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Precession of Schiff's Proposed Gyroscope in an Arbitrary Force Field.

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Schiff's proposed gyroscope experiment (1) is of great current interest because of the possibilities it offers for testing Einstein's general theory of relativity. In a recent publication (2) we showed that there is a contribution to the precession of the gyroscope due to the quadrupole moment of the earth. In the case of *equatorial* orbits around the earth, at a moderate altitude, we showed that the magnitude of this precession exceeds the experimental error for observations which cover a span of about one year or greater. The purpose of this communication is to extend our previous results to *arbitrary orbits* and *arbitrary force fields*. Proceeding as before (2), within the framework of Einstein's theory of general relativity, the metric of the space-time which represents an arbitrary static distribution of matter whose Newtonian gravitational potential is φ, correct to lowest order in φ (in our units $c=1$), may be written (3)

$$\mathrm{d}s^2 = (1-2\varphi)\,\mathrm{d}t^2 - (1+2\varphi)(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2)\,. \tag{1}$$

The Newtonian gravitational force, $\boldsymbol{F}_G$ say, at the position of the gyroscope, is related to φ in the usual manner

$$\boldsymbol{F}_G = \boldsymbol{\nabla}\varphi\,. \tag{2}$$

At this initial stage of the calculation, two methods of attack are possible, *viz.* *a*) Schiff's procedure (1) and *b*) the Lagrangian method of Pustovoit and Bautin (PB) (4). We have already demonstrated (2) the application of method *a*)—it is straightforward but somewhat tedious. It turns out that method *b*) is more direct and also more physically enlightening. Now PB (4) wrote down the Lagrangian (5) for a point mass dm

(1) L. I. Schiff: *Proc. Nat. Acad. Sci.*, **46**, 871 (1960); in *Proceedings on the Theory of Gravitation*, ed. by L. Infeld (Paris and Warsaw, 1964), p. 71.
(2) R. F. O'Connell: *Astrophys. Space Sci.* (in press).
(3) G. C. McVittie: *General Relativity and Cosmology*, II ed. (Urbana, Ill., 1965), p. 61.
(4) V. I. Pustovoit and A. V. Bautin: *Sov. Phys. JETP*, **19**, 937 (1964).
(5) L. D. Landau and E. M. Lifshitz: *The Classical Theory of Fields*, revised II ed. (Reading, Mass., 1965).

in the gravitational field g_{ik}, and for the Newtonian potential φ appearing in their expression they took the value $\varphi = GM/r$, where M is the mass of the earth. The analysis which leads to the Lagrangian used by BP is based on the Schwarzschild metric. However, starting with the more general line element given by eq. (1), it is possible to show that φ in the Lagrangian can be interpreted more generally as the Newtonian potential of an arbitrary static gravitational field. Using this more powerful expression for the Lagrangian, and extending the method of PB [1], enables us to calculate the precession of a gyroscope in a force field F_G produced by any static distribution of matter. We write the total angular velocity of precession, $\mathbf{\Omega}_E$ say, in the form

$$\mathbf{\Omega}_E = \mathbf{\Omega}_G + \mathbf{\Omega}_T + \mathbf{\Omega}_{LT}\,, \tag{3}$$

where $\mathbf{\Omega}_T$ and $\mathbf{\Omega}_{LT}$ are the familiar Thomas and Lense-Thirring contributions [1-6], and where $\mathbf{\Omega}_G$ is the contribution due to the static gravitational forces. We find that

$$\mathbf{\Omega}_G = -\tfrac{3}{2}(\boldsymbol{f}_G \times \boldsymbol{v})\,, \tag{4}$$

where $\boldsymbol{f}_G$ is the acceleration of the gyroscope due to $\boldsymbol{F}_G$ ($\boldsymbol{F}_G = m\boldsymbol{f}_G$, where m is the mass of the gyroscope) and $\boldsymbol{v}$ is the velocity of the gyroscope. An important point we wish to emphasize is that $\boldsymbol{F}_G$ includes not only the total gravitational force due to the earth but also all other perturbing forces, of a gravitational origin, acting on the gyroscope. It is interesting to note that eq. (3) may thus be written in the form

$$\mathbf{\Omega}_E = \tfrac{1}{2}(\boldsymbol{f}_{NG} - 3\boldsymbol{f}_G) \times \boldsymbol{v} + \mathbf{\Omega}_{LT}\,, \tag{5}$$

where $\boldsymbol{f}_{NG}$ is the acceleration arising from any nongravitational forces.

In the case of a spherically symmetric source, $\boldsymbol{f}_G = -(GM/r^3)\boldsymbol{r}$ and $\mathbf{\Omega}_G$ reduces to $\mathbf{\Omega}_{DS}$, the so-called De Sitter contribution [1-6]. As previously noted [2], the greatest gravitational perturbing force acting on the gyroscope is caused by the quadrupole moment of the earth. Writing

$$\varphi = \frac{GM}{r} + Q\,\frac{GM}{r^3}\,(1 - 3\cos^2\theta)\,, \tag{6}$$

where θ is the polar angle, it follows that for equatorial orbits

$$\mathbf{\Omega}_G = \left(1 + \frac{3Q}{r^2}\right)\mathbf{\Omega}_{DS}(\theta = \pi/2) = \left(1 + \frac{3Q}{r^2}\right)\frac{3GM}{2r^3}\,(\boldsymbol{r}\times\boldsymbol{v})\,, \tag{7}$$

in agreement with our previous analysis [2]. In the more general case of arbitrary θ, we get

$$\mathbf{\Omega}_G = \frac{3GM}{2r^3}\,(\boldsymbol{A}\times\boldsymbol{v})\,, \tag{8}$$

where

$$\boldsymbol{A} = \boldsymbol{r}\left\{1 + \frac{3Q}{r^2}\,(1 - 3\cos^2\theta)\right\} - \hat{\theta}\,\frac{3Q}{r}\sin 2\theta \tag{9}$$

(2) R. F. O'Connell: *Phys. Rev. Lett.*, **20**, 69 (1968).

and $\hat{\theta}$ denotes the unit vector in the θ-direction. We would also like to draw attention to the fact that the earth's quadrupole moment Q will not cause any secular perturbations (7) in the angle θ; the principal effects (7,8) of Q on a satellite orbit are that the longitude of the ascending node regresses, the perihelion may advance or decline, depending on the value of θ, and the mean anomaly is perturbed.

In order of importance, the next perturbing contribution to be considered is that due to the third harmonic J_3 in the expansion of the earth's potential (it arises from the «pear-shape» of the earth (9)). However, since $J_3 \simeq 10^{-6}$ compared to $J_2 \simeq 10^{-3}$ it is clear that it makes a negligible contribution to $\boldsymbol{\Omega}_E$, though it has a well-established effect on the orbit (the orbit is also affected, to a measureable extent, by still higher harmonics). Of course, for a gyroscope at rest with respect to the rotating earth, $\boldsymbol{f}_G$ is given exactly by the value of the local gravitational acceleration $\boldsymbol{g}$.

We have previosuly pointed out (6) that the gyroscope experiment offers perhaps the best possibility for deciding between the Einstein and Brans-Dicke (BD) (10) theories of gravitation. For the general force field considered here, we find that the angular velocity of precession in BD theory, $\boldsymbol{\Omega}_{BD}$ say, is given by

$$\boldsymbol{\Omega}_{BD} = \boldsymbol{\Omega}_T - [(4-3\omega)/(6+3\omega)]\boldsymbol{\Omega}_G + [(3+2\omega)/(4+2\omega)]\boldsymbol{\Omega}_{LT}, \tag{10}$$

where ω is the dimensionless coupling constant appearing in BD theory (10). In the case of a spherically symmetric earth, this expression reduces of course to our previous result (6). With regard to linear theories of gravitation, it is clear that the analysis of PB may be extended to treat the case of an arbitrary force field. Analogous to what occurs in the Einstein and BD theories, as outlined above, the new result is the same as that obtained for the spherically symmetric force field (4), except that $-(GM/r^3)\boldsymbol{r}$ is replaced by $\boldsymbol{f}_G$ in the De Sitter term.

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(7) J. M. A. DANBY: *Fundamentals of Celestial Mechanics* (New York, 1962), p. 261.

(8) R. F. O'CONNELL: *Astrophys. Journ. Lett.*, **152**, L11 (1968).

(9) J. A. O'KEEFE, A. ECKELS and R. K. SQUIRES: *Science*, **129**, 565 (1959); *Astron. Journ.*, **64**, 245 (1959).

(10) C. BRANS and R. H. DICKE: *Phys. Rev.*, **124**, 925 (1961).