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PRODUCTION OF HELIUM IN THE BIG-BANG EXPANSION OF A MAGNETIC UNIVERSE

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ABSTRACT

The effects of a magnetic field on the reaction rates for the processes $n + e^+ \rightleftarrows p + \bar{\nu}$, $n + \nu \rightleftarrows p + e^-$, and $n \rightleftarrows p + e^- + \bar{\nu}$ are calculated. The implications for the production of helium in the big-bang expansion of a magnetic universe are discussed. We also consider the influence of a magnetic field on the expansion rate of the Universe and find that its effect on the rate of helium production dominates over the effects arising from the changed reaction rates.

I. INTRODUCTION

The possibility of a large primordial magnetic field has been cited by Hoyle (1958). Woltjer (1964) has suggested that fields as large as $H_c = 4.4 \times 10^{13}$ gauss may exist in neutron stars, and there are indications (Brownell and Callaway 1969; Silverstein 1969; Rice 1969) that neutron stars may be ferromagnetic. In addition, evidence is rapidly accumulating which favors Gold's hypothesis (1968, 1969*a*) that pulsars are rotating neutron stars with strong magnetic fields. The existence of fields as large as H_c is also an essential ingredient of theories which speculate that pulsars are the main source of cosmic rays (Gold 1968, 1969*a*, *b*; Gunn and Ostriker 1969).

These suggestions have prompted us to consider the effect of a uniform magnetic field H on the production of helium in the big-bang expansion of the Universe. Thus we are led to consider the effect of H on the weak reactions

$$n + e^+ \rightleftarrows p + \bar{\nu}\,, \tag{1a}$$

$$n + \nu \rightleftarrows p + e^-\,, \tag{1b}$$

$$n \rightleftarrows p + e^- + \bar{\nu}\,. \tag{1c}$$

In recent calculations (O'Connell and Matese 1969*a*, *b*; Matese and O'Connell 1969) we have shown that the rate of neutron β-decay is substantially affected by a field of the order of H_c. We now wish to determine the extent to which a large magnetic field affects the rates of reaction (1a)–(1c).

The neutron population at the epoch $T \approx 5 \times 10^9$ ° K essentially determines the amount of helium ultimately formed in the big bang. Calculations of the neutron population evolving from a dense high-temperature phase of the Universe by means of the reactions (1a)–(1c) have been performed by Alpher, Follin, and Herman (1953) and by Hoyle and Tayler (1964). They conclude that the neutron-proton ratio, n/p, at $T \sim 5 \times 10^9$ ° K is approximately 1/5.5.

If it is assumed that all neutrons are burned into ^{4}He, it follows that the Universe should contain approximately 30 percent helium by mass. This result is in good agreement with that obtained by the more elaborate calculations of Peebles (1966*a*, *b*) and of Wagoner, Fowler, and Hoyle (1967). We shall discuss the way in which these conclusions are altered if a large primordial field existed.

In § II we derive expressions for the reaction rates of processes (1a)–(1c), and in § III we present numerical results. A discussion of the problem of helium production is given in § IV.

II. THE REACTION RATES

The reaction rates for the processes (1a)–(1c) are computed as a function of temperature under the assumption that a uniform magnetic field is present. The extreme nondegenerate approximation is made in which the chemical potentials in the distribution functions for electrons and positrons are set equal to zero. This is equivalent to neglecting the difference between the number of electrons and positrons compared with their sum (Hayashi 1950). The calculation proceeds in a manner analogous to the work of Alpher, Follin, and Herman (1953), the distinctions being that the electron (positron) wave function and phase space are modified by the presence of the field.

Electron states are specified by their component of momentum along the field direction, which we denote by k, and by their principal quantum number, n, such that the energy of an electron state is given by (Matese and O'Connell 1969)

$$E(n, k) = (1 + k^2 + 4\gamma n)^{1/2}\,, \tag{2}$$

where $\gamma = H/2H_c$ (we work in units $\hbar = m_e = c = 1$). The squared matrix element for each of the reactions (1a)–(1c) is the same when the spin of the initial nucleon is averaged and the spins of the remaining particles are summed. We find (Matese and O'Connell 1969)

$$\Sigma_{\text{spins}}|\langle f|H_\beta|i\rangle|^2 = \frac{g_V^2(1+3\lambda^2)\gamma}{\pi L V}[\![1 - \tfrac{1}{2}\delta_{n0}\{1 - [k/E(n, k)]\}]\!]\,. \tag{3}$$

In equation (3) L and V are the normalization length and volume, and (Wu and Moszkowski 1966)

$$g_V^2(1+3\lambda^2) = 2\pi^3/\tau\,, \qquad \tau = ft_{1/2}/\ln 2 \approx 1700\,. \tag{4}$$

The various rates are then determined by the density of final states and the distribution of initial states

$$\lambda_{i\to f} = 2\pi\Sigma_i n_i \int_f dE_f \frac{dn_f}{dE_f}\delta(E_f - E_i)\Sigma_{\text{spins}}|\langle f|H_\beta|i\rangle|^2\,, \tag{5}$$

where n_i is the distribution of initial states and dn_f/dE_f is the density of final states.

a) $n + e^+ \to p + \bar{\nu}$

The distribution of initial states for the reaction $n + e^+ \to p + \bar{\nu}$ is given by

$$n_i = \{1 + \exp[E(n, k)/KT]\}^{-1}\,, \tag{6}$$

while the density of final states takes the form

$$\frac{dn_f}{dE_f} = \frac{V}{2\pi^2}E_{\bar{\nu}}^2\{1 - [1 + \exp(E_{\bar{\nu}}/KT_{\bar{\nu}})]^{-1}\}\,. \tag{7}$$

To a good approximation, the neutrino temperature T_ν may be taken to be the same as the temperature T of the remainder of the medium, for the epoch of interest (Alpher, Follin, and Herman 1953). The sum over the initial state can be written

$$\Sigma_i = \sum_{n=0}^{\infty}\Sigma_k = \sum_{n=0}^{\infty}\frac{L}{2\pi}\int_{-\infty}^{\infty} dk\,. \tag{8}$$

Using equations (5)–(8), we obtain

$$\lambda_{n+e^+\to p+\bar\nu} = \frac{\gamma}{\tau}\sum_{n=0}^{\infty}(1-\tfrac{1}{2}\delta_{n0})\int_{-\infty}^{\infty} dk[W_0+E(n,k)]^2 \times \frac{\exp\{[W_0+E(n,k)]/KT\}}{[\![1+\exp\{[W_0+E(n,k)]/KT\}]\!]\{1+\exp[E(n,k)/KT]\}}, \tag{9}$$

where $W_0 = m_n - m_p = 2.54$.

b) $n + \nu \to p + e^-$

For the reaction $n + \nu \to p + e^-$ the distribution of initial states is given by

$$n_i = [1+\exp(E_\nu/KT)]^{-1}, \tag{10}$$

and the density of final states by

$$\frac{dn_f}{dE_f} = \frac{L}{2\pi}\frac{dk}{dE_f}\{1+\exp[E(n,k)/KT]\}^{-1}. \tag{11}$$

The sum over initial neutrino states is represented by

$$\Sigma_i = \Sigma_{p_\nu} = \frac{V}{(2\pi)^3}\int dp_\nu = \frac{V}{2\pi^2}\int dE_\nu E_\nu^2, \tag{12}$$

and the sum over final states consists of a sum over n and an integration over k. Energy conservation requires that $E(n, k) \geq W_0$, or $k^2 \geq W_0^2 - 1 - 4\gamma n = p_0^2 - 4\gamma n \equiv p_n^2$. Then the available k-space is

$$0 \leq k^2 \leq \infty \quad \text{for} \quad n > N_c, \qquad p_n^2 \leq k^2 \leq \infty \quad \text{for} \quad n \leq N_c$$

where N_c is the largest integer in $p_0^2/4\gamma$. Therefore,

$$\begin{aligned}\int_f dk &= \sum_{n=0}^{N_c}\int_{-\infty}^{-p_n} dk + \sum_{n=0}^{N_c}\int_{p_n}^{\infty} dk + \sum_{n=N_c+1}^{\infty}\int_{-\infty}^{\infty} dk \\ &= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dk - \sum_{n=0}^{N_c}\int_{-p_n}^{p_n} dk.\end{aligned} \tag{13}$$

Using equations (10)–(13) in equation (5), we obtain

$$\begin{aligned}\lambda_{n+\nu\to p+e^-} &= \frac{\gamma}{\tau}\sum_{n=0}^{\infty}(1-\tfrac{1}{2}\delta_{n0}) \\ &\times \int_{-\infty}^{\infty}\frac{dk[E(n,k)-W_0]^2\exp[E(n,k)/KT]}{[\![1+\exp\{[E(n,k)-W_0]/KT\}]\!]\{1+\exp[E(n,k)/KT]\}} \\ &- \frac{\gamma}{\tau}\sum_{n=0}^{N_c}(1-\tfrac{1}{2}\delta_{n0}) \\ &\times \int_{-p_n}^{p_n}\frac{dk[E(n,k)-W_0]^2\exp[E(n,k)/KT]}{[\![1+\exp\{[E(n,k)-W_0]/KT\}]\!]\{1+\exp[E(n,k)/KT]\}}.\end{aligned} \tag{14}$$

c) $n \rightarrow p + e^- + \bar{\nu}$

For neutron β-decay we have only a single initial state to sum over. The density of final states is

$$\frac{dn_f}{dE_f} = \frac{LV}{4\pi^3}[W_0 - E(n,k)]^2[\![1 - \{1 + \exp[E(n,k)/KT]\}^{-1}]\!] \times (1 - [\![1 + \exp\{[W_0 - E(n,k)]/KT\}]\!]^{-1}) . \tag{15}$$

The restriction of energy conservation, $E(n, k) \leq W_0$, requires that

$$k^2 \leq p_n^2 \quad \text{and} \quad n \leq N_c .$$

Our summation over final states is then

$$\int_f = \sum_{n=0}^{N_c} \int_{-p_n}^{p_n} dk . \tag{16}$$

Using equations (15) and (16) in equation (5), we find the rate for β-decay

$$\lambda_{n\rightarrow p+e^-+\bar{\nu}} = \frac{\gamma}{\tau}\sum_{n=0}^{N_c}(1 - \tfrac{1}{2}\delta_{n0}) \int_{-p_n}^{p_n} dk[W_0 - E(n,k)]^2 \times \frac{\exp[E(n,k)/KT]}{\{1 + \exp[E(n,k)/KT]\}[\![1 + \exp\{[W_0 - E(n,k)]/KT\}]\!]} . \tag{17}$$

We note that if $H/H_c \geq \frac{1}{2}p_0^2 = 2.7$, then $N_c = 0$ and $\lambda_{n\rightarrow p+e^-+\bar{\nu}}$ behaves linearly with H. However, it is seen that the rate of neutron loss from β-decay is exactly canceled by a term in the rate $\lambda_{n+\nu\rightarrow p+e^-}$. A similar cancellation occurs in the zero-field rates (Alpher, Follin, and Herman 1953).

d) The Inverse Reactions

The inverse reactions to those discussed above can readily be obtained by invoking statistical balance and comparing phase-space factors. We illustrate this for the reaction $p + \bar{\nu} \rightarrow n + e^+$. We now have for the distribution of initial states

$$n_i = [1 + \exp(E_{\bar{\nu}}/KT)]^{-1} , \tag{18}$$

and for the density of final states

$$\frac{dn_f}{dE_f} = \frac{L}{2\pi}\frac{dk}{dE_f}[\![1 - \{1 + \exp[E(n,k)/KT]\}^{-1}]\!] . \tag{19}$$

The sum over initial states is

$$\Sigma_i = \Sigma_{p_{\bar{\nu}}} = \frac{V}{2\pi^2}\int dE_{\bar{\nu}}E_{\bar{\nu}}^2 , \tag{20}$$

and the final state integration is

$$\int_f dk = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} dk . \tag{21}$$

Inserting equations (18)–(21) into equation (5) and using energy conservation, $E_{\bar{\nu}} = W_0 + E(n, k)$, we obtain

$$\lambda_{p+\bar{\nu}\to n+e^+} = \exp(-W_0/KT)\lambda_{n+e^+\to p+\bar{\nu}} .$$

Similarly, the remaining inverse reactions are given by

$$\lambda_{p+e^-\to n+\nu} = \exp(-W_0/KT)\lambda_{n+\nu\to p+e^-} ,$$

$$\lambda_{p+e^-+\bar{\nu}\to n} = \exp(-W_0/KT)\lambda_{n\to p+e^-+\bar{\nu}} .$$

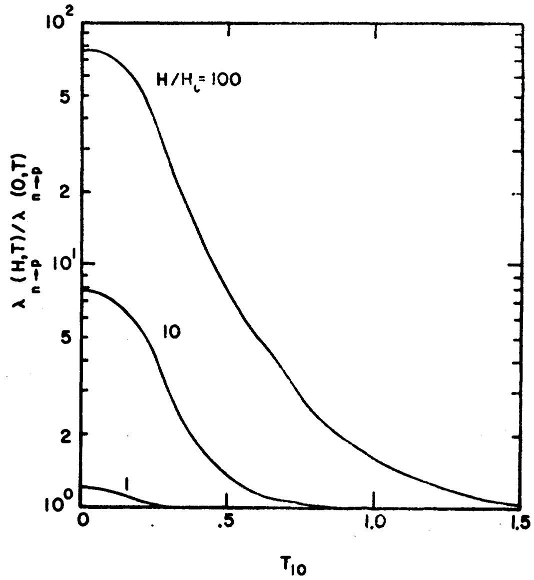

FIG. 1.—Neutron-depletion rate $\lambda_{n\to p}(H, T)$, normalized to the free-field rate, plotted as a function of $T_{10} = T/10^{10}$ ° K for constant values of $H/H_c = 1$, 10, and 100.

III. RESULTS

Since we are interested in the total reaction rate for converting neutrons into protons, we sum equations (9), (14), and (17) to obtain

$$\lambda_{n\to p} = \frac{\gamma}{\tau}\sum_{n=0}^{\infty}(1 - \tfrac{1}{2}\delta_{n0}) \int_{-\infty}^{\infty} dk\{1 + \exp[E(n, k)/KT]\}^{-1}$$
$$\times \left[\left[\frac{[W_0 + E(n, k)]^2 \exp\{[W_0 + E(n, k)]/KT\}}{1 + \exp\{[W_0 + E(n, k)]/KT\}} + \frac{[W_0 - E(n, k)]^2 \exp[E(n, k)/KT]}{1 + \exp\{[E(n, k) - W_0]/KT\}}\right]\right] . \qquad (22)$$

Whereas the individual rates $\lambda_{n\to p+e^-+\bar{\nu}}$ and $\lambda_{n+\nu\to p+e^-}$ depend upon a critical value of the Landau quantum number N_c, the total rate $\lambda_{n\to p}$ is independent of N_c and therefore is less sensitive to the value of the magnetic field.

Equation (22) has been numerically evaluated for various values of H and T. The results are shown in Figure 1. We plot $\lambda_{n\to p}(H, T)/\lambda_{n\to p}(0, T)$ as a function of T for various values of H. The effects of a magnetic field are seen to be negligible until the temperature drops to a point where the reactions $\lambda_{n+e^+\rightarrow p+\bar{\nu}}$ and $\lambda_{n+\nu\rightarrow p+e^-}$ begin to freeze out ($W_0/KT \gtrsim 1$) and neutron β-decay begins to dominate.

In the Appendix we show that equation (22) reduces to the free-field reaction rate (Alpher, Follin, and Herman 1953) in the limit $H \rightarrow 0$.

IV. HELIUM PRODUCTION

The neutron population at the epoch $T \approx 5 \times 10^9$ ° K, along with the expansion rate of the Universe, essentially determines the amount of helium ultimately formed in the big bang. Letting n and p represent the neutron and proton densities and defining

$$r = n/(n + p) ,$$

we have (Alpher, Follin, and Herman 1953)

$$\frac{dr}{dt} = (1 - r)\lambda_{p\rightarrow n} - r\lambda_{n\rightarrow p} . \tag{23}$$

Since $\lambda_{p\rightarrow n} = \exp(-W_0/KT)\lambda_{n\rightarrow p}$,

$$\frac{dr}{dt} = \{\exp(-W_0/KT) - [1 + \exp(-W_0/KT)]r\}\lambda_{n\rightarrow p} . \tag{24}$$

In order to obtain the neutron-proton ratio at a given temperature from equation (24), one must assume a cosmological model. We adopt the simple model of Greenstein (1969) in which the geometry is still described by a Robertson-Walker metric. Now Greenstein has applied this model to give qualitative estimates of the effect of a magnetic field on helium formation. However, his arguments are based on the assumption that the total rate for neutron depletion $\lambda_{n\rightarrow p}$ depends on the field H as strongly as does the β-decay rate $\lambda_{n\rightarrow p+e^-+\bar{\nu}}$. We have seen that this is only correct for temperatures which are too low to be of interest. Further, he has not taken into account the change in the value of the magnetic field as the Universe expands. We have taken both of these effects into account. However, it turns out that our overall conclusions are essentially in agreement with those of Greenstein (1969).

Flux conservation gives a simple temperature dependence for H in this model, $H \sim T^2$. Therefore, the energy density of the magnetic field has the same temperature dependence as the energy density of the leptons and of the radiation field. We define

$$A \equiv \frac{\rho_{\rm H}}{\rho_\gamma + \rho_e + \rho_\nu} \tag{25}$$

and obtain

$$\frac{H}{H_c} \sim 2.1 A^{1/2} T_{10}^2 , \tag{26}$$

$$T_{10} = 1.04(1 + A)^{-1/4} t^{-1/2} , \tag{27}$$

where $T_{10} = T/10^{10}$ ° K.

Using equation (27), one can rewrite equation (24) in the form

$$\frac{dr}{dT_{10}} = \frac{2.16 T_{10}^3}{(1 + A)^{1/2}} \{-\exp(-W_0/KT) + [1 + \exp(-W_0/KT)]r\}\lambda_{n\rightarrow p} . \tag{28}$$

In this model, then, the existence of a magnetic field has two competing effects, viz., (a) an increase in the expansion rate by a factor $(1 + A)^{1/2}$ and (b) an influence in the reaction rate $\lambda_{n\to p}$. In Figure 2 we plot $\lambda_{n\to p}(A, T)/\lambda_{n\to p}(0, T)$, using equation (26) to specify the dependence of the field on the temperature. In particular,

$$\frac{1}{(1+A)^{1/2}}\frac{\lambda(A, T)}{\lambda(0, T)} < 1$$

for all values of A, and so the existence of a primordial field will increase the neutron population at a given temperature from the value it would have if the field were not

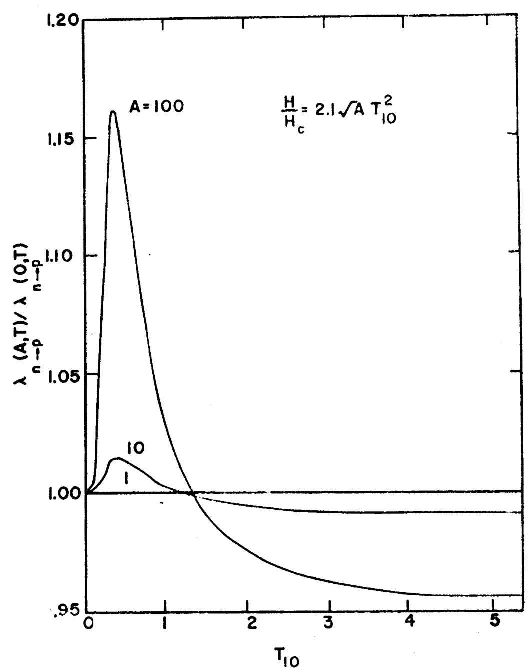

FIG. 2.—Neutron-depletion rate $\lambda_{n\to p}(A, T)$, normalized to the free-field rate, plotted as a function of $T_{10} = T/10^{10}\,^\circ$ K for constant values of $A = 1$, 10, and 100.

present, if the expansion is assumed to be isotropic. The curve labeled $A = 1$ is seen to be indistinguishable from $H = 0$ for the scale given. Thus, within the framework of the assumed model, we conclude that the overall effect of a primordial magnetic field is to increase the rate of helium production.

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APPENDIX

We show that equation (22) reduces to the correct expression in the limit $H \to 0$. Changing the integration variable from k to E,

$$k^2 = E^2 - 1 - 4\gamma n \equiv E^2 - u_n ,$$

we have

$$\lambda_{n\to p} = \frac{\gamma}{\tau}\sum_{n=0}^{\infty}(2 - \delta_{n0}) \int_{\sqrt{u_n}}^{\infty} dE \frac{G(E)}{(E^2 - u_n)^{1/2}} , \tag{A1}$$

where

$$G(E) = E[1 + \exp(E/KT)]^{-1}\left\{\frac{(W_0 + E)^2 \exp[(W_0 + E)/KT]}{1 + \exp[(W_0 + E)/KT]} + \frac{(W_0 - E)^2 \exp(E/KT)}{1 + \exp(E/KT)}\right\}. \tag{A2}$$

In the limit $H \to 0$ ($\gamma \to 0$), the summation over n can be replaced by an integration over a continuous variable u

$$\lambda_{n\to p} = \frac{2\gamma}{\tau}\frac{1}{4\gamma}\int_1^\infty du \int_{\sqrt{u}}^\infty dE \frac{G(E)}{(E^2 - u)^{1/2}} = \frac{1}{2\tau}\int_1^\infty du \int_{\sqrt{u}}^\infty dE \frac{G(E)}{(E^2 - u)^{1/2}} \quad (H \to 0)\,. \tag{A3}$$

The order of integration can be interchanged to give

$$\lambda_{n\to p} = \frac{1}{2\tau}\int_1^\infty dEG(E)\int_1^{E^2} \frac{du}{(E^2 - u)^{1/2}} = \frac{1}{\tau}\int_1^\infty dEG(E)(E^2 - 1)^{1/2} \qquad (H \to 0)\,. \tag{A4}$$

which is the desired result (Alpher, Follin, and Herman 1953).

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