On the Origin of Magnetic Fields in White Dwarfs and Neutron Stars. - II

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Please note the correction in the title of this paper; also eq. (12) should read

$$M(\mu, B) = -\frac{B^{\frac{1}{2}}}{2\pi^{2}}\,\alpha^{\frac{1}{2}}\,\mu F(\mu)\,\Sigma_2\,.$$

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It has been pointed out (1,2) that, even when the external magnetic field H is zero, solutions of the equation (3)

$$B = H + 4\pi M(B) \tag{1}$$

exist with nonzero values for the magnetic induction B, in addition to the $B = 0$ solution. The suggestion was also made (1) that these $B \neq 0$ solutions (the so-called (1) LOFER solutions) could explain the origin of magnetic fields in white dwarfs and neutron stars. However, before one can accept this suggestion it is necessary to examine whether or not these solutions correspond to minima in $G(B)$, the Gibbs free energy and, in addition, calculate which state has the lowest minimum value of $G(B)$. We have already examined this question in the case of a nonrelativistic electron gas (4) and we concluded that the minimum with the lowest value of $G(B)$ corresponds to the $B = 0$ solution from which it followed that it is very unlikely that the LOFER states correspond to physically realizable configurations. These conclusions are immediately applicable to metals but, in the case of white dwarfs and neutron stars, a relativistic treatment is required. It is our purpose here to provide such a treatment and, as we shall see, exactly the same conclusions are reached as in the nonrelativistic case. We take the temperature $T = 0$ (complete degeneracy) and use units $\hbar = c = m_e$ (mass of the electron) $= 1$, so that $B_q = \alpha^{-1}$, where α is the fine-structure constant, and B_q, which is numerically equal to $4.4 \cdot 10^{13}$ G, is the so-called quantum critical magnetic field, beyond which the theory of quantum electrodynamics breaks down (5). We define

$$b = 2\pi\mu/(B/B_q) , \tag{2}$$

where μ denotes the chemical potential, and we consider only $b \gg 1$, which is physically the most interesting situation. Now the energy eigenvalues of a relativistic electron

(1) H. J. LEE, V. CANUTO, H. Y. CHIU and C. CHIUDERI: *Phys. Rev. Lett.*, **23**, 290 (1969); *Nature*, **225** 47 (1970).
(2) R. F. O'CONNELL: *Lett. Nuovo Cimento*, **3**, 218 (1970).
(3) D. SHOENBERG: *Phil. Trans.*, A **255**, 85 (1962).
(4) R. F. O'CONNELL and K. M. ROUSSEL: *Nature* (in press).
(5) L. D. LANDAU and E. M. LIFSHITZ: *The Classical Theory of Fields*, second edition (Reading, Mass., 1965), footnote on p. 235.

in a magnetic field H oriented along the z-axis are (6)

$$(3) \qquad E = \{1 + P_z^2 + 2n(H/H_0)\}^{\frac{1}{2}} ,$$

where $n = 0, 1, 2, \ldots$. The Lifshitz-Kosevich (7) analysis is now applied (with suitable modifications to take spin effects into account in a relativistic way) to this particular dispersion law and enables us to calculate $G_0(\mu, H)$, the Gibbs free energy for a noninteracting (denoted by the subscript zero) electron gas. Thus, using familiar thermodynamical relations, we obtain expressions for the magnetic moment $M(\mu, H)$ and the number of particles $N(\mu, H)$. Proceeding now as in the nonrelativistic case (1), the magnetic moment for interacting electrons is got (8) by simply replacing H by B in $M(\mu, H)$ to give $M(\mu, B)$. However, the Gibbs free energy of the interacting system, $G(\mu, B)$ say, is not $G_0(\mu, B)$ but is given by (8)

$$(4) \qquad G(\mu, B) = G_0(\mu, B) + 2\pi M^2(B) .$$

Recalling that $b \gg 1$, in our subsequent equations we will, generally, only write down those terms which contribute to the final result. In addition, all quantities are calculated for unit volume. Hence,

$$(5) \qquad G_0(\mu, B) = -\frac{1}{24\pi^2} f(x) - \frac{(2\mu)^{\frac{5}{2}}}{4\pi^{\frac{5}{2}}} D(\mu) b^{-\frac{5}{2}} \Sigma_1 ,$$

where

$$(6) \qquad x = \{(\mu + 1)^2 - 1\}^{\frac{1}{2}} ,$$

$$(7) \qquad f(x) = x(x^2 + 1)^{\frac{1}{2}}(2x^2 - 3) + 3 \sinh^{-1} x ,$$

$$(8) \qquad \Sigma_1 = \sum_{r=1}^{\infty} \frac{\cos[A(\mu) br - \pi/4]}{r^{\frac{5}{2}}} ,$$

$$(9) \qquad A(\mu) = 1 + \frac{\mu}{2}$$

and

$$(10) \qquad D(\mu) = (1 + \mu)^{-1} .$$

It follows that

$$(11) \qquad N = \frac{x^3}{3\pi^2} \left\{ 1 + \frac{3\pi^{\frac{1}{2}}}{2} \left(1 + \frac{\mu}{2}\right)^{-\frac{1}{2}} b^{-\frac{3}{2}} \Sigma_2 \right\}$$

and

$$(12) \qquad M(\mu, B) = -\frac{B^{\frac{1}{2}}}{2\pi^2} \alpha^{\frac{3}{2}} \mu F(\mu) \Sigma_2 ,$$

(6) M. H. JOHNSON and B. A. LIPPMANN: *Phys. Rev.*, **76**, 828 (1949); **77**, 702 (1950).
(7) E. M. LIFSHITZ and A. M. KOSEVICH: *Sov. Phys. JETP*, **2**, 636 (1956).
(8) A. B. PIPPARD: *Proc. Roy. Soc.*, A **272**, 192 (1963).

where

$$\Sigma_2 = \sum_{r=1}^{\infty} \frac{\sin[A(\mu)br - \pi/4]}{r^{\frac{3}{2}}}$$

and

$$F(\mu) = A(\mu)D(\mu) = \left[\left(1 + \frac{\mu}{2}\right)\Big/(1+\mu)\right]. \tag{13}$$

Our expression for M is the same as we obtained [2,4] in the nonrelativistic case ($\mu \ll 1$) except for the factor $F(\mu)$. Thus, substituting eq. (12) into eq. (1) and taking $H = 0$, we find that

$$B + dB^{\frac{1}{2}} = 0\,, \tag{14}$$

where

$$d = 2\pi^{-2}\alpha^{\frac{1}{2}}\mu F(\mu)\Sigma_2\,. \tag{15}$$

It follows from eq. (14) that $B = 0$ or $B^{\frac{1}{2}} = -d$. The latter implies $B = d^2$ and hence

$$\frac{B}{B_0} = \frac{4}{\pi^4}|\Sigma_2|^2 F^2(\mu)\alpha^2\mu^2\,. \tag{16}$$

In the nonrelativistic ($\mu \ll 1$) and extreme relativistic ($\mu \gg 1$) limits we see from eq. (13) that $F(\mu)$ takes the values 1 and 0.5, respectively.

We now examine the vital question of thermodynamical stability by calculating $G(\mu, B)$. Using eqs. (4), (5), (12) and (16) we obtain, to the order required,

$$G(\mu, B) = -\frac{1}{24\pi^2}f(x) + \frac{(2\mu)^{\frac{1}{2}}}{8\pi^{\frac{3}{2}}}|\Sigma_2|F(\mu)b^{-\frac{3}{2}}\,. \tag{17}$$

It is clear from eq. (17) that a nonzero value of B results in a larger value for $G(\mu, B)$, relative to the $B = 0$ case. Thus, as in the nonrelativistic case [1], *we conclude that the most stable configuration is that for which* $B = 0$. In other words, the LOFER states (even though they may correspond to minima in $G(B)$, as many of them do) always give higher values for $G(B, \mu)$ than the $B = 0$ configuration. We conclude that LOFER *states are unlikely to exist in white dwarfs and neutron stars*. This is actually a welcome conclusion since if the LOFER states were associated with the ground-state energy of the system then one would expect large magnetic fields in *all* white dwarfs, which is contrary to observations [9]. The fact that large magnetic fields are found in *some* white dwarfs [10,11] might be due to the occurrence of some unusual circumstance resulting in the system going into a *metastable* LOFER state instead of the thermodynamically more stable $B = 0$ state.

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[9] J. R. P. Angel and J. D. Landstreet: *Astrophys. Journ. Lett.*, **162**, L61 (1970).
[10] J. C. Kemp, J. B. Swedlund, J. D. Landstreet and J. R. P. Angel: *Astrophys. Journ. Lett.*, **161**, L77 (1970).
[11] J. R. P. Angel and J. D. Landstreet: *Astrophys. Journ. Lett.*, **164**, L15 (1971); **165**, L67, L71 (1971).