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Simple Derivation of Schwinger's Quantization Relation between Electric and Magnetic Charges.

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Dirac [1] has shown that the existence of magnetic charge (monopoles) g implies the quantization of electric charge e, *i.e.* $2ge = n\hbar c$, where n is an integer. However, Schwinger [2] has recently claimed that the correct relation is actually

$$ge = n\hbar c\,. \tag{1}$$

Schwinger's derivation is quite complicated so, with the intention of obtaining a straightforward derivation of the quantization condition, Efinger [3] analysed the solution of Schrödinger's equation for a charge e in a uniform magnetic field H produced by an infinite plane sheet of total magnetic charge g. He obtained $ge = (n+\frac{1}{2})\hbar c$. However, his analysis is incomplete because he neglected electron spin (*i.e.* he considered the Landau [4] contribution to the quantization of electron orbits in a plane perpendicular to H, but he neglected the Pauli [5] contribution).

Consider the motion of a relativistic electron in a magnetic field $H = H_z$ produced by an infinite plane sheet of total magnetic charge g in the xy-plane. The energy eigenvalues resulting from a solution of Dirac's equation are [6,7] (henceforth we take $\hbar = c = m = 1$, where m is the mass of the electron)

$$E^2 = 1 + P_z^2 + 2enH\,, \qquad n = 0, 1, 2, \dots\,. \tag{2}$$

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(1) P. A. M. Dirac: *Proc. Roy. Soc.*, A **133**, 60 (1931); *Phys. Rev.*, **74**, 817 (1948).
(2) J. Schwinger: *Phys. Rev.*, **144**, 1087 (1966).
(3) H. J. Efinger: *Note on the quantum theory of elastic and magnetic charges* (preprint).
(4) L. Landau: *Zeits. Phys.*, **64**, 629 (1930).
(5) W. Pauli: *Zeits. Phys.*, **44**, 81 (1927).
(6) M. H. Johnson and B. A. Lippmann: *Phys. Rev.*, **76**, 828 (1949); **77**, 702 (1950).
(7) A. A. Sokolov: *Introduction to quantum electrodynamics* (U.S. Atomic Energy Commission: AEC-tr-4322, 1960).

Thus, the momentum $P_\perp$ in the xy-plane is quantized according to the relation $P_\perp^2 = 2enH$. Now, in contrast to the classical motion, we cannot say that the electron trajectory in the xy-plane is a circle of radius R. This arises from the fact that there is an uncertainty in locating the centre of the electron orbit ([6]) with the result that there is a quadratic fluctuation superimposed upon the circular motion ([6,7]). However, we can still talk about the radius R of the equilibrium circle about which the fluctuations take place ([8]). This radius corresponds to the position of maximum charge density ([9]) and is given by ([8,9])

$$R = (2n/eH)^{\frac{1}{2}} . \tag{3}$$

It follows that $R = P_\perp/eH$, in agreement ([8,9]) with the classical result. Thus, the magnetic flux F enclosed within these quantized electron orbits is

$$F = \pi R^2 H = \pi 2n/e . \tag{4}$$

But $F = 2\pi g$. Therefore,

$$ge = n , \tag{5}$$

which is Schwinger's ([3]) quantization relation. In addition, it is interesting to note that the magnetic flux is quantized in units of $2\pi\hbar c/e$, the original unit derived by LONDON ([10]) in his discussion of quantized flux through a superconducting ring.

([8]) Ref. ([7]), eq. (28.23).

([9]) J. J. KLEIN: *Rev. Mod. Phys.*, **40**, 523 (1968).

([10]) F. LONDON: *Superfluids*, vol. **1** (New York, 1950), p. 152.