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Origin of Magnetic Fields in White Dwarfs and Neutron Stars

THE origin of magnetic fields in degenerate matter is an interesting question at present because of the discovery of large magnetic fields in some white dwarfs[1,2], and because of the even higher fields of the order of the critical magnetic field $B_c \equiv (m^2c^3/e\text{h}) = 4.4 \times 10^{13}$ G that are thought to exist in pulsars (accepting the rotating neutron star theory[3]). An equally intriguing question is why some—but not all[4]—white dwarfs are magnetic. The fact that some supernovae do not have associated pulsars might also lead one to speculate that perhaps the collapse of all supernovae does indeed result in a neutron star, but that all neutron stars are not magnetic.

The incisive suggestion of Shoenberg[5] that the magnetic moment M of an electron gas is a function of the magnetic induction B and not the external field H, that is

$$B = H + 4\pi M(B) \qquad (1)$$

immediately leads to the conclusion that even when H is zero, solutions of equation (1) exist with non-zero values for B, in addition to the $B=0$ solution. It was shown numerically[6], and later analytically[7], that in fact in the case $H=0$, these values obtained for B increase with density and are consistent with the values found in white dwarfs and neutron stars. Two important questions remain to be answered, however: (*a*) do these solutions correspond to minima in $G(B)$, the Gibbs free energy—a necessary condition for the existence of a magnetically stable state, and (*b*), for the states which do correspond to a minimum if $G(B)$, which state has the lowest value of $G(B)$? As emphasized by Azbel[8], equation (1) alone "... can give not only equilibrium solutions, but also metastable and even absolutely unstable solutions". We will show here that the condition for the so-called LOFER states[6] (solutions of equation (1) with $B \neq 0$) to correspond to a maximum or minimum value of $G(B)$ depends very sensitively on the density, such that slight changes in the density (either an increase or a decrease) can switch the configuration from, say, a stable to an unstable LOFER state. But another and more important conclusion is that the minimum with the lowest value of $G(B)$ corresponds to the $B=0$ solution. As a minimum with a lower value of $G(B)$ is thermodynamically more stable than a minimum with a larger value of $G(B)$, it is thus very probable that the system will have $B=0$. In other words, it is very probable that the LOFER states do not correspond to physically realizable configurations. On the other hand, if some (as yet unknown) mechanism exists which could sometimes put the configuration into a LOFER state, then, depending on the density, the system may exist in a metastable state with a non-zero B value—which would be consistent with the results of the recent observations of white dwarfs[1,2,4] which show that some, but only a small percentage of them, have large magnetic fields (because if any LOFER state had a value of $G(B)$ lower than the $B=0$ state, then all white dwarfs and neutron stars should be magnetic).

As we have previously pointed out[7], a convenient starting-point for deriving the magnetic moment of a relativistic electron gas is the article of Lifshitz and Kosevich[9], who considered an arbitrary dispersion law. We have applied these results, with some necessary modifications, to treat the case of the energy eigenvalues of a relativistic electron. It turns out, however, that exactly the same basic conclusions are reached in the case of a non-relativistic electron gas. Thus, with the purpose of making the algebra less cumbersome and the ideas more transparent, we will concentrate here on the non-relativistic electron gas; this also has the advantage of using what is probably a more familiar starting point[10] than ref. 9. The detailed relativistic computations will be presented elsewhere. We take the temperature $T=0$ (complete degeneracy) and use units $h=c=m_e$ (mass of the electron)$=1$ so that $B_c=\alpha^{-1}$, where α is the fine structure constant. We define

$$b \equiv [2\pi\mu/(B/B_c)] \qquad (2)$$

where μ denotes the chemical potential (Fermi energy), and we consider only $b \gg 1$, which is physically the most interesting situation. The basic quantity we calculate is $G_0(\mu,H)$, that is, the Gibbs free energy for a non-interacting (denoted by the subscript zero) electron gas, from which the expressions for the magnetic moment $M(\mu,H)$ and the number of particles $N(\mu,H)$ are derived (all quantities are calculated for unit volume). The magnetic moment for interacting electrons is now obtained, following Schoenberg, simply by replacing H by B in $M(\mu,H)$ to give $M(\mu,B)$. As Pippard points out, however[11], the Gibbs free energy of the interacting system, $G(\mu,B)$ say, is not $G_0(\mu,B)$ but is given by

$$G(\mu,B)=G_0(\mu,B)+2\pi M^2(B) \tag{3}$$

Recalling that $b \gg 1$, in our subsequent equations we will usually only write down those terms which contribute to the final result (thus, for example, we omit the non-oscillatory terms in M). Hence

$$G_0(\mu,B) = -\frac{(2\mu)^{5/2}}{15\pi^2}\left\{1+\frac{5\pi^2}{4}b^{-2}-\frac{15\pi^{\frac{1}{2}}}{4}b^{-5/2}\Sigma_1\right\} \tag{4}$$

$$M(\mu,B) = -\frac{B^{\frac{1}{2}}}{2\pi^3}\alpha^{3/4}\mu\Sigma_2 \tag{5}$$

and

$$N=\frac{(2\mu)^{3/2}}{3\pi^2}\left\{1+\frac{3}{2}\pi^{\frac{1}{2}}b^{-3/2}\Sigma_2\right\} \tag{6}$$

where

$$\Sigma_1 \equiv \sum_{r=1}^{\infty}\frac{\cos\left[br-\frac{\pi}{4}\right]}{r^{5/2}} \tag{7}$$

and

$$\Sigma_2 \equiv \sum_{r=1}^{\infty}\frac{\sin\left[br-\frac{\pi}{4}\right]}{r^{3/2}} \tag{8}$$

Thus, substituting equation (5) into equation (1) and taking $H=0$, we find that

$$B+dB^{\frac{1}{2}}=0 \tag{9}$$

where

$$d \equiv 2\pi^{-2}\alpha^{3/4}\mu\Sigma_2 \tag{10}$$

It follows from equation (9) that $B=0$ or $B^{1/2}=-d$. The latter implies $B=d^2$ and hence

$$\frac{B}{B_c} = 4\pi^{-4}|\Sigma_2|^2\alpha^2\mu^2 \tag{11}$$

As we have already shown[7], this result agrees with the numerical calculations[6].

We come now to the all-important question of whether or not the $B \neq 0$ solutions are stable, that is, whether they correspond to a minimum value of $G(\mu,B)$, and, in the case where they do correspond to a minimum, whether the corresponding value of $G(\mu,B)$ is greater or less than the value of $G(\mu,B)$ associated with the $B=0$ solution. It will turn out that these questions can actually be answered without explicitly evaluating the value of Σ_2. Combining equation (11) with equations (3), (4) and (5) enables us to write, to the order required,

$$G(\mu,B) = -\frac{(2\mu)^{5/2}}{15\pi^2}\left\{1 - \frac{15\pi^{\frac{1}{2}}}{8}|\Sigma_2|\, b^{-3/2}\right\} \qquad (12)$$

It is clear from this equation that a non-zero value of B results in a larger value for $G(\mu,B)$, relative to the $B=0$ case. Thus we conclude that the most stable configuration is that for which $B=0$. Hence the LOFER states are unlikely to exist in white dwarfs and neutron stars. We have verified this result by deriving an expression for the Helmholtz free energy

$$F(N,B) = G(\mu,B) + N\mu \qquad (13)$$

and then minimizing F with respect to variations in B, keeping N constant (whereas in the minimization of G one keeps μ constant while varying B). We would also like to emphasize that taking the temperature to be non-zero will not change our conclusions, because the most favourable conditions[6] for the existence of LOFER exist at $T=0$.

It should be pointed out, however, that, because of the oscillatory nature of Σ_2, the value of $G(\mu,B)$ given by equation (12) goes through successive maxima and minima as the density (and hence μ and B) is slightly changed. This results from the fact that the actual value of Σ_2 is determined not so much by the magnitude of b (or B) but by the amount by which $(b/2\pi)$ differs from an integral number. Although these minima correspond to larger values of $G(B)$ than that given in the $B=0$ case, perhaps there is some way in which the system can get into these metastable states which, as already pointed out, may offer an explanation for the fact that large magnetic fields do not seem to exist in most white dwarfs[4] but certainly do exist in some[1,2]. But even if one accepts the possibility of the existence of this metastable state, a further objection to the LOFER hypothesis is the fact that the lowest energy state might correspond to the formation of magnetic domains[8,12] with the result that even if B were non-zero internally, it would probably be zero externally. We are grateful to Dr A. K. Rajagopal for stressing the importance of the minimization condition and for many enlightening conversations to do with this work.

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