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Relativity Gyroscope Experiment at Arbitrary Orbit Inclinations

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We carry out a calculation of the "indirect" quadrupole moment effect (given by "averaging" the de Sitter term, $\vec{\Omega}_{DS}$, over a period of the motion) for "distorted" elliptical orbits of small eccentricity at arbitrary inclination to the earth's equator. Combining this with other effects enables us to calculate the total gyro precession. We then discuss the question of what is the optimum orbit for the relativity gyroscope experiment.

I. INTRODUCTION

A modern test of general relativity, proposed by Schiff,[1] is the measurement of the precession of the spin of a gyroscope in orbit about the earth. This test is the only one thus far proposed which is likely to measure the off-diagonal Lense-Thirring terms in the metric tensor. In the near future Everitt and Fairbank[2,3] expect to carry out this experiment by means of a satellite containing two pairs of superconducting gyroscopes in a polar orbit[4] around the earth, and they predict[3] an ultimate accuracy of 0.001″/yr.

The rate of change of the spin angular momentum, $\vec{S}_0$, of the gyroscope in the gyroscope rest frame is given by[1]

$$d\vec{S}_0/dt=\vec{\Omega}\times\vec{S}_0\ , \tag{1}$$

where $\vec{\Omega}$ is the angular velocity of precession and we write $\vec{\Omega}\equiv\vec{\Omega}_E$ and $\vec{\Omega}\equiv\vec{\Omega}_{BD}$ in the Einstein[1,5-7] and Brans-Dicke[8,9] theories, respectively. We have

$$\vec{\Omega}_E=\vec{\Omega}_T+\vec{\Omega}_{DS}+\vec{\Omega}_{LT}+\vec{\Omega}_Q+\vec{\Omega}_{DS}^S\ , \tag{2}$$

$$\vec{\Omega}_{BD}=\vec{\Omega}_T+\frac{4+3\omega}{6+3\omega}(\vec{\Omega}_{DS}+\vec{\Omega}_Q+\vec{\Omega}_{DS}^S)+\frac{3+2\omega}{4+2\omega}\vec{\Omega}_{LT}\ , \tag{3}$$

where $\vec{\Omega}_T$, $\vec{\Omega}_{DS}$, $\vec{\Omega}_{LT}$, $\vec{\Omega}_Q$, and $\vec{\Omega}_{DS}^S$ are the Thomas, de Sitter, Lense-Thirring, quadrupole-moment,[5,6] and sun[7] contributions, respectively, and ω is the dimensionless coupling constant in the Brans-Dicke theory. By putting the gyroscope in a satellite it is possible to have $\vec{\Omega}_T$ essentially zero.[1] We shall be interested in the secular results in which case the $\vec{\Omega}$'s are averaged over a period of the motion, in which case we have

$$(d\vec{S}_0/dt)_{av}=\vec{\Omega}_{av}\times\vec{S}_0\ . \tag{4}$$

In Sec. II of this paper we shall calculate the quadrupole moment term of order J_2 in $\vec{\Omega}_{DSav}$ for a distorted elliptical[10] orbit (with eccentricity $e\ll 1$) inclined to the equator at an arbitrary angle α and use this result in Sec. III where we discuss the gyroscope precession for arbitrary angle α using all the terms in $\vec{\Omega}$ given by Eq. (2). Finally in Sec. IV we discuss which orbit is the most desirable.

II. INDIRECT QUADRUPOLE-MOMENT EFFECT

The Lagrangian for a satellite of mass m in orbit about the earth of mass M, quadrupole moment J_2, and spin angular momentum $\vec{S}^{(2)}$ (let $\vec{n}^{(2)}$ be a unit vector in the $\vec{S}^{(2)}$ direction) can be written as

$$\mathcal{L} = \tfrac{1}{2}mv^2 + \frac{GmM}{r} + \frac{GJ_2 mM}{2r^3}\left(1 - 3\,\frac{(\vec{n}^{(2)}\cdot\vec{r})^2}{r^2}\right). \quad (5)$$

Let us consider a Cartesian coordinate system such that the x-y plane is inclined at an angle α to the equatorial plane of the earth (see Fig. 1) and let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be unit vectors in the x, y, and z directions, respectively. Thus, α is also the angle between the vectors $\vec{k}$ and $\vec{n}^{(2)}$ such that

$$\vec{n}^{(2)} = \vec{k}\cos\alpha + \vec{i}\sin\alpha \,. \quad (6)$$

Using the polar coordinates r, θ, and φ corresponding to the above Cartesian coordinate system the Lagrangian of Eq. (5) can be written as

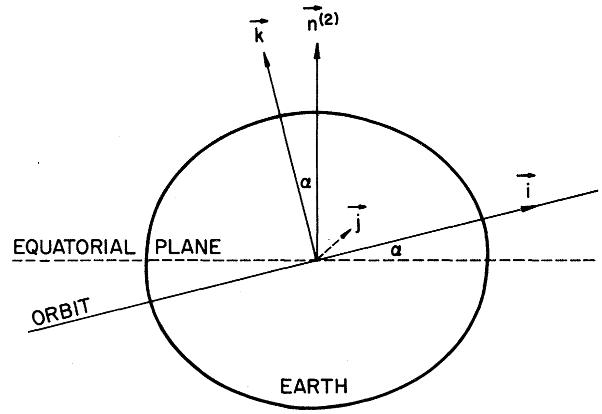

FIG. 1. The unit vectors $\vec{i}$, $\vec{n}^{(2)}$, and $\vec{k}$ are in the plane of the paper while the unit vector $\vec{j}$ points into the paper. Also $\vec{n}^{(2)} = \vec{k}\cos\alpha + \vec{i}\sin\alpha$.

$$\mathcal{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2) + GmM/r$$
$$+ (GJ_2 mM/2r^3)\{1 - 3[\sin^2\alpha\,\sin^2\theta\cos^2\varphi + \cos^2\alpha\,\cos^2\theta + \tfrac{1}{2}\sin(2\alpha)\sin(2\theta)\cos\varphi]\} \,. \quad (7)$$

The Euler-Lagrange equations from the above Lagrangian, for the variables r, θ, and φ, respectively, are given by

$$\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2 + GM/r^2 = -(3GJ_2M/2r^4)\{1 - 3[\sin^2\alpha\,\sin^2\theta\cos^2\varphi + \cos^2\alpha\,\cos^2\theta + \tfrac{1}{2}\sin(2\alpha)\sin(2\theta)\cos\varphi]\}, \quad (8)$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - \tfrac{1}{2}r^2\sin(2\theta)\dot{\varphi}^2 = (3GJ_2M/2r^3)[(\cos^2\alpha - \sin^2\alpha\cos^2\varphi)\sin(2\theta) - \sin(2\alpha)\cos(2\theta)\cos\varphi] \,, \quad (9)$$

$$r^2\sin\theta\,\ddot{\varphi} + 2r^2\cos\theta\,\dot{\theta}\dot{\varphi} + 2r\sin\theta\,\dot{r}\dot{\varphi} = (3GJ_2M/2r^3)[\sin^2\alpha\,\sin\theta\sin(2\varphi) + \sin(2\alpha)\cos\theta\sin\varphi] \,. \quad (10)$$

A solution to the above Eqs. (8)–(10) can be written as

$$r = a + R \,, \quad (11)$$

$$\theta = \tfrac{1}{2}\pi + \Theta \,, \quad (12)$$

$$\varphi = \bar{\omega}t + \Phi \,, \quad (13)$$

where a and $\bar{\omega}$ are constants and R/a, Θ, and Φ are regarded as small quantities whose absolute values are $\ll 1$. To lowest order we have Kepler's relation

$$\bar{\omega} = (GM/a^3)^{1/2} \,. \quad (14)$$

It is convenient to demand that this is true in general and this will in effect constitute a definition for a. Using Eqs. (11)–(14) in (8)–(10) we obtain the three simpler equations

$$\ddot{R} - 2\bar{\omega}a\dot{\Phi} - 3\bar{\omega}^2R = -3\bar{\omega}^2a(J_2/2a^2)$$
$$\times[1 - 3\sin^2\alpha\cos^2(\bar{\omega}t)] \,, \quad (15)$$

$$\ddot{\Theta} + \bar{\omega}^2\Theta = 3\bar{\omega}^2\sin(2\alpha)(J_2/2a^2)\cos(\bar{\omega}t) \,, \quad (16)$$

$$\ddot{\Phi} + 2\bar{\omega}\dot{R}/a = 3\bar{\omega}^2\sin^2\alpha(J_2/2a^2)\sin(2\bar{\omega}t) \,. \quad (17)$$

Using the general solutions of Eqs. (15)–(17) in (11)–(13) we obtain

$$r = a[1 \pm e\cos(\bar{\omega}t - \delta)] + (J_2/2a)\cos^2\alpha$$
$$- (J_2/2a)\sin^2\alpha\cos^2(\bar{\omega}t) \,, \quad (18)$$

$$\theta = \tfrac{1}{2}\pi + c_1\cos(\bar{\omega}t) + c_2\sin(\bar{\omega}t)$$
$$+ (3J_2/4a^2)\sin(2\alpha)\,\bar{\omega}t\sin(\bar{\omega}t) \,, \quad (19)$$

$$\varphi = \bar{\omega}t \mp 2e\sin(\bar{\omega}t - \delta) - \zeta$$
$$- (J_2/8a^2)\sin^2\alpha\sin(2\bar{\omega}t) \,, \quad (20)$$

where e, δ, c_1, c_2, and ζ are constants of integration which, except for δ, must be $\ll 1$ in absolute value for our approximations. An extra term on the right-hand side of Eq. (18) of c_3a and an extra term on the right-hand side of Eq. (20) of $-\frac{3}{2}c_3\bar{\omega}t$ were omitted through the definition of $\bar{\omega}$ as the average angular velocity; thus

$$\bar{\omega} = 2\pi/T \,, \quad (21)$$

where T is the period (defined to be the time taken for φ to change by 2π). The constant a (or $\bar{\omega}$) is to be regarded as the sixth arbitrary constant along with e, δ, c_1, c_2, and ζ to be determined by the initial conditions of the variables r, θ, φ, $\dot{r}$, $\dot{\theta}$, and $\dot{\varphi}$. The quantity $e \geq 0$ is the eccentricity of the distorted ellipse due to the quadrupole moment J_2, i.e., the orbit of Eqs. (18)–(20). Since e^2 must be regarded as zero in our approximation, the "undistorted" ellipse is a circle with the focus displaced a distance ae from the center of the ellipse toward the perihelion, i.e., the orbit of Eqs. (18)–(20) with $J_2 = 0$.

The instantaneous de Sitter term can be written as[1]

$$\vec{\Omega}_{\mathrm{DS}} = (3GM/2c^2)(\vec{r}\times\vec{v}/r^3) , \tag{22}$$

which, upon using the solutions for r, θ, and φ, gives

$$\vec{\Omega}_{\mathrm{DS}} = (3GM/2c^2)(\vec{k}\dot{\varphi}/r + (3J_2\overline{\omega}/4a^3)\sin(2\alpha) \times\{-\vec{i}\sin^2(\overline{\omega}t) + \vec{j}[\overline{\omega}t + \tfrac{1}{2}\sin(2\overline{\omega}t)]\} + (c_1\overline{\omega}/a)\vec{i} + (c_2\overline{\omega}/a)\vec{j}) . \tag{23}$$

The "average" value for $\vec{\Omega}_{\mathrm{DS}}$ over N orbits is given by

$$\vec{\Omega}_{\mathrm{DS\,av}} = \frac{3GM}{2c^2}\frac{1}{NT}\int_0^{NT}\frac{\vec{r}\times\vec{v}}{r^3}\,dt , \tag{24}$$

which gives the final result

$$\vec{\Omega}_{\mathrm{DS\,av}} = (3GM\overline{\omega}/2c^2a) \times\{[1 + (1 - 3\cos^2\alpha)(J_2/4a^2)]\vec{k} + 3\sin(2\alpha)(J_2/8a^2)(-\vec{i} + 2\pi N\vec{j}) + c_1\vec{i} + c_2\vec{j}\} . \tag{25}$$

For the initial conditions at $t = 0$ corresponding to $c_1 = c_2 = 0$, we have $\theta_{t=0} = \frac{1}{2}\pi$ and $\dot{\theta}_{t=0} = 0$ so that $\vec{\Omega}_{\mathrm{DS}(t=0)}$ will be in the $\vec{k}$ direction. Except for the polar or equatorial orbits [in which case $\sin(2\alpha) = 0$] the plane of the orbit will change, and hence $\vec{\Omega}_{\mathrm{DS\,av}}$ will *not* be in the $\vec{k}$ direction. If the initial conditions are such that $c_1 = 3(J_2/8a^2)\sin(2\alpha)$ then $\vec{\Omega}_{\mathrm{DS\,av}}$ will have no component in the $\vec{i}$ direction. It is to be noted that only the component of $\vec{\Omega}_{\mathrm{DS\,av}}$ in the $\vec{j}$ direction depends on the number of orbits averaged over. The coordinate system can always be chosen such that $c_1 = c_2 = 0$ by orientating the coordinate system, so that $\theta = \frac{1}{2}\pi$ and $\theta = 0$ at $t = 0$. Thus, no matter what orbit the satellite is put into we can always have our coordinate system such that $c_1 = c_2 = 0$. This seems the most preferable thing to do as we can then forget about other choices of the constants c_1 and c_2.

Let us now consider in more detail the limits of validity of Eqs. (18)–(20) and (25), so that Eq. (25) will not introduce errors even as much as 0.001″/yr. If the constants e and ζ are of the order 5×10^{-3} or less (note that J_2/a^2 is of the order 10^{-3}); and if the constants c_1 and c_2 are of the order $5\times10^{-3} \times|\sin(2\alpha)|$ or less [note that in Eq. (19) we do not want the terms involving c_1 and c_2 to be of larger order than the last term of Eq. (19), hence the factor of $|\sin(2\alpha)|$ for c_1 and c_2]; and finally if the time t does not exceed $T/|\sin(2\alpha)|$, i.e., N must be less than $1/|\sin(2\alpha)|$, then Eqs. (18)–(20) and (25) will be of the required accuracy. The worst possible case occurs when $\alpha = \frac{1}{4}\pi$. In this case Eqs. (18)–(20) and (25) are only valid for *one* revolution. If α is near 0 or $\frac{1}{2}\pi$, then Eqs. (18)–(20) and (25) are valid for *many* revolutions. In Sec. IV we discuss the case where α is near $\frac{1}{2}\pi$. The estimates for the allowed values of e, ζ, c_1, c_2, t, and N are based on the error in Eq. (25) being of the order of $(5\times10^{-3})^2(3GM\overline{\omega}/2c^2a) = 0.000\,175''/\mathrm{yr}$ for a circular orbit 300 miles above the earth.

III. GYRO ORBIT WITH ARBITRARY INCLINATION

In addition to $\vec{\Omega}_{\mathrm{DS\,av}}$ given by Eq. (24), we also have the following terms[5–7] which can affect the precession of the gyroscope by more than 0.005″/yr:

$$\vec{\Omega}_{\mathrm{LT\,av}} = (GS^{(2)}/2c^2a^3)[\vec{n}^{(2)} - 3\cos\alpha\,\vec{k}] , \tag{26}$$

$$\vec{\Omega}_{\mathrm{Q\,av}} = (3GM\overline{\omega}/2c^2a)(3J_2/2a^2) \times[\tfrac{1}{2}(5\cos^2\alpha - 1)\vec{k} - \cos\alpha\,\vec{n}^{(2)}] , \tag{27}$$

$$\vec{\Omega}^{S}_{\mathrm{DS\,av}} = [3GM_\odot\overline{\omega}_E/2c^2a_E(1 - e_E^2)]\vec{n}^{(S)} , \tag{28}$$

where Eqs. (26) and (27) hold for the orbit Eqs. (18)–(20) and the notation is the same as in Sec. II. Equation (28) is the de Sitter term[11] for the earth's orbit about the sun where $M_\odot$, $\overline{\omega}_E$, a_E, e_E, and $\vec{n}^{(S)}$ are the mass of the sun, the average angular velocity of the earth about the sun, the semimajor axis of the earth's orbit, the eccentricity of the earth's orbit, and a unit vector in the direction of the earth's orbital angular momentum, respectively. The numerical value for J_2 is given by[12]

$$J_2 = (1082.64 \pm 0.08)\times10^{-6}R^2, \tag{29}$$

where R is the earth's equatorial radius.

Equation (4) can be put in the form

$$(d\vec{n}^{(1)}/dt)_{\mathrm{av}} = \vec{\Omega}_{\mathrm{av}}\times\vec{n}^{(1)} , \tag{30}$$

where $\vec{n}^{(1)}$ is a unit vector in the $\vec{S}_0$ direction. It is the quantity $\vec{\Omega}_{\mathrm{av}}\times\vec{n}^{(1)}$ that is used in quoting numerical values in seconds of arc per year. To this end it is convenient to calculate $\vec{\Omega}_{\mathrm{av}}\times\vec{n}^{(1)}$ for $\vec{n}^{(1)} = \vec{n}^{(1)}_{\mathrm{DS}}$ and $\vec{n}^{(1)} = \vec{n}^{(1)}_{\mathrm{LT}}$, where

$$\vec{n}^{(1)}_{\mathrm{DS}} \equiv \hat{\Omega}_{\mathrm{LT\,av}} = \frac{\vec{n}^{(2)} - 3\cos\alpha\,\vec{k}}{(1 + 3\cos^2\alpha)^{1/2}} , \tag{31}$$

$$\vec{n}^{(1)}_{LT} \equiv \vec{k} \approx \hat{\Omega}_{DS\,av} \ . \tag{32}$$

The gyroscope measuring the *de Sitter* term will have its spin in the $\vec{n}^{(1)}_{DS}$ direction so that $\vec{\Omega}_{LT\,av}\times\vec{n}^{(1)}_{DS} = 0$ and the gyroscope measuring the *Lense-Thirring* term will have its spin in the $\vec{n}^{(1)}_{LT}$ direction[13] so that $\vec{\Omega}_{DS\,av}\times\vec{n}^{(1)}_{LT} \approx 0$. We thus have for the *de Sitter* gyro

$$\vec{\Omega}_{DS\,av}\times\vec{n}^{(1)}_{DS} = \{[A_{DS}+\tfrac{1}{2}(1-3\cos^2\alpha)A_Q]\sin\alpha\,\vec{j} - \tfrac{3}{2}A_Q\sin(2\alpha)[\cos\alpha\,\vec{j}+\pi N(2\cos\alpha\,\vec{i}+\sin\alpha\,\vec{k})] + [2c_1\cos\alpha\,\vec{j} - c_2(2\cos\alpha\,\vec{i}+\sin\alpha\,\vec{k})]A_{DS}\}(1+3\cos^2\alpha)^{-1/2} \ , \tag{33}$$

$$\vec{\Omega}_{LT\,av}\times\vec{n}^{(1)}_{DS} = 0 \ , \tag{34}$$

$$\vec{\Omega}_{Q\,av}\times\vec{n}^{(1)}_{DS} = -\tfrac{3}{2}A_Q(1+3\cos^2\alpha)^{-1/2}(1+\cos^2\alpha)\sin\alpha\,\vec{j} \ , \tag{35}$$

$$\vec{\Omega}^{S}_{DS\,av}\times\vec{n}^{(1)}_{DS} = A^{S}_{DS}(1+3\cos^2\alpha)^{-1/2}(\vec{n}^{(S)}\times\vec{n}^{(2)} - 3\cos\alpha\,\vec{n}^{(S)}\times\vec{k}) \ ; \tag{36}$$

and for the *Lense-Thirring gyro,*

$$\vec{\Omega}_{DS\,av}\times\vec{n}^{(1)}_{LT} = \tfrac{3}{4}A_Q\sin(2\alpha)(\vec{j}+2\pi N\vec{i}) + A_{DS}(c_2\vec{i} - c_1\vec{j}) \ , \tag{37}$$

$$\vec{\Omega}_{LT\,av}\times\vec{n}^{(1)}_{LT} = -A_{LT}\sin\alpha\,\vec{j} \ , \tag{38}$$

$$\vec{\Omega}_{Q\,av}\times\vec{n}^{(1)}_{LT} = \tfrac{3}{2}A_Q\sin(2\alpha)\,\vec{j} \ , \tag{39}$$

$$\vec{\Omega}^{S}_{DS\,av}\times\vec{n}^{(1)}_{LT} = A^{S}_{DS}\,\vec{n}^{(S)}\times\vec{k} \ , \tag{40}$$

where

$$A_{DS} = 3GM\varpi/2c^2a \ , \tag{41}$$

$$A_{LT} = GS^{(2)}/2c^2a^3 \ , \tag{42}$$

$$A_Q = (3GM\varpi/2c^2a)(J_2/2a^2) \ , \tag{43}$$

$$A^{S}_{DS} = 3GM_{\odot}\varpi_E/2c^2a_E(1-e_E^{\,2}) \ . \tag{44}$$

For a circular orbit 300 miles above the earth we have the following numerical values:

$$A_{DS} = 7.0''/\text{yr} \ , \tag{45}$$

$$A_{LT} = 0.0438''/\text{yr} \ , \tag{46}$$

$$A_Q = 0.0033''/\text{yr} \ , \tag{47}$$

$$A^{S}_{DS} = 0.0192''/\text{yr} \ , \tag{48}$$

which give us an idea of the relative importance of the terms $\vec{\Omega}_{DS}$, $\vec{\Omega}_{LT}$, $\vec{\Omega}_Q$, and $\vec{\Omega}^{S}_{DS}$, respectively.

IV. DISCUSSION AND CONCLUSION

The most important terms for the de Sitter gyro and Lense-Thirring gyro are given by Eqs. (33) and (38), respectively, as

$$\vec{\Omega}_{DS\,av}\times\vec{n}^{(1)}_{DS} \approx A_{DS}(1+3\cos^2\alpha)^{-1/2}\sin\alpha\,\vec{j} \ , \tag{49}$$

$$\vec{\Omega}_{LT\,av}\times\vec{n}^{(1)}_{LT} = -A_{LT}\sin\alpha\,\vec{j} \ . \tag{50}$$

Clearly both terms of Eqs. (49)–(50) will be maximum for $\alpha = \frac{1}{2}\pi$. Thus in this respect the polar orbit is preferred.

Next let us consider which orbit gives us the cleanest separation of the de Sitter effect and the Lense-Thirring effect. Since the right-hand side of Eq. (34) is zero, the Lense-Thirring effect does not contribute to the de Sitter gyro for all angles α. Since the right-hand side of Eq. (37) is *not* in general zero, the de Sitter effect *will* contribute to the Lense-Thirring gyro unless the gyro is in a polar orbit,[14] for which case $\sin(2\alpha) = 0$ and we use $c_1 = c_2 = 0$. In addition, for the polar orbit the right-hand side of Eq. (33) is greatly simplified and the right-hand side of Eq. (39) is zero. Thus again the polar orbit is preferred.

For the *de Sitter gyro* in *polar orbit* with $c_1 = c_2 = 0$, we have

$$\vec{\Omega}_{DS\,av}\times\vec{n}^{(1)}_{DS} = (A_{DS}+\tfrac{1}{2}A_Q)\,\vec{j} \ , \tag{51}$$

$$\vec{\Omega}_{LT\,av}\times\vec{n}^{(1)}_{DS} = 0 \ , \tag{52}$$

$$\vec{\Omega}_{Q\,av}\times\vec{n}^{(1)}_{DS} = -\tfrac{3}{2}A_Q\,\vec{j} \ , \tag{53}$$

$$\vec{\Omega}^{S}_{DS\,av}\times\vec{n}^{(1)}_{DS} = A^{S}_{DS}\,\vec{n}^{(S)}\times\vec{n}^{(2)} \tag{54}$$

and for the *Lense-Thirring gyro* in *polar orbit* with $c_1 = c_2 = 0$ we have

$$\vec{\Omega}_{DS\,av}\times\vec{n}^{(1)}_{LT} = 0 \ , \tag{55}$$

$$\vec{\Omega}_{LT\,av}\times\vec{n}^{(1)}_{LT} = -A_{LT}\,\vec{j} \ , \tag{56}$$

$$\vec{\Omega}_{Q\,av}\times\vec{n}^{(1)}_{LT} = 0 \ , \tag{57}$$

$$\vec{\Omega}^{S}_{DS\,av}\times\vec{n}^{(1)}_{LT} = A^{S}_{DS}\,\vec{n}^{(S)}\times\vec{k} \ . \tag{58}$$

It is to be noted that the terms of Eqs. (53) and (54) are much less than that of Eq. (51); hence we *will* get a clean test of the de Sitter effect. On the other hand the term of Eq. (58) is of the same order of magnitude as that of Eq. (56), hence we will *not* get a very clean test of the Lense-Thirring effect.

We now ask whether it is possible to have an orbit that would give a cleaner test of the Lense-Thirring effect. The effect of $\vec{\Omega}^{S}_{DS}$ on the Lense-Thirring gyro can be eliminated by placing the orbit of the gyro in the plane of the ecliptic such that $\vec{k} = \vec{n}^{(S)}$ and $\alpha = 23.44°$. Unfortunately the effects given in Eqs. (37) and (39) that were eliminated by using a polar orbit will now contribute to the

Lense-Thirring gyro. If the measurements could be made during a very small number of orbits (N small) then these effects would be smaller than that of Eq. (58) for a polar orbit. As the experiment will be carried out over many revolutions of the satellite about the earth the number N will be large and the polar orbit is to be preferred.

The results for $\alpha = \frac{1}{2}\pi + \epsilon$ where $|\epsilon| \ll 1$ will be of use in the case where the satellite does not go exactly into the preferred polar orbit.

For the *de Sitter gyro* with $\alpha = \frac{1}{2}\pi + \epsilon$ and initial conditions such that $c_1 = c_2 = 0$, we have

$$\vec{\Omega}_{\mathrm{DSav}} \times \vec{n}^{(1)}_{\mathrm{DS}} = [A_{\mathrm{DS}} + \tfrac{1}{2}A_{\mathrm{Q}}]\vec{j} + 3\pi N\epsilon A_{\mathrm{Q}}\vec{k}\ , \tag{59}$$

$$\vec{\Omega}_{\mathrm{LTav}} \times \vec{n}^{(1)}_{\mathrm{DS}} = 0\ , \tag{60}$$

$$\vec{\Omega}_{\mathrm{Qav}} \times \vec{n}^{(1)}_{\mathrm{DS}} = -\tfrac{3}{2}A_{\mathrm{Q}}\vec{j}\ , \tag{61}$$

$$\vec{\Omega}^{S}_{\mathrm{DSav}} \times \vec{n}^{(1)}_{\mathrm{DS}} = A^{S}_{\mathrm{DS}}\vec{n}^{(S)} \times \vec{n}^{(2)}\ , \tag{62}$$

and for the *Lense-Thirring gyro* with $\alpha = \frac{1}{2}\pi + \epsilon$ and initial conditions such that $c_1 = c_2 = 0$ we have

$$\vec{\Omega}_{\mathrm{DSav}} \times \vec{n}^{(1)}_{\mathrm{LT}} = -3\pi N\epsilon A_{\mathrm{Q}}\vec{i}\ , \tag{63}$$

$$\vec{\Omega}_{\mathrm{LTav}} \times \vec{n}^{(1)}_{\mathrm{LT}} = -A_{\mathrm{LT}}\vec{j}\ , \tag{64}$$

$$\vec{\Omega}_{\mathrm{Qav}} \times \vec{n}^{(1)}_{\mathrm{LT}} = 0\ , \tag{65}$$

$$\vec{\Omega}^{S}_{\mathrm{DSav}} \times \vec{n}^{(1)}_{\mathrm{LT}} = A^{S}_{\mathrm{DS}}\vec{n}^{(S)} \times \vec{k}\ . \tag{66}$$

After N orbits the quantity $2\pi N(3J_2/4a^2)|\sin(2\alpha)|$ must be $\leqslant 5\times10^{-3}$ in order for Eq. (19) to be valid. In the case we are considering, this means that $3\pi N(J_2/2a^2)|2\epsilon|$ must be $\leqslant 5\times10^{-3}$, which will be the case if $|2\epsilon| \leqslant 1/N$. If one carries out the experiment in a 300-mile circular orbit for a year then $N = 5580$ and $|\epsilon| \leqslant 10^{-4}$ rad $= 20''$.

We note that if we use the polar orbit Eqs. (51)–(58), instead of the more exact nonpolar orbit Eqs. (59)–(66), the inclination of the orbit must not deviate from polar orbit by more than $1''$ in order that the magnitude of the right-hand side of Eq. (63) should not be more than $0.001''$/yr for a 300 mile circular orbit with $N = 5580$.

In conclusion we wish to point out that in Secs. III and IV we have chosen the orientation of the gyros in the $\vec{n}^{(1)}_{\mathrm{DS}}$ and $\vec{n}^{(1)}_{\mathrm{LT}}$ directions [see Eqs. (31) and (32)] as these seem to be the most appropriate directions since they are useful in separating the de Sitter and Lense-Thirring effects and leave the final results in as simple a form as possible. However, in actual practice the gyroscopes will be aimed at stars[15] whose location may not be exactly in the $\vec{n}^{(1)}_{\mathrm{DS}}$ and $\vec{n}^{(1)}_{\mathrm{LT}}$ directions. This should cause no problem as the de Sitter and Lense-Thirring effects can still be unambiguously determined by the two pairs of gyros for all orbit inclinations except ones near the equatorial orbit. The general result for $\vec{\Omega}_{\mathrm{av}} \times \vec{n}^{(1)}$, where $\vec{n}^{(1)}$ is an arbitrary unit vector, i.e.,

$$\vec{n}^{(1)} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k} \quad \text{with} \quad a_1^2 + a_2^2 + a_3^2 = 1\ , \tag{67}$$

can readily be obtained from Eqs. (25)–(28) and (67) and is what is required for a realistic experiment. The general result is, however, rather lengthy and no useful purpose is served in writing it down. In practice it is better to work out the results when the choice of stars at which the gyros will be pointed is made.

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along $\vec{\Omega}_{DS\ av}$ initially, so that $\vec{\Omega}_{DS\ av} \times \vec{n}_{LT}^{(1)} = 0$, but it is not as convenient to do this.

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Best-Fit Estimate of Relativistic Effects in Time-Delay Experiments*

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Time-delay experiments are analyzed within the frame of a curved space-time. Residuals from Newtonian best fits of relativistic data are used as a measure of the "relativistic effects." Radial transponder trajectories are considered. If the motion is towards the sun, the relativistic residuals are of the order of 100 m. If the motion is away from the sun, they are at the 10-km level and the fraction due to the second-order curvature of the metric is at the 1-km level. Those effects are significantly smaller than those calculated from the divergence of the Newtonian and relativistic predictions after exact fit of the initial measurements.

I. INTRODUCTION

In preceding papers,[1-4] time-delay experiments performed from the earth to a natural or to an artificial planet have been analyzed within the frame of a curved space-time. In Refs. 2, 3, and 4, the method used is essentially the following: If the motion has n constants of integration, it is assumed that an equal number of time-delay measurements are used to determine them, thus providing a perfect fit of hypothetical data at these points. Then one calculates the divergence of Newtonian and relativistic predictions for the other measurements. These divergences are interpreted as "relativistic effects." In some cases, this method gives results which are highly sensitive to the particular and arbitrary choice of the measurements used to determine the constants of the motion. The results can in fact vary by orders of magnitude.

In the present paper, a best-fit-analysis approach is followed in an attempt to simulate more closely the actual experiment. Essentially, one examines to which extent relativistic effects in the data could equally be explained within Newtonian theory by appropriate increments to the constants of the motion. More precisely a best fit of the relativistic terms in the relativistic expression for the time delay is performed with increments to the classical parameters of the expression. Residuals from this best fit are "relativistic effects" which cannot be explained in classical theory within the limitation of the problem. Such residuals can be compared with the expected accuracy of the measurements for an estimate of the possibility of determining the components of the curvature of the metric and thus test general relativity and other theories of gravitation.

This approach is applied here to time-delay experiments carried from the earth to artificial planets moving on radial trajectories towards the sun or away from it. The results for this simple model should suggest an upper limit to the order of magnitude of the relativistic effects to be seen on quasiradial sections of a "grand tour" trajectory or of a very high eccentricity orbit of comparable energy.

For simplicity, it is assumed that the earth is on a circular orbit and that the artificial planet trajectory is contained in the ecliptic plane. Classical perturbations such as the oblateness of the sun and the gravitational field of other planets are neglected. Relativistic effects due to the rotation of the sun (Lense-Thirring effects) are also neglected (they are at the cm level or lower and undetectable at the present time). The field of the sun is assumed to have spherical symmetry and is described by a generalized metric. Thus relativity corrections for theories which absorb the gravitational field in the curvature of space-time and where test particles travel along geodesics can be evaluated and compared.

In Sec. II, a solution to first order in GM/rc^2 is given for the motion of a test mass along a radial orbit. Previous results on the propagation of photons and on circular orbits are also included. In Sec. III, the Newtonian best fit of relativistic data and the calculation of relativistic residuals