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Another Effect of the Earth's Quadrupole Moment on the Precession of a Gyroscope.

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An experiment to measure the precession of the spin of a gyroscope in an Earth satellite, proposed by SCHIFF (1) as a test of Einstein's theory of relativity, will be carried out in the near future by EVERITT and FAIRBANK (2,3). One of us (4) has recently shown that there is a contribution to the precession due to the quadrupole moment of the Earth of the order of 0".01/yr. We refer to this effect as the *direct* quadrupole-moment effect and it is our purpose here to point out that there is also an *indirect* effect due to the quadrupole moment, which manifests itself only when the principal term (*i.e.* $\boldsymbol{\Omega}_{DS}$, the de Sitter term (4)) is *averaged* over a period of the motion. This indirect effect results from the fact that the orbit of the satellite is no longer purely elliptical but is in fact *distorted* because of the presence of the quadrupole term in the potential.

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(1) L. I. SCHIFF: *Proc. Nat. Acad. Sci.*, 46, 871 (1960); in *Proceedings on the Theory of Gravitation*, edited by L. INFELD (Paris and Warsaw, 1964), p. 71.

(2) C. W. F. EVERITT and W. M. FAIRBANK: in *Proc. X Int. Conf. on Low-Temperature Physics*, Vol. 2 B (Moscow, 1966), p. 337.

(3) W. M. FAIRBANK: *London Award Lecture*, in *Proc. XI Int. Conf. on Low-Temperature Physics*, Vol. 6 (St. Andrews, 1968), p. 14 and 15.

(4) R. F. O'CONNELL: *Astrophys. Space Sci.*, 4, 119 (1969); *Lett. Nuovo Cimento*, 1, 933 (1969).

The Newtonian potential Φ at the gyroscope due to a body of mass M (with axis of symmetry along the z direction, say) and to the quadrupole moment Q is

$$\Phi = \frac{GM}{r} + Q\frac{GM}{r^3}(1 - 3\cos^2\theta)\,, \tag{1}$$

where r is the distance from the center of M to the gyroscope and θ is the angle between $\boldsymbol{r}$ and the z-axis. Thus, the Lagrangian of the Earth-gyroscope system (where M and m are the respective masses and $M \gg m$) is

$$\mathscr{L} = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\varphi}^2) + \frac{GMm}{r} + \frac{GMm}{r^3}Q(1 - 3\cos^2\theta)\,, \tag{2}$$

where the dot denotes differentiation with respect to the time t, and φ is the azimuthal angle. We now apply the Euler-Lagrange equations to obtain the equations of motion and the latter are then integrated to give the orbit equation. In general, numerical methods are necessary, but we have succeeded in finding an analytic solution in the case of a distorted circular *polar* orbit. The polar orbit is of prime importance as this is the orbit selected for the Standford experimental test (2,3) primarily because it enables the de Sitter and Lense-Thirring contributions (1) to the precession to be measured separately. For the polar orbit we find the solution

$$r = a - \frac{Q}{a}\cos^2\bar{\omega}t\,, \tag{3}$$

$$\theta = \bar{\omega}t - \frac{Q}{4a^2}\sin(2\bar{\omega}t)\,, \tag{4}$$

where a is a constant and $\bar{\omega}$ is the average angular velocity. From eqs. (3) and (4), it is clear that a is identical with the equatorial radius. Note that the polar radius is a factor $(1 - Q/a^2)$ times the equatorial radius. In addition,

$$\bar{\omega} = \frac{2\pi}{T} = \left(\frac{GM}{a^3}\right)^{\frac{1}{2}}, \tag{5}$$

where T is the period. It is clear that

$$\bar{r} = a - \frac{Q}{2a} \tag{6}$$

and

$$\bar{\dot{\theta}} = \bar{\omega}\,, \tag{7}$$

where the bar denotes the average over a period.

Now the de Sitter contribution to the precession, $\boldsymbol{\Omega}_{DS}$ say, is given by (1)

$$\boldsymbol{\Omega}_{DS} = \frac{3GM}{2c^2}\,\frac{\boldsymbol{r}\times\boldsymbol{v}}{r^3}\,. \tag{8}$$

Thus, for the distorted polar orbit

$$\overline{\boldsymbol{\Omega}}_{\mathrm{DS}} = \frac{3GM}{2c^2}\frac{\bar{\omega}}{a}\left(1+\frac{Q}{2a^2}\right)\boldsymbol{n}\,, \tag{9}$$

where $\boldsymbol{n}$ is a unit vector along the direction of the satellite's orbital angular momentum. The second term on the right-hand side of eq. (9) is the indirect quadrupole-moment contribution. It is of the same order of magnitude as the direct contribution, which is given by (4)

$$\overline{\boldsymbol{\Omega}}_{Q} = \frac{3GM}{2c^2}\frac{\bar{\omega}}{a}\left(-\frac{3Q}{2a^2}\right)\boldsymbol{n}\,. \tag{10}$$

The overall effect of the quadrupole moment, which we will denote by $\overline{\boldsymbol{\Omega}}_{QT}$, is given simply by the addition of the direct and indirect contributions. From eqs. (9) and (10) it is clear that these two contributions are acting in opposite directions so that

$$\overline{\boldsymbol{\Omega}}_{QT} = -\frac{Q}{a^2}\overline{\boldsymbol{\Omega}}_{\mathrm{DS}}^{(0)}\,, \tag{11}$$

where $\overline{\boldsymbol{\Omega}}_{\mathrm{DS}}^{(0)}$ is the averaged de Sitter contribution for $Q=0$. Now (5)

$$2Q = 1.083\cdot10^{-3}R^2\,, \tag{12}$$

where R is the Earth's equatorial radius.

Therefore

$$\overline{\boldsymbol{\Omega}}_{QT} = -0.541\cdot10^{-3}\left(\frac{R}{a}\right)^2\overline{\boldsymbol{\Omega}}_{\mathrm{DS}}^{(0)}\,. \tag{13}$$

Thus, for a satellite in a *polar* orbit 300 miles above the Earth at the equator (6), the magnitude of $\bar{\Omega}_{\mathrm{DS}}^{(0)}$ is the oft-quoted value of $7''.0$/yr and so $\bar{\Omega}_{QT}$ equals $-3.3\cdot10^{-3}{}''$/yr, *i.e.* it is 3.3 times larger than what can be measured (2). We have carried out a similar calculation for a satellite in an *equatorial* orbit, 300 miles above the Earth, and we find that $\bar{\Omega}_{QT}$ is exactly twice as large as in the case of a polar orbit with the opposite sign.

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(5) D. E. SMITH: *Planet. Space Sci.*, **13**, 1151 (1965).

(6) Taking into account the fact that the satellite is 1.99 miles closer to the *centre* of the Earth when above the poles and the fact that the polar radius of the Earth is 13.28 miles smaller than the equatorial radius, it follows that the satellite is 311 miles above the Earth at the poles. In other words, the satellite at the equator is 11 miles closer to the Earth's surface than when it is at the poles.