Gyroscope Test of Gravitation: An Analysis of the Important Perturbations

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We discuss two important perturbations due to (1) the earth's quadrupole moment (Refs. 1 and 2) (say Q) and (2) the earth's revolution around the sun (Ref. 3), which must be taken into account in an analysis of Schiff's (Ref. 4) proposed gyroscope test of gravitation. The rate of change of the spin axis $\vec{S}$ of a gyroscope may be written (Ref. 4)

$$\frac{d\vec{S}}{dt} = \vec{\Omega} \times \vec{S} \qquad (1)$$

where $\vec{\Omega}$ is the angular velocity of precession.

Explicitly, in Einstein theory ($\vec{\Omega} \equiv \vec{\Omega}_E$), we have (Refs. 1-4)

$$\vec{\Omega}_E = \vec{\Omega}_T + \vec{\Omega}_{DS} + \vec{\Omega}_{LT} + \vec{\Omega}_Q + \vec{\Omega}^S_{DS} \qquad (2)$$

where $\vec{\Omega}_T$, $\vec{\Omega}_{DS}$, $\vec{\Omega}_{LT}$, $\vec{\Omega}_Q$, and $\vec{\Omega}^S_{DS}$ are the so-called Thomas, de Sitter, Lense-Thirring, quadrupole-moment (Refs. 1 and 2) and sun (Ref. 3) contributions, respectively. The corresponding quantity in the Brans-Dicke (BD) scalar-tensor theory (Ref. 5), say $\vec{\Omega}_{BD}$, is given by (Ref. 6)

$$\vec{\Omega}_{BD} = \vec{\Omega}_T + \frac{4+3\omega}{6+3\omega}\left[\vec{\Omega}_{DS} + \vec{\Omega}_Q + \vec{\Omega}^S_{DS}\right]$$

$$+ \frac{3+2\omega}{4+2\omega}\vec{\Omega}_{LT} \qquad (3)$$

where ω is the dimensionless coupling constant of the BD theory. It is possible to have $\vec{\Omega}_T$ essentially zero (Ref. 4) by putting the gyroscope in a satellite. Henceforth, we will regard the $\vec{\Omega}$ as being averaged over a period of the motion and confine our discussion to orbits which would be circular in the absence of the earth's quadrupole moment Q (although elliptic orbits have also been treated; Ref. 2). The term $\vec{\Omega}_Q$ is referred to as the direct quadrupole moment effect, but there is also an indirect effect which manifests itself only when the principal term $\vec{\Omega}_{DS}$ is averaged over a period of the motion (Ref. 1).

We define a length "a" such that Kepler's law holds in its normal form for the distorted (due to Q) circular orbit; i.e.,

$$\omega = \frac{2\pi}{T} = \left(\frac{GM}{a^3}\right)^{1/2} \qquad (3a)$$

where T is the period and ω is the average angular velocity (Ref. 2) of the gyroscope in the field of the earth of mass M, spin angular momentum $\vec{S}^{(2)}$, and quadrupole moment Q, given by (Ref. 7)

$$2Q = (1082.64 \pm 0.08) \times 10^{-6} R^2 \qquad (4)$$

where R is the earth's equatorial radius. It follows (Refs. 1-4) that, to the accuracy required,

$$\vec{\Omega}_{DS} = \frac{3GM\omega}{2c^2 a}\left(1 + k\frac{Q}{a^2}\right)\vec{n} \qquad (5)$$

$$\vec{\Omega}_{LT} = \frac{GS^{(2)}}{2c^2 a^3}\left[\vec{n}^{(2)} - 3\cos\theta\,\vec{n}\right] \qquad (6)$$

$$\vec{\Omega}_Q = \frac{3GM\omega}{2c^2 a}\left(\frac{3Q}{a^2}\right)\left[\frac{1}{2}(5\cos^2\theta - 1)\vec{n} - \cos\theta\,\vec{n}^{(2)}\right] \qquad (7)$$

$$\vec{\Omega}^S_{DS} = \frac{3GM_\odot \omega_E}{2c^2 r}\vec{n}^{(S)} \qquad (8)$$

where k is a number of order unity which depends on the inclination of the orbit (Ref. 2) (it equals 0.5 for a polar orbit and -1.0 for an equatorial orbit), $\vec{n}$ and $\vec{n}^{(2)}$ are unit vectors along the orbital angular momentum of the gyro and $\vec{S}^{(2)}$ directions, respectively, and θ is the angle between the $\vec{n}$ and $\vec{n}^{(2)}$ directions. In addition, $M_\odot$, ω_E, and $\vec{n}^{(S)}$ denote the mass of the sun, the average angular velocity of the earth around the sun, and the unit vector along the earth's orbital angular momentum; r is the earth-sun distance.

Perhaps the most unique feature of Schiff's gyro test is that it is the only experiment thus far proposed which is likely to measure the off-diagonal Lense-Thirring terms in the metric tensor. Everitt and Fairbank (Ref. 8) and Fairbank (Ref. 9) expect to carry out this experiment in the near future by launching a satellite containing two pairs of superconducting gyroscopes into a polar orbit around the earth; the spin of one pair (gyro 1) will be parallel to the earth's axis and the spin of the other pair (gyro 2) will be perpendicular to the plane of the orbit.

A polar orbit ($\theta = \pi/2$) was selected because (from Eqs. 5 and 6) $\vec{\Omega}_{DS}$ and $\vec{\Omega}_{LT}$ are at right angles for such an orbit (and, in addition, precession of the gyro due to a possible gyro quadrupole moment is zero; Ref. 10). For definiteness, consider the earth's angular velocity to be in the z-direction and the polar orbit to be in the xz-plane so that the orbital angular momentum of the satellite points in the y-direction. Then $\vec{\Omega}_{DS}$ lies along y and $\vec{\Omega}_{LT}$ along z. Thus, gyro 1 (with spin along z) will not be affected by $\vec{\Omega}_{LT}$, and gyro 2 (with spin along y) will not be affected by $\vec{\Omega}_{DS}$. Therefore, it was thought that gyro 2 would provide a "clean" test of the Lense-Thirring terms -- clean in the sense of being sensitive to $\vec{\Omega}_{LT}$ only. However, as we now make clear, this possibility is ruled out because of the sensitivity of gyro 2 to the sun's perturbation.

The magnitude* of $\vec{\Omega}_{LT}$ for a satellite in a circular polar orbit 300 miles above the earth is 43.8×10^{-3} sec/yr (at this altitude the magnitude of $\vec{\Omega}_{DS}$ is the oft-quoted value of 7.0 sec/yr). Using BD theory, this value is reduced (Ref. 6) by a factor of 1/16, i.e., by 2.7×10^{-3} sec/yr. As before (Ref. 6), we take $\omega = 6$. Thus, to distinguish between the Einstein and BD theories, the experiment should be capable of measuring such small precession angles. In fact, measurement accurate to 10^{-3} sec/yr will be possible (Ref. 9).

The question we wish to consider is whether there are any perturbations of magnitude greater than 10^{-3} sec/yr along the z-axis, in addition to $\vec{\Omega}_{LT}$. With regard to $\vec{\Omega}_Q$, as is clear from Eq. (7), this contribution turns out to be in the same direction as $\vec{\Omega}_{DS}$ for a polar orbit (though this is not true in general), and thus it has no component along z. However, the magnitude** of $\vec{\Omega}^S_{DS}$ is 19.2×10^{-3} sec/yr, and since the earth's equator is inclined at an angle ϕ of 23.44 deg to the ecliptic, the z-component is $0.917\,\vec{\Omega}^S_{DS}$, i.e., 17.6×10^{-3} sec/yr. It is more than 6 times as large as the difference between the Lense-Thirring contributions arising from the Einstein and BD theories and 17.6 times larger than what can be measured! Actually, the best orbit to select for the purpose of obtaining the "cleanest" test of $\vec{\Omega}_{LT}$ is one inclined at an angle θ to the equator so that $\vec{n}$ and $\vec{n}^{(S)}$ point in the same direction (i.e., $\theta = \phi$). Gyro 2 is again placed in the $\vec{n}$ direction so that it will now be insensitive to both $\vec{\Omega}_{DS}$ and $\vec{\Omega}^S_{DS}$. Unfortunately, a price must be paid; that is, the Lense-Thirring contribution to the precession of gyro 2 is reduced by a factor $\sin\phi = 0.398$ to a value 17.4×10^{-3} sec/yr, and the difference between the Einstein and BD contributions in this case is 1.1×10^{-3}, very close to the limits of the expected experimental accuracy. In addition, there is now a contribution from $\vec{\Omega}_Q$ amounting to 3.6×10^{-3} sec/yr. For this configuration, similar to the polar orbit case, a possible gyro quadrupole moment does not contribute to the precession.

To summarize, we wish to emphasize that, although the perturbations discussed will certainly make the analysis of the observations more complex, they have all been calculated precisely to the accuracy desired, with the result that the experiment should be capable of deciphering each separate contribution to the angular velocity of precession of the spin.

References

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*The value of $\vec{\Omega}_{LT}$ is usually calculated by assuming that the earth is homogeneous in density. However, for our present purposes, it is necessary to use a value for the angular momentum of the earth which is based on a more realistic mass distribution for the earth (Ref. 11). This is a significant modification because it reduces $\vec{\Omega}_{LT}$ by about 17%, which is considerably larger than the approximately 6% difference between the Einstein and Brans-Dicke predictions, the quantity which the experiment proposes to measure.

**From Eq. (76a), Ref. 2, it follows that $\vec{\Omega}^S_{DS}$ is simply one-half of the perihelion precession of the earth around the sun.

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