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POLARIZED RADIATION FROM MAGNETIC WHITE DWARFS: EXACT SOLUTION OF KEMP'S MODEL

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ABSTRACT

An exact quantum-mechanical calculation of Kemp's magnetoemission process, for magnetic white dwarfs at low temperatures, is presented and the implications discussed.

I. INTRODUCTION

Some white dwarfs have been observed to emit circularly polarized light (Kemp *et al.* 1970). It is deduced from Kemp's (1970*a*, *b*) magnetoemission theory that this is due to the presence of magnetic fields B of the order of 10^7 gauss. In this theory a model gray-body radiating system in a magnetic field B was shown to emit radiation of frequency ω with a fractional circular polarization given, to first order in B, by $q(\omega) \sim -eB/m\omega$.

Since Kemp's prediction that q is proportional to λ (wavelength) is at variance with the observations, Shipman (1971) extended Kemp's theory to take into account radiative transfer in the atmosphere of Grw+70°8247 and found a λ dependence more in conformity with observations, though discrepancies remained with the observations in the ultraviolet and particularly in the infrared. However, Kemp's assumption that $eB/m\omega \ll 1$ breaks down in the infrared (Shipman 1971), which leaves open the possibility that an exact quantum-mechanical calculation might lead to better agreement with the infrared observations. Motivated, therefore, by a need to extend Kemp's calculations to higher orders in B, we consider his gray-body quantum-mechanical harmonic-oscillator model and solve it exactly for low temperatures.

II. FORMULATION

The Hamiltonian for a three-dimensional harmonic oscillator of charge e in a magnetic field B along the z-axis is

$$H = \frac{1}{2m}[p_x^2 + p_y^2 + m^2\omega_c^2(x^2 + y^2)] + \frac{1}{2m}[p_z^2 + m^2\omega_0^2 z^2] + \Omega L_z\,, \tag{1}$$

where we have taken the vector potential $\boldsymbol{A} = \frac{1}{2}\boldsymbol{B} \times \boldsymbol{r}$, $\Omega = eB/2mc$, ω_0 the natural frequency of the oscillator and $\omega_c \equiv (\omega_0^2 + \Omega^2)^{1/2}$. If we now make the transformations

$$a_x = \left(\frac{m\omega_c}{2\hbar}\right)^{1/2} x + i(2m\hbar\omega_c)^{-1/2} p_x\,, \quad a_y = \left(\frac{m\omega_c}{2\hbar}\right)^{1/2} y + i(2m\hbar\omega_c)^{-1/2} p_y\,,$$

$$a_z = \left(\frac{m\omega_0}{2\hbar}\right)^{1/2} z + i(2m\hbar\omega_0)^{-1/2} p_z\,,$$

so that $[a_i, a_j] = 0$, $[a_i, a_j^\dagger] = \delta_{ij}$ for $i, j = x, y, z$, and the further transformation

$$A_\pm = 2^{-1/2}(a_x \mp i a_y)\,, \quad A_\pm^\dagger = 2^{-1/2}(a_x^\dagger \pm i a_y^\dagger)\,,$$

so that

$$[A_r, A_s] = 0\,, \quad [A_r, A_s^\dagger] = \delta_{rs} \qquad (\text{for } r, s = +, -)\,,$$

and

$$[a_z, a_z^\dagger] = 1\,,$$

we find that the Hamiltonian can be written in the diagonal form:

$$H = \hbar(\omega_c + \Omega)(N_+ + \tfrac{1}{2}) + \hbar(\omega_c - \Omega)(N_- + \tfrac{1}{2}) + \hbar\omega_0(n_z + \tfrac{1}{2}) . \qquad (2)$$

Here $N_\pm = A_\pm{}^\dagger A_\pm$ are the number operators for the quanta of type $\pm$ and $n_z = a_z{}^\dagger a_z$ is the number operator for the quanta of type z. If one now includes the radiation terms in the Hamiltonian, the interaction term H_I which causes transitions of circularly polarized radiation is given by

$$H_I = \frac{ieEe^{-i\omega t}}{\omega}\left\{\frac{1}{m}(p_x \pm ip_y) - \Omega(y \mp ix)\right\} ,$$

where the $\pm$ signs correspond to left and right circularly polarized radiation. It follows that

$$H_I = \frac{ieEe^{-i\omega t}}{2\omega}\left(\frac{\hbar}{m\omega_c}\right)^{1/2}[\omega_c(A_\mp - A_\pm{}^\dagger) \mp \Omega(A_\mp + A_\pm{}^\dagger)] .$$

Following Kemp we shall assume that the temperature is sufficiently low that only the ground state n and the lowest excited states need be considered.

Let us denote a general state vector by the notation $|n_+n_-n_z\rangle$. We emphasize that these are eigenstates of the Hamiltonian to all orders in B. Then we have

$$\langle 1\,0\,0|H_I|0\,0\,0\rangle = \frac{ieEe^{-i\omega t}}{2\omega}\left(\frac{\hbar}{m\omega_c}\right)^{1/2}(\Omega + \omega_c) . \qquad (3)$$

Similarly, we note that the matrix element corresponding to right circularly polarized radiation is given by

$$\langle 0\,1\,0|H_I|0\,0\,0\rangle = \frac{ieEe^{-i\omega t}}{2\omega}\left(\frac{\hbar}{m\omega_c}\right)^{1/2}(\Omega - \omega_c) . \qquad (4)$$

Now from equation (2) we note that $E_{100} - E_{000} = \hbar(\omega_c + \Omega)$ while $E_{010} - E_{000} = \hbar(\omega_c - \Omega)$, so that in equation (3) $\omega = \omega_c + \Omega$ if $2\Omega < \omega$, while in equation (4) $\omega = \omega_c - \Omega$ for all Ω. Hence if $2\Omega < \omega$, radiation of frequency ω comes from two different oscillators (here following Kemp's quantum-mechanical treatment we assume low temperatures) of natural frequencies $\omega_0 = (\omega^2 \mp 2\omega\Omega)^{1/2}$ giving left and right circularly polarized radiation, respectively, the square of the appropriate matrix elements being proportional to $(\omega \mp \Omega)^{-1}$. It is remarkable that this exact result agrees with Kemp's first-order theory. Hence, the exact fractional circular polarization is $q(\omega) = -\Omega/\omega$, which is the same as Kemp's approximate result. For $2\Omega > \omega$, $q(\omega) = 1$.

The above can be extended to discuss linear polarization. In this case if we look in a plane perpendicular to the z-axis and choose the x-axis along the line of sight, the interaction Hamiltonian can be reduced to the form

$$H_I = \frac{ieEe^{-i\omega t}}{\omega}\Bigg[\left(\frac{\hbar}{4m\omega_c}\right)^{1/2}[\omega_c(A_+ - A_- + A_+{}^\dagger - A_-{}^\dagger) + \Omega(A_+ + A_- + A_+{}^\dagger + A_-{}^\dagger)] - i\left(\frac{\hbar\omega_0}{2m}\right)^{1/2}(a_z - a_z{}^\dagger)\Bigg].$$

In this case we find that the fractional linear polarization of the π and σ components is given by

$$q^* = \frac{\Omega^2}{2\omega^2 - \Omega^2} \qquad \text{for } 2\Omega \leq \omega .$$

For small Ω/ω we find that $q^* \sim \frac{1}{2}\Omega^2/\omega^2$, in agreement with the result of Kemp (1970*b*).

For $\omega < 2\Omega$, $q^* = -(\omega + 2\Omega)/(\omega + 3\Omega)$.

III. DISCUSSION

Since our exact result for the $q(\omega)$ agrees with Kemp's first-order result, it would appear that the discrepancy between theory and observations in the infrared would have to be sought elsewhere. However, it must be emphasized that our results are only valid at low temperatures ($kT \ll \hbar\omega$) and hence not applicable in the infrared to a hot or moderately hot white dwarf such as Grw+70°8247.

Finally, we note that it should be possible to extend our results to higher temperatures. Kemp has solved the classical problem analytically at high temperatures ($kT \gg \hbar\omega$). We are presently investigating the quantum-mechanical problem at all temperatures.

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Note added in proof.—The effect of the magnetic field on the distribution of oscillator frequencies was taken into account by Kemp in his classical calculation. The inclusion of this effect in the quantum-mechanical case leads to a value of q in agreement with Kemp's classical results. We are grateful to Professor J. C. Kemp for pointing this out, as well as for other valuable comments. The corresponding value for q^* is given by

$$q^* = [(1+t)^{1/2} + (1-t)^{1/2} - 2(1-t^2)^{1/2}]/[(1+t)^{1/2} + (1-t)^{1/2} + 2(1-t^2)^{1/2}] \quad \text{for } t \leq 1$$

and

$$q^* = [s^{1/2} - 2(1+s)^{1/2}]/[s^{1/2} + 2(1+s)^{1/2}] \quad \text{for } 1 > s > 0\,,$$

where $s \equiv t^{-1} \equiv \omega/2\Omega$.