(*Reprinted from Nature Physical Science, Vol. 232, No. 35, pp. 193–195, August 30, 1971*)

Lense-Thirring Type Gravitational Forces Between Disks and Cylinders

BECAUSE of the widespread interest in measuring forces between spinning cylinders and disks[1,2] and the speculative nature of the conclusions reached in previous discussions, we have made a quantitative calculation of the magnitude of the forces. As in earlier calculations[3], our approach is based on a weak field low velocity (EIH) approximation to general relativity. By methods familiar from potential theory in electrodynamics we first compute the first order corrections to a flat space metric due to the presence of a spinning cylinder.

Let $h_{\alpha\beta}$ ($\alpha, \beta, \ldots = 0, \ldots, 3$) denote the small corrections to a flat space metric in a cartesian coordinate system, that is:

$$h_{\alpha\beta} = g_{\alpha\beta} - \eta_{\alpha\beta} \tag{1}$$

where $\eta_{\alpha\beta}$ is the Lorentz metric with diagonal elements given by $(1, -1, -1, -1)$. In weak field approximation[4]

$$h_{00}(x) = -2\int \frac{T_{00}(x')\, d^3x'}{|x - x'|} \tag{2a}$$

$$h_{0i}(x) = -4\int \frac{T_{0i}(x')\, d^3x'}{|x - x'|} \tag{2b}$$

$$h_{ij} = h_{00}\,\delta_{ij} \tag{2c}$$

where $T_{\alpha\beta}$ is the stress-energy tensor of the rotating-cylinder ($i, j, \ldots = 1,2,3$). We take $G = c = 1$. Keeping terms to first order in the velocity, then

$$T_{00} = \varepsilon \tag{3a}$$

and

$$T_{0i} = -\varepsilon\, v_i \tag{3b}$$

The mass density of the cylinder material is ε and v is the velocity of a mass element. For our cartesian coordinate system we choose the z-axis parallel to the cylinder spin axes in Fig. 1 and place the origin at the centre of cylinder 1. (The presence of cylinder 2 is ignored in the field calculation.) A mass element of cylinder 1 with coordinates (x, y, z) then has velocity

$$v = \hat{x}\,(-\,\rho\,\omega\,\sin\varphi) + \hat{y}\,(\rho\,\omega\,\cos\varphi) \qquad (4)$$

where $\rho = (x^2 + y^2)^{\frac{1}{2}}$ and $\tan\varphi = y/x$. The denominator of the integrand in equation (2) is expanded in terms of Bessel functions:

$$\frac{1}{|x - x'|} = \sum_{m=-\infty}^{\infty} \int_0^{\infty} dk\; e^{im(\varphi-\varphi')}\, J_m(k\rho)\, J_m(k\rho')\, e^{-k(z_> - z_<)} \qquad (5)$$

where $z_>$ is either the z-coordinate for the field point, x, or the z-coordinate for the source point, x', whichever is the larger. $z_<$ is correspondingly the smaller of the two[5].

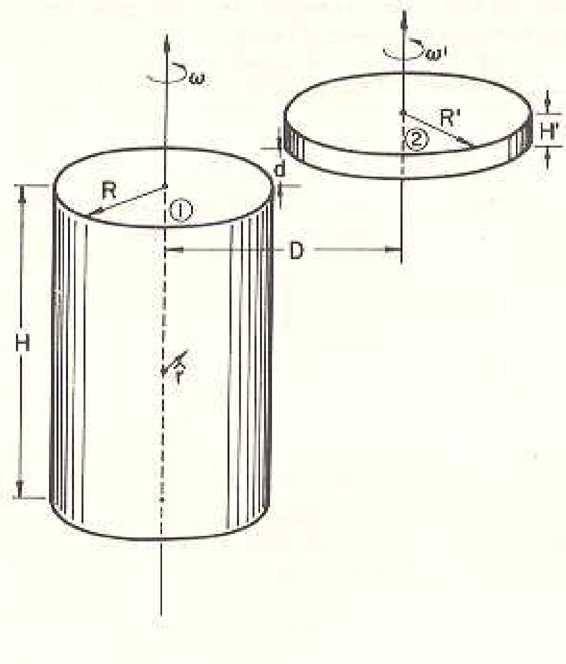

Fig. 1 The spinning disk and cylinder.

Equations (3)–(5) when substituted into equations (2) result in

$$h_{oo}(x) = -8\pi\varepsilon R \int_0^\infty \frac{dk}{k^2} f(k,z)\, J_o(k\rho)\, J_1(kR) \tag{6a}$$

and

$$h_{oi}(x) = \frac{16\,\pi R^2\,\varepsilon\,\omega}{\rho}(-y,x,o) \int_0^\infty \frac{dk}{k^2} f(k,z)\, J_1(k\rho)\, J_2(kR) \tag{6b}$$

where

$$f(k,z) = \begin{cases} 1-e^{-kH/2}\cosh kz;\ |z| \leq H/2 \\ e^{-k|z|}\sinh kH/2;\ |z| \geq H/2 \end{cases} \tag{6c}$$

To compute the interaction force between the two cylinders of Fig. 1 we obtain the force on a mass element of cylinder 2 generated by the field of cylinder 1 (equations (6)) and then integrate over the volume of cylinder 2. The force on the mass elements is computed from the usual Euler–Lagrange equations[6]:

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} = F_i \tag{7}$$

where the "kinetic energy" T is obtained from the metric

$$T = -1/2\, dm\,\{(1+h_{oo})\,\dot{t}^2 + 2I\rho\dot{\varphi}\dot{t} - (1-h_{oo})(\dot{x}^2+\dot{y}^2+\dot{z}^2)\} \tag{8a}$$

where

$$I = 16\pi\, R^2\,\varepsilon\,\omega \int_0^\infty \frac{dk}{k^2} f(k,z)\, J_1(k\rho)\, J_2(kR) \tag{8b}$$

A dot denotes differentiation with respect to the path parameter s. Substitution of equations (8) in equation (7) then gives the equations of motion for the mass element dm.

Rather than using these equations to find the trajectory in the given coordinate system we prescribe the path and compute the force. The force, F_i, in equation (7) is viewed as an applied force necessary to counteract the forces between the cylinders. At the current level of approximation for these forces proper time and coordinate time may be considered equal. As an explicit example the force to lowest order in the z direction is given by

$$F_z = \mathrm{d}m\, \ddot{z} + 1/2\, \mathrm{d}m \frac{\partial h_{00}}{\partial z} + \mathrm{d}m\, \rho\, \dot{\varphi} \frac{\partial I}{\partial z} \qquad (9)$$

The first term on the right hand side is the inertial term and the second quantity is just Newtonian gravity. The third term is the force of present interest and results from the rotations. The forces of equation (7) are applied forces necessary to maintain the prescribed trajectory and are consequently opposite in sign to the forces between the cylinders themselves. Analogous expressions to equation (9) are obtained for the forces in the other directions.

A straightforward integration over the volume of the second cylinder in the specified configuration depicted in Fig. 1, using standard results from the theory of Bessel functions[7], gives the spin–spin force, F_{ss} (acts parallel to rotation axes), and the Lense-Thirring type force, F_{LT} (acts perpendicular to rotation axes):

$$F_{ss} = -\frac{64\, SS'}{R^2 R'^2 HH'} \frac{G}{c^2} \int_0^\infty \frac{\mathrm{d}k}{k^3} g_2(k)\, J_0(kD) \qquad (10a)$$

and

$$F_{LT} = -\frac{64\, SS'}{R^2 R'^2 HH'} \frac{G}{c^2} \int_0^\infty \frac{\mathrm{d}k}{k^3} g_2(k)\, J_1(kD) \qquad (10b)$$

where

$$g_v(k) = \mathrm{e}^{-kd} (1-\mathrm{e}^{-kH}) (1-\mathrm{e}^{-kH'})\, J_v(kR)\, J_v(kR') \qquad (10c)$$

S and S' are the spin angular momentums of cylinders 1 and 2 respectively.

For comparison we give the Newtonian forces between the cylinders when the spin axes coincide. In such a case

$$F_N = 4\, \pi^2\, \varepsilon^2\, RR' \int_0^\infty \frac{\mathrm{d}k}{k^3} g_1(k) \qquad (11)$$

Frehland's result[2] for the Lense-Thirring force can be obtained by considering F_{LT} to be doubled by placing above cylinder 2 in Fig. 1 an additional cylinder aligned and identical to cylinder 1. Then one considers the doubled value of equation (10b) in the case $R = R' = D$, with the limiting values $H \to \infty$, $d \to 0$, $H' \to 0$. This result may also be obtained by letting $H \to \infty$ equations (6) and then integrating over a thin disk as described by Frehland[2].

In the limit of large d equation (10a) reduces to the spin–spin force term between two spinning spheres given by[3]

$$F_{ss}\,(\text{sphere}) = -\frac{6\,SS'}{d^4}\frac{G}{c^2} \tag{12}$$

where d is the separation distance between their centres and S and S' are their spins.

Fig. 2 contains graphs of these forces for the following cases: (*a*) F_{ss} and F_{LT} are plotted as functions of D where we choose $H=20.0$ cm, $H'=1.0$ cm, $d=0.1$ cm. (*b*) F_{ss} is plotted as a function of d with $H=H'=5.0$ cm and $D=0$. (*c*) $|F_{ss}/F_N|$ is plotted as a function of d with same parameter choices as in (*b*). In all cases the rotations are assumed to be near the breaking strength of the material with $\varepsilon=8$ g cm^{-3} and tensile strength of 1.0×10^9 dynes cm^{-2}—values representative of steel. The radii of all cylinders are taken to be 10.0 cm—consistent with the largest disks reported to have been suspended by a Beams's magnetic suspension[9]. The integrals in equation (10) and (11) are evaluated numerically.

We note first that $F_{ss}(d)$ is not proportional to any inverse power of d for small d. Rather than behaving as d^{-1} as postulated by Salisbury[1] or as d^{-4} (see equation (12)) the force attains a maximum value as the separation between the disks goes to zero. In the specific example chosen $F_{ss}(d)\to 2.955\times10^{-14}$ dynes as $d\to0$.

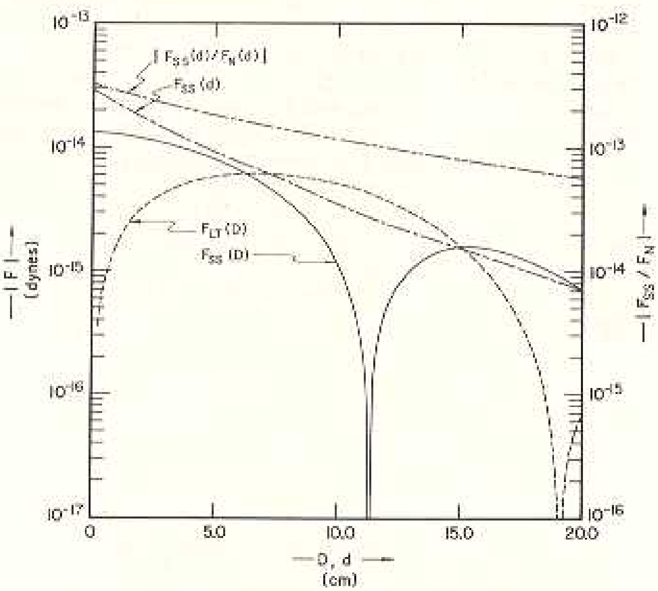

Fig. 2 The force between the disk and cylinder for the various configurations discussed in the text.

Second, the maximum values for F_{ss} and F_{LT} are of the same order of magnitude with the F_{ss} generally larger than F_{LT}. That this is true in general can be seen from equations (10). F_{ss} reaches its maximum when $D=0$ because $J_0(\zeta)\leq 1$ for $0\leq\zeta<+\infty$ and they are equal only when $\zeta=0$. Because $J_1(\zeta)=0$ for $\zeta=0$, F_{LT} attains maximum absolute value for some non-zero value of D. Also, because $J_1(\zeta)<1$ for $0\leq\zeta<+\infty$ we see that F_{ss} (max) $>$ F_{LT} (max). This does not imply that F_{ss} is the more amenable to measurement but only shows that if one force is measurable but the other one is not then this implies a basic difference in character and not a large difference in magnitude. In the example of Fig. 1, F_{LT} (max) $=$ 6.16×10^{-15} dynes at $D=6.6$ cm and F_{ss} (max) $=1.32\times 10^{-14}$ dynes at D=0. These forces are a few orders of magnitude smaller than the force calculated by Frehland[2] because of the size of the objects considered.

Third, Fig. 2 also shows how the forces between the cylinders change from repulsive to attractive as the separation distance D increases. This result may be anticipated from the force between two spinning spheres. If $\hat{r}$ is a unit vector along the line joining the centres of the cylinders then the force between them (for parallel spins) changes from repulsive to attractive as $\hat{r}$ changes from being parallel to the spins to nearly perpendicular to them. This is precisely analogous to the spinning sphere result[3].

Fourth, the numerical values for the ratio F_{ss}/F_N show that a measurement of spin–spin forces between disks in the manner suggested by Salisbury[1] requires an accuracy in the Newtonian force on the order of 1 part in 10^{13} which demands an improvement by 9 orders of magnitude over previous measurements[9].

An interesting, but exceedingly small, effect which might be considered is the strain in a cylinder spinning about its axis due to its self induced spin–spin force. Similar analysis to that described above leads to the following result for the spin–spin force induced change in a length of a cylinder spinning at breaking strength:

$$\Delta l \simeq 3.0\times 10^{-26}\left(\frac{\varepsilon T}{E}\right)\left\{R^3\int_0^\infty \frac{d\zeta}{\zeta^4}e^{-\zeta\delta/2}\left(\sinh\frac{\zeta\delta}{2}-\frac{\zeta\delta}{2}\right)[J_2(\zeta)]^2\right\}\text{cm} \qquad (13)$$

where $\delta=H/R$, $T=$ tensile strength of material and E is Young's modulus. Equation (13) represents a strain only due to the spin–spin force and ignores contributions from stresses perpendicular to the spin axis. Numerical integration shows that about the best one could hope to do in choosing the dimensional parameters of the disk would be to make $\{\ \} \simeq 10^4$. The factor involving the material parameters of the disk ($\varepsilon T/E$), for metals is of the order of 10^{-2} or 10^{-3} and even for rubber cannot be made much greater than about 1.

A quantum analogue of these spin forces would be an interaction with a body whose nuclear spins are aligned[10]. The magnitude of this effect is even smaller than for the classical effects considered here.

This measurement of the spin–spin gravitational force between laboratory size disks or cylinders as suggested by Salisbury[1] is presently not possible because it requires a measurement of exceedingly small forces. The Lense-Thirring type force between such objects discussed by Frehland[2] is not any larger.

One of us (R. F. O'C.) was a participant in the relativity workshop at Montana State University. R. F. O'C. thanks Dr K. Nordtvedt, for his hospitality. S. N. R. thanks W. O. Hamilton and P. B. Pipes for numerous discussions.

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Received August 16, 1971.

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Printed in Great Britain by Flarepath Printers Limited, Watling Street, Colney Street, St. Albans, Herts.