$$\Lambda(a)-\Lambda_0=\left\{\lim_{a^*\to\delta}\left[J_k(a^*)-\tilde{J}_k(a^*)\right]-J_k(a_0)\right\}+\tilde{J}_k(a). \tag{18}$$

In Eqs. (17) and (18) the bracketed terms assure us that the limiting solution matches properly to the exact solution. It is to be noted that the limit $a^*\to\delta$ is to be taken after the exact integration is performed. Thus, the exact solutions are required, in general, to evaluate these limits.

Using this procedure, we can immediately write down the asymptotic dependence of the solutions for $a\to 0$, viz.,

$$t(a)=\lim_{a^*\to 0}\left[I_k(a^*)-(\tfrac{1}{3}\beta B^2)^{-1/2}\tfrac{1}{3}a^{*3}\right]+(\tfrac{1}{3}\beta B^2)^{-1/2}(\tfrac{1}{3}a^3).$$

Since a^* must drop out of the solution and, moreover, since we have chosen $a=0$ at $t=0$ the limit term above must vanish. Similarly, as $a\to 0$ we have for λ

$$\lambda=\lambda_0\lim_{a^*\to 0}\left\{\exp[J_k(a^*)](a^*)^{-(3/\beta)^{1/2}}\right\}\exp[-J_k(a_0)]a^{(3/\beta)^{1/2}}.$$

Here the limit term must be determined from the exact solution as given in the text. The other limits are computed similarly and agree in form with those of the text.

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[1]R. E. Morganstern, preceding paper, Phys. Rev. D 4, 278 (1971).

[2]R. H. Dicke, Phys. Rev. 125, 2163 (1962).

[3]It is important to emphasize that the scalar field, expansion parameter, and density are here expressed as functions of the cosmic time associated with the particular units (α) under consideration. The reason for this is that only the cosmic time t_α corresponds to a Robertson-Walker metric (see Ref. 1). Therefore, our results must be expressed as functions of the cosmic time if they are to be meaningful.

[4]$\alpha=0$, 1, -1 correspond, respectively, to units for which G, m, or Gm is constant (see Ref. 1).

[5]Note that we do not choose the lower limit a_0 to be zero in Eq. (4), since as $a\to 0$ the integral is logarithmically divergent and the resulting logarithmic ratio $\ln(a/a_0)$ would become undefined as both a and $a_0\to 0$. The net result is that the limiting behavior of λ/λ_0 becomes uncertain to within a multiplicative constant. This circumstance is avoided by our choice of $a_0\neq 0$.

[6]The constant cannot be determined by this method, but only from the exact solutions.

[7]This UT differs from Eqs. (4) of Ref. 1 because here we are transforming from the $\alpha=0$ formalism to the general α formalism, whereas there we transformed from $\alpha=1$ (unbarred) to the general α (barred formalism). It might also be in order to add the rather obvious comment that the UT does not affect the equation of state $p=\epsilon\rho$, since ϵ is dimensionless and p and ρ have essentially the same dimensions.

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Effect of Gravitational Light Deflection on the Proposed Gyroscope Test of the Lense-Thirring Effect

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From a previous analysis of the proposed gyroscope test of the Lense-Thirring effect, it is known that all perturbations that contribute more than 10^{-3} sec/yr to the precession of the gyroscope must be taken into account. A frame of reference is obtained by rigidly attaching the gyro housing to a telescope which is trained on some reference star. Here we point out that the deflection of the light from the reference star due to the sun's gravitational field can give rise to an apparent precession of the gyroscope equal to $(4.1\times 10^{-3}\cot\frac{1}{2}\theta)$ sec, where θ is the angle between the earth-sun direction and the earth-star direction ($\pi\geq\theta>0$). Thus, over a six-month period, this could amount to as much as 1.75 sec, depending on the angle which the sun-star line makes with the ecliptic.

In a recent analysis of the proposed gyroscope test of the Lense-Thirring effect, Barker and O'Connell[1] pointed out that all perturbations that contribute more than 10^{-3} sec/yr to the precession of the gyroscope must be taken into account. In particular, with regard to the Lense-Thirring contribution to the precession of a gyroscope in a satellite in a circular polar orbit 300 miles above

the earth and with its initial spin direction oriented along the orbital angular momentum direction of the satellite, the difference in the precession angles calculated in Einstein and Brans-Dicke theories amounts to only 2.7×10^{-3} sec/yr. However, it is expected that measurements accurate to 10^{-3} sec/yr will be possible.[2,3] A frame of reference is obtained by rigidly attaching the gyro housing to a telescope which is trained on some reference star and thus one is interested in difference measurements between the gyro and telescope readings.[2]

It is our purpose here to point out that the deflection of the light from the reference star due to the sun's gravitational field can give rise to an apparent precession of the gyroscope which, over a six-month period, could amount to as much as 1.75 sec, depending on the angle, λ say, which the sun-star line makes with the ecliptic plane.

The general-relativistic deflection of light from a star due to the sun's gravitational field may be calculated very simply by the use of a general relativistic potential.[4] To order G, it was shown *rigorously*[5] that light is deflected by a heavy object by twice the amount predicted by Newtonian theory, which of course explains the well-known fact that Soldner's Newtonian corpuscular-theory calculation of the deflection of light *grazing* the sun is one-half the value calculated by Einstein. A recent discussion of Soldner's method has been given by Weber,[6] and we wish to generalize this treatment to include all possible orientations of the earth, sun, and star. Let θ be the angle between the earth-sun direction and the earth-star direction ($\pi \geq \theta > 0$), R the distance of closest approach of the light trajectory to the sun, and d the distance between the earth and the sun. The distance along the light trajectory, from the point (P say) at which it is nearest to the sun, to the earth will be denoted by a. Thus

$$\cos\theta = \pm a/d, \tag{1}$$

the + and − signs corresponding to $\frac{1}{2}\pi \geq \theta > 0$ and $\pi \geq \theta \geq \frac{1}{2}\pi$, respectively.

For the purpose of calculating the light deflection, Δ say, it is convenient to follow Weber[6] and choose a coordinate system such that the path of light is in the xy plane and parallel to the x axis at the origin, which for our present purposes we choose to be at the point P. Then

$$\frac{d^2y}{dt^2} = -\frac{2GM}{r^2}\frac{y}{r}, \tag{2}$$

where M is the mass of the sun and $r^2 = x^2 + y^2$. Consider that the star is at $x = -\infty$. Thus

$$\Delta = \left(\frac{dy}{dx}\right)_{x=-\infty} - \left(\frac{dy}{dx}\right)_{x=\pm a}, \tag{3}$$

the ± signs corresponding to the same θ values as above. It follows that

$$\Delta = \frac{2GM}{Rc^2}\left(1 \pm \frac{a}{d}\right). \tag{4}$$

Thus, for all values of θ, use of Eqs. (1) and (4) enable us to write

$$\Delta = \frac{2GM}{Rc^2}(1 + \cos\theta). \tag{5}$$

Since $R = d\sin\theta$, we can thus write

$$\Delta = \frac{2GM}{c^2 d}\cot\tfrac{1}{2}\theta$$

$$= (4.1\times10^{-3}\cot\tfrac{1}{2}\theta) \text{ sec}. \tag{6}$$

In Brans-Dicke theory[7] this value is reduced by a factor of $(3+2\omega)/(4+2\omega)$, equal to $\frac{15}{16}$ if we make the common choice of $\omega = 6$ for the dimensionless coupling constant ω.

The minimum value of θ for which the star is within view of the telescope is $R_\odot/d$, where $R_\odot$ is the radius of the sun. The corresponding value of Δ, say $\Delta_{\max}$, is thus

$$\Delta_{\max} = \frac{4GM}{c^2 R_\odot} = 1.75 \text{ sec}, \tag{7}$$

which is the well-known result of Einstein for light grazing the sun. Thus for $\pi \geq \theta \geq \theta_{\min}$ we see that

$$0 \leq \Delta \leq 1.75 \text{ sec}. \tag{8}$$

For the intermediate value of $\theta = \frac{1}{2}\pi$, we obtain $\Delta = 4.1\times10^{-3}$ sec.

The change in θ as the earth revolves around the sun depends on the angle λ. Of course, after exactly one year, the over-all change in θ is zero, for all values of λ, but even though it is customary to quote precession angles per year the gleaning of maximum information from the experiment demands a continuous monitoring of the output of the orientation of the gyroscope.[8] If the star chosen for the experiment is such that the sun-star line is at right angles to the ecliptic ($\lambda = \frac{1}{2}\pi$), θ remains fixed. On the other hand, for $\lambda = 0$, θ will change by π in six months with a corresponding change of 1.75 sec in Δ. It is clear that such a change cannot be ignored in the calculation of the *true* angle of precession of the gyroscope.

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Radiation from a Beam of Modulated Electrons (The Schwarz-Hora Effect)

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The mechanism of radiation from a beam of modulated electrons (Schwarz-Hora effect) is discussed. It is shown that the effect should be quadratic (at a given velocity of the electrons) in the current arriving to the screen, being due to a coherent emission from the electrons in the beam; no emission linear in the current, specific to the modulation, is expected. In the conditions of the experiment, however, the calculated radiated power turns out to be at least 10^3 times smaller than the observed power.

I. INTRODUCTION

The purpose of this paper is to discuss the interesting effect recently observed by Schwarz and Hora[1,2]: When a beam of electrons (Fig. 1) passes through a thin crystal with a superimposed laser beam, it is observed (this is the Schwarz-Hora effect) that the electrons produce light of the same color of the laser light when they impinge on a *nonluminescent* screen.

This effect poses two problems: (a) to calculate the modulation of the wave function of each electron when it crosses the crystal in the presence of the laser light, and (b) to try to clarify the mechanism by which the modulated electrons produce light when they impinge on the nonluminescent screen.

Problem (a) is a standard though complicated problem in quantum mechanics[3]; although we have solved it quite generally (in the eikonal approximation), in this note we shall only report some schematic formulas – by now rather well known[2,4] – which are necessary for the discussion of problem (b). It is indeed to this problem that we shall focus our attention here. We can anticipate the results of this note as follows:

(1) The rate of radiation at the laser frequency $\bar{\omega}$ by an *individual electron* is substantially independent of whether the wave function of such an electron is or is not modulated at frequency $\bar{\omega}$; in other words, if we call "incoherent" the radiation emitted by the individual electrons impinging on the screen, the modulation of the wave function of the electrons does not produce an enhancement at frequency $\bar{\omega}$ above the ordinary transition or bremsstrahlung radiation in the incoherent radiation rate.

(2) There is the possibility of a preferential emission at frequency $\bar{\omega}$ only when the cooperative, or more precisely, coherent effect of all the modulated electrons is taken into account.

(3) If the radiation observed by Schwarz and Hora is due to the above coherent emission, the power emitted from the screen should be proportional (for a given electron velocity) to the square of the electron current; it becomes in our opinion very important to establish this point experimentally.

(4) The power emitted at frequency $\bar{\omega}$ from the screen by the above coherent mechanism is, however, according to our estimates, *at least* 10^3 times smaller than the power of 10^{-10} W observed by Schwarz and Hora.

II. THE MODULATED WAVE FUNCTION

For simplicity we consider here a one-dimensional schematization; the electron is assumed to