\omega - \omega_0)\right].$$

We then find that the only nonzero matrix elements are given by

$$\langle n_+n_-n_z|H_I|n_+ + 1, n_-, n_z\rangle = \frac{X(n_+ + 1)^{1/2}}{\omega - \Omega}\,, \tag{12}$$

where $\omega_c = \omega - \Omega$. This matrix element is zero if $\omega < 2\Omega$.

$$\langle n_+n_-n_z|H_I|n_+, n_- + 1, n_z\rangle = \frac{X}{\omega + \Omega}(n_- + 1)^{1/2}\,, \tag{13}$$

where $\omega_c = \omega + \Omega$. Also,

$$\langle n_+n_-n_z|H_I|n_+, n_-, n_z + 1\rangle = \frac{2^{1/2}X}{\omega}(n_z + 1)^{1/2} \tag{14}$$

with $\omega = \omega_c$.

Equations (12) and (13) correspond to the two σ-components σ_1, σ_2, and equation (14) corresponds to the π-component. Let the corresponding intensities of radiation be denoted by I_{σ_1}, I_{σ_2}, I_π. Then proceeding as before (cf. eqs. [4], [5]), we find that

$$\begin{aligned} I_{\sigma_1}(\omega) &= \hbar\omega X^2 S/(\omega + \Omega) && \text{if } \omega \geq 2\Omega\,, \\ &= 0 && \text{if } \omega < 2\Omega\,; \\ I_{\sigma_2}(\omega) &= \hbar\omega X^2 S/(\omega - \Omega)\,, \\ I_\pi(\omega) &= 2\hbar\omega X^2 S/\omega && \text{for all } \omega\,. \end{aligned} \tag{15}$$

The fractional linear polarization is then given by

$$q^* \equiv \frac{I_{\sigma_1} + I_{\sigma_2} - I_\pi}{I_{\sigma_1} + I_{\sigma_2} + I_\pi}\,, \tag{16}$$

so that

$$\begin{aligned} q^* &= \Omega^2/(2\omega^2 - \Omega^2) && \text{if } \omega \geq 2\Omega\,, \\ &= -(\omega + 2\Omega)/(\omega + 3\Omega) && \text{if } \omega < 2\Omega\,. \end{aligned} \tag{17}$$

If one includes the effects of the magnetic field on the distribution of oscillators, we find that

$$\begin{aligned} q^* &= [(1 + t)^{1/2} + (1 - t)^{1/2} - 2(1 - t^2)^{1/2}]/[(1 + t)^{1/2} + (1 - t)^{1/2} + 2(1 - t^2)^{1/2}] \\ &\qquad\qquad \text{for } t \equiv S^{-1} \leq 1\,, \\ &= [S^{1/2} - 2(1 + S)^{1/2}]/[S^{1/2} + 2(1 + S)^{1/2}] \qquad \text{for } 1 > S > 0\,. \end{aligned} \tag{18}$$

We emphasize that, similar to the case of the circular polarization, the q^* value is temperature independent (cf. Paper I, Note added in proof). This property is a feature of the harmonic-oscillator model.

IV. DISCUSSION

It is remarkable that the results of Paper I for q are unchanged when we relax the low-temperature assumption and allow population of higher levels. This means that an explanation of the discrepancy between the infrared circular polarization observations and Kemp's model must be sought elsewhere.

A feature of Kemp's model is that one has to have a distribution of oscillators with essentially a continuum of natural frequencies to account for the continuous emission. Furthermore, one has to make an additional assumption regarding the number distribution of oscillators. Thus, while Kemp's model has been extremely valuable in providing insights into the nature of the polarized radiation, complete agreement with observations will have to await either (*a*) modifications in Kemp model or (*b*) a new approach which will discuss the behavior of atoms in magnetic white dwarfs.

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