Neutron Beta Decay in a Uniform Constant Magnetic Field

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The β-decay rate of a polarized neutron source in a constant, uniform magnetic field of arbitrary strength is calculated. We show that the decay rate is substantially changed if the magnetic field is of order 10^{13} G or larger. Fields of such strength have been speculated to exist in the early universe or in neutron stars.

I. INTRODUCTION

RECENT advances in the production of large magnetic fields in the laboratory[1,2] have generated interest in the effect of intense magnetic fields on various phenomena.[3,4] The largest field that can presently be produced in the laboratory[1,2] is of the order of 10^6 G, which is considerably lower than the quantum critical field value[5] of $H_e = m^2c^3/e\hbar = 4.4\times10^{13}$ G. However, the "cosmic laboratory" may be a source of much stronger fields; in fact, it has been suggested[6] that magnetic fields as large as 10^{14}–10^{16} G may exist in neutron stars (and white dwarfs may have fields as large as 10^{10} G). Hoyle[7] has cited the possibility of a large primordial magnetic field, and Brownell and Callaway[8] speculate that neutron stars and the dense early universe may be ferromagnetic. In previous papers,[9] one of us examined various effects of a large magnetic field and also drew attention to an "effect of magnetic fields which has been often ignored in astrophysical investigations, viz., that the rates of all elementary particle processes will be affected." Pursuing this idea, it is our purpose here to examine the effect of a magnetic field on the β-decay rate of a neutron source of arbitrary polarization. This is a fundamental process in many astrophysical phenomena; in particular it has a crucial bearing[10] on a problem of current interest, the production of helium in the "big-bang" expansion of the universe.[11] The astrophysical implications of our results will be discussed elsewhere.

Further, we call attention to additional phenomena that would be influenced by the existence of large magnetic fields. An extensive calculation of the cooling rate of neutron stars has been carried out by Bahcall and Wolf.[12] The cooling is essentially caused by the emission of neutrinos in the reactions

$$n+n \to n+p+e^- + \bar{\nu}_e,$$
$$n+\pi^- \to n+e^- + \bar{\nu}_e,$$

similar reactions with e^- replaced by μ^- and $\bar{\nu}_e$ by $\bar{\nu}_\mu$, as well as the inverse of all of these processes. The importance of these calculations stems from the restrictions they place on the observability of neutron stars. In particular, they indicate that the discrete x-ray sources in the direction of the galactic center are unlikely to be neutron stars.

Now if magnetic fields as large as 10^{14}–10^{16} G exist in neutron stars, as has been speculated by Woltjer,[6] they will affect the rates of the relevant reactions listed above and therefore the cooling rate of neutron stars.

In Sec. II we develop the general formalism for neutron decay in a uniform, constant magnetic field. Section III includes the calculation of transition rates, and various limits of these expressions are given in Sec. IV. The results are discussed in Sec. V.

II. GENERAL FORMALISM

Our starting point is the well-known V-A Hamiltonian density[13,14]

$$\mathcal{H} = \sqrt{2}^{-1} g_V [\bar{p}\gamma_\mu (1-\lambda\gamma_5) n][\bar{e}\gamma_\mu (1+\gamma_5)\nu], \qquad (1)$$

where $\lambda \equiv g_A/g_V \simeq 1.18$. Here p, n, e, and ν stand for the proton, neutron, electron, and neutrino operators with $\bar{\psi} = \psi^\dagger\gamma_4$.

The antineutrino wave function is given by[13]

$$\psi_{\bar{\nu}} = \frac{1}{(2V)^{1/2}} e^{-i\mathbf{q}\cdot\mathbf{r}} \begin{pmatrix} \chi_{\bar{\nu}} \\ -\chi_{\bar{\nu}} \end{pmatrix},$$
$$\chi_{\bar{\nu}} = \begin{pmatrix} \sin\frac{1}{2}\theta_{\bar{\nu}} \\ -e^{i\phi_{\bar{\nu}}} \cos\frac{1}{2}\theta_{\bar{\nu}} \end{pmatrix}, \qquad (2)$$

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[1] *Proceedings of the International Conference on High Magnetic Fields, Cambridge, Mass.*, (M.I.T. Press, Cambridge, Mass., 1961); D. H. Parkinson and B. E. Mulhall, *The Generation of High Magnetic Fields* (Plenum Press, Inc., New York, 1967).
[2] T. Erber and H. G. Latal, Bull. Am. Phys. Soc. **10**, 1103 (1965).
[3] T. Erber [Rev. Mod. Phys. **38**, 626 (1966)] gives a very extensive coverage of this general area.
[4] J. J. Klein, Rev. Mod. Phys. **40**, 523 (1968).
[5] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Inc., Reading, Mass., 1965), revised 2nd ed., Chap. 9. Note that H_c itself is actually about 137 times smaller than the field beyond which classical electrodynamics breaks down.
[6] L. Woltjer, Astrophys. J. **140**, 1309 (1964).
[7] F. Hoyle, in *La Structure et l'Evolution de l'Univers* (Editions Stoops, Brussels, 1958).
[8] D. H. Brownwell and J. Callaway (to be published).
[9] R. F. O'Connell, Phys. Rev. Letters **21**, 397 (1968); Phys. Letters **27A**, 391 (1968); Phys. Rev. **176**, 1433 (1968).
[10] R. J. Taylor, Nature **217**, 433 (1968).
[11] F. Hoyle and R. J. Tayler, Nature **203**, 1108 (1964); R. V. Wagoner, Science **155**, 1369 (1967); R. V. Wagoner, W. A. Fowler, and F. Hoyle, Astrophys. J. **148**, 3 (1967).
[12] J. N. Bahcall and R. A. Wolf, Phys. Rev. **140**, B1445 (1965); **140**, B1452 (1965).
[13] C. S. Wu and S. A. Moszkowski, *Beta Decay*, (Wiley-Interscience, Inc., New York, 1966).
[14] Y. Smorodinskii, Soviet Phys. Usp. Fiz. Nauk. **67**, 43 (1959) [English transl.: Soviet Phys.—Usp. **2**, 1 (1959)] gives a very lucid review of the zero-field results.

where $\theta_{\bar{\nu}}, \phi_{\bar{\nu}}$ refer to the antineutrino momentum $\mathbf{q}$. The wave function is normalized to

$$\int dV\, \psi_{\bar{\nu}}^\dagger \psi_{\bar{\nu}} = 1. \tag{3}$$

We neglect the effects of the proton's charge and magnetic moment on the electron since, from an analysis of neutron decay in the absence of a magnetic field, they are known to be small. Thus, we can use the exact wave functions for an electron in a constant, uniform magnetic field (we choose $H=H_z$). In cylindrical coordinates r, ϕ, z, these may be written[15,16] ($\hbar=c=m_e=1$)

$$\psi_{n,s,k}=\frac{e^{ikz}}{(2\pi L)^{1/2}}\left(\frac{\gamma E}{1+E}\right)^{1/2} \times \begin{Bmatrix} C_1[(1+E)/E]I_{n-1,s}(\rho)e^{i(l-1)\phi} \\ iC_2[(1+E)/E]I_{n,s}(\rho)e^{il\phi} \\ [[(4\gamma n)^{1/2}/E]C_2+(k/E)C_1]I_{n-1,s}(\rho)e^{i(l-1)\phi} \\ i[[(4\gamma n)^{1/2}/E]C_1-(k/E)C_2]I_{n,s}(\rho)e^{il\phi} \end{Bmatrix}, \tag{4}$$

where

$$\rho=\gamma r^2, \quad \gamma=\tfrac{1}{2}(H/H_c), \quad k=p_z, \tag{5a}$$

the energy is given by

$$E=E(k,n)=(1+k^2+4\gamma n)^{1/2}, \tag{5b}$$

and

$$I_{n,s}(\rho)=(n!s!)^{-1}e^{-\rho/2}\rho^{(n-s)/2}Q_s^{n-s}(\rho), \tag{5c}$$

and where $Q_s^{n-s}(\rho)$ is the generalized Laguerre polynomial,

$$Q_s^{n-s}(\rho)=(-1)^s\sum_{j=0}^{s}(-1)^j\frac{s!n!\rho^{s-j}}{j!(s-j)!(n-j)!}, \tag{5d}$$

with $l=n-s$. The quantum numbers n, s take on the values $n=1, 2\cdots$ for spin-up states and $n=0, 1, 2\cdots$ for spin-down states; $s=0, 1, 2\cdots$. The coefficients C_1 and C_2 are associated with the two spin states. We can extend the values that n can take on for the spin-up states to $n=0, 1, 2\cdots$ by taking

$$\begin{aligned} &C_1=1-\delta_{n0}, \quad C_2=0, \quad \text{spin up}, \\ &C_1=0, \qquad\quad C_2=1, \quad \text{spin down}. \end{aligned} \tag{6}$$

The length along the z axis of the normalization volume is denoted by L and, in the form Eq. (4), each Landau level n, s is nondegenerate. The normalization is one particle in all space.

Sokolov[16] shows that the radius of the classical orbit is given by

$$R\simeq(n/\gamma)^{1/2}, \tag{7a}$$

[15] M. H. Johnson and B. A. Lippmann, Phys. Rev. **76**, 828 (1949); **77**, 702 (1950).
[16] A. A. Sokolov, *Introduction to Quantum Electrodynamics*, (U. S. Atomic Energy Commission: AEC-tr-4322, 1960).

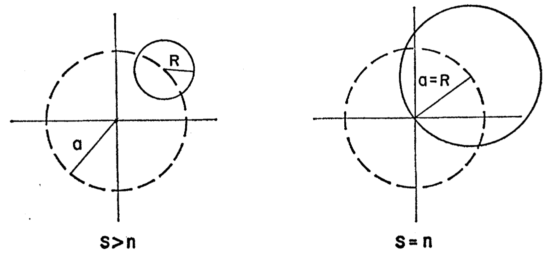

FIG. 1. Illustration of the classical orbit radius and center of a charged particle in a uniform, constant magnetic field.

while the center of this orbit lies on the circle whose center is the origin and whose radius is given by

$$a\simeq(s/\gamma)^{1/2}. \tag{7b}$$

These results are illustrated in Fig. 1.

In the usual manner, the lepton wave functions in Eq. (1) are to be evaluated at $r=R_0\approx$ nucleon radius. Defining $\rho_{R_0}=\gamma R_0^2$ we find that for all fields of interest $\rho_{R_0}\ll 1$. Now

$$I_{n,s}(\rho)\xrightarrow[\rho\to 0]{}\frac{\rho^{|n-s|/2}}{|n-s|!}\left(\frac{n!}{s!}\right)^{1/2}. \tag{8}$$

Substituting Eq. (8) into Eq. (4) we see that only for $s=n$, $n-1$ are the wave functions nonvanishing as $\rho\to 0$. From Fig. 1 we see that this admits a simple physical interpretation. Only those states whose classical orbits pass through the origin (the location of the decaying neutron) will contribute to the transition rate to lowest order in ρ_{R_0}. In the allowed approximation of ordinary beta decay ($H=0$) one retains only the s-wave part of the expansion of the plane waves into a sum over spherical harmonics. Higher partial waves introduce terms depending on the nuclear radius which are neglected. The analog here is to retain only the $s=n$, $n-1$ Landau states since they will dominate as $\rho\to 0$. Therefore, in the allowed approximation, the electron wave function is given by

$$\psi_{n,s,k}{}^{\uparrow}\xrightarrow[r\to 0]{}\frac{1}{(2\pi L)^{1/2}}\left[\frac{\gamma}{E(1+E)}\right]^{1/2} \times \begin{Bmatrix} (1+E)\delta_{s,n-1} \\ 0 \\ k\delta_{s,n-1} \\ i(4\gamma n)^{1/2}\delta_{s,n} \end{Bmatrix}(1-\delta_{n,0}), \tag{9}$$

$$\psi_{n,s,k}{}^{\downarrow}\xrightarrow[r\to 0]{}\frac{1}{(2\pi L)^{1/2}}\left[\frac{\gamma}{E(1+E)}\right]^{1/2}\begin{Bmatrix} 0 \\ i(1+E)\delta_{s,n} \\ (4\gamma n)^{1/2}\delta_{s,n-1} \\ -ik\delta_{s,n} \end{Bmatrix},$$

where $\psi^{\uparrow}$ and $\psi^{\downarrow}$ indicate spin up and spin down, respectively.

The neutron and proton are treated nonrelativistically as is usual. Since we are concerned with extremely large magnetic fields, a discussion of the effect of these fields on the neutron and proton is in order. We first consider the energy associated with the interaction between the magnetic field and the nucleon moment

$$\epsilon_{n,p}=\mu_{n,p}H\,,$$
$$|\epsilon_n|+|\epsilon_p|\approx(1/800)H/H_c. \qquad (10)$$

This is to be compared with the rest mass energy difference $W_0=M_n-M_p=2.53$. Thus, for all fields of interest the contribution of $\epsilon_{n,p}$ to the energy of the electron-antineutrino system is negligible. Therefore, we will neglect this effect and use W_0 as the transition energy independent of the spin states of the neutron and proton.

Finally, we comment on the fact that Landau quantization of proton orbits in a plane perpendicular to H is neglected. Since $H_c(\text{proton})=H_c(\text{electron})/1836$, we see that using free-particle nonrelativistic spinor wave functions for the proton is an approximation which is consistent with the above discussion on energy differences.

III. CALCULATION OF MATRIX ELEMENTS

Matrix elements can now be constructed in a manner similar to ordinary neutron β decay. The distinction is simply one of replacing the plane-wave states for the electron in the lepton matrix element by the wave functions given in Eq. (9). Squaring the matrix element, $\mathcal{H}_{fi}$, summing over proton spins and the electron quantum number s, and integrating over neutrino variables, we find for a neutron source of polarization P,

$$|\mathcal{H}_{fi}(P,k,n,e\uparrow)|^2=(g_V{}^2\gamma/2\pi LV)(1+3\lambda^2)(1-\delta_{n,0})$$
$$\times\{1-k/E-P\Lambda[1-k/E-4\gamma n/E(1+E)]\}\,, \qquad (11a)$$

$$|\mathcal{H}_{fi}(P,k,n,e\downarrow)|^2=(g_V{}^2\gamma/2\pi LV)(1+3\lambda^2)$$
$$\times\{1+k/E+P\Lambda[1+k/E-4\gamma n/E(1+E)]\}\,, \qquad (11b)$$

where $e\uparrow$ and $e\downarrow$ refer to the electron spin states and where

$$\Lambda\equiv 2\lambda(1-\lambda)/(1+3\lambda^2)\,. \qquad (12)$$

The polarization P is equal to the probability of neutron spin up minus the probability of neutron spin down. If the electron spin is not measured, we sum Eqs. (11a) and (11b) to obtain

$$|\mathcal{H}_{fi}(P,k,n)|^2$$
$$=(g_V{}^2\gamma/\pi LV)(1+3\lambda^2)\{1-\tfrac{1}{2}\delta_{n,0}(1-k/E)$$
$$+P\Lambda[k/E+\tfrac{1}{2}\delta_{n,0}(1-k/E)]\}\,. \qquad (13)$$

The transition rate is then obtained from

$$dw(P,k,n)/dk=2\pi\rho|\mathcal{H}_{fi}(P,k,n)|^2\,, \qquad (14)$$

where the density of states is

$$\rho=(LV/4\pi^3)(W_0-E)^2\,. \qquad (15)$$

If we want the transition rate to all momentum states for a given n we integrate over k consistent with

$$k\binom{\max}{\min}=\pm(W_0{}^2-1-4\gamma n)^{1/2} \qquad (16)$$

and obtain

$$w(P,n)=(g_V{}^2\gamma/\pi^3)(1+3\lambda^2)[1-\tfrac{1}{2}\delta_{n,0}(1-P\Lambda)]$$
$$\times\Bigg\{(p_0{}^2-4\gamma n)^{1/2}(1+\tfrac{1}{3}p_0{}^2+(8/3)\gamma n)$$
$$-W_0(1+4\gamma n)\sinh^{-1}\left[\left(\frac{p_0{}^2-4\gamma n}{1+4\gamma n}\right)^{1/2}\right]\Bigg\}\,, \qquad (17)$$

where $p_0{}^2=(W_0{}^2-1)$. Thus, the total transition rate is simply

$$w(P)=\sum_{n=0}^{N}w(P,n)\,, \qquad (18)$$

where N is the largest integer occurring in $p_0{}^2/4\gamma$. We discuss the interesting features of these expressions in Sec. IV.

IV. LIMITING CASES

We now obtain exact analytic expressions for $w(P)$ in the limits (a) $H/H_c\ll 1$ and (b) $H/H_c>\tfrac{1}{2}p_0{}^2$.

(*a*) $H/H_c\ll 1$. In obtaining analytic results for small fields it is easier to return to the differential transition rate Eq. (14) rather than use the total transition rate. We anticipate the k integration and set all terms that are odd functions of k equal to zero. In that case we have

$$dw(P,k,n)/dk=(g_V{}^2\gamma/2\pi^3)(1+3\lambda^2)$$
$$\times[1-\tfrac{1}{2}\delta_{n,0}(1-P\Lambda)][W_0-E(k,n)]^2\,. \qquad (14')$$

Summing over Landau levels n, we obtain

$$dw(P,k)/dk=(g_V{}^2\gamma/2\pi^3)(1+3\lambda^2)$$
$$\times\{-\tfrac{1}{2}(1-P\Lambda)[W_0-(1+k^2)^{1/2}]^2$$
$$+\sum_{n=0}^{N'}[W_0-E(k,n)]^2\}\,. \qquad (19)$$

We write

$$\sum_{n=0}^{N'}[W_0-E(k,n)]^2=4\gamma\sum_{n=0}^{N'}[c-(a^2+n^2)^{1/2}]^2\,, \qquad (20)$$

where N' is the largest integer occurring in $(p_0{}^2-k^2)/4\gamma$,

$$c^2=W_0{}^2/4\gamma\,,\quad a^2=(1+k^2)/4\gamma\,. \qquad (21)$$

The Euler summation formula[17] and the binomial expansion formula can be employed to rearrange terms

[17] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun, (U. S. Government Printing Office, National Bureau of Standards, Washington, D. C., 1964), p. 806.

and collect them in groups with equal powers of c and a. The result is

$$\sum_{n=0}^{N'} [W_0-E(k,n)]^2$$

$$=4\gamma\left\{[\tfrac{1}{6}c^4-3c^2a^2-\tfrac{1}{2}a^4+\tfrac{4}{3}ca^3] +[\tfrac{1}{2}(c-a)^2]+\left[\frac{1}{12}\left(\frac{c}{a}-1\right)\right]+O(\gamma)\right\}. \quad (22)$$

Inserting Eq. (22) into Eq. (19) and integrating over k, we obtain

$$w(P)=(g_V^2/8\pi^3)(1+3\lambda^2) \times\{-p_0-\tfrac{1}{3}p_0^3+2/15p_0^5+W_0\sinh^{-1}p_0 +[2P\Lambda(p_0+\tfrac{1}{3}p_0^3-W_0\sinh^{-1}p_0)](H/H_c) +[\tfrac{2}{3}W_0\sinh^{-1}p_0-\tfrac{2}{3}p_0](H/H_c)^2+O((H/H_c)^3)\}. \quad (23)$$

The field-independent term is exactly equal to the result obtained using plane waves for the electron and we denote it by w_0. Evaluating this equation, we obtain

$$w(P)=w_0[1-0.063P(H/H_c)+0.17(H/H_c)^2] \quad (23')$$

(*b*) $H/H_c>\frac{1}{2}p_0^2$. Referring to Eq. (18) we see that if $H/H_c>\frac{1}{2}p_0^2\simeq 2.7$, then $N=0$ and the sum reduces to a single term

$$w(P)=(g_V^2/4\pi^3)(1+3\lambda^2)(1+P\Lambda) \times(p_0+\tfrac{1}{3}p_0^3-W_0\sinh^{-1}p_0)(H/H_c), \quad (24)$$

or

$$w(P)=0.77w_0(1-0.082P)(H/H_c). \quad (24')$$

V. DISCUSSION AND CONCLUSIONS

A neutron source in a magnetic field is always positively polarized. The value of the polarization will of course depend upon the temperature and the magnetic field. In Fig. 2 we have plotted Eq. (18) for $P=0$ (unpolarized source) and $P=1$ (completely polarized source).

We see that for small fields an unpolarized source has its transition rate increase quadratically in a magnetic field while the effect on a polarized source is to decrease the transition rate linearly. Both effects are impossible to measure with the laboratory fields that can be produced today ($H/H_c\approx 10^{-7}$).

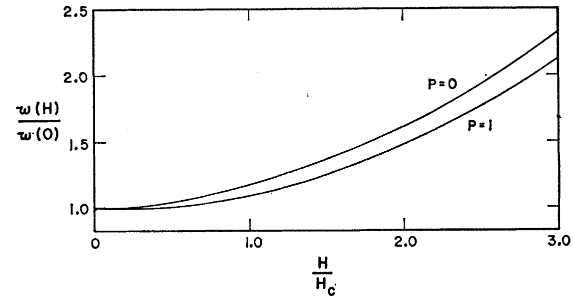

FIG. 2. β-decay transition probability per unit time in a magnetic field, normalized to the free-field result, for an unpolarized ($P=0$) and a completely polarized ($P=1$) source of neutrons.

However, for fields of the order of those speculated to exist in the early universe and in neutron stars we see from Fig. 2 that the effect on the decay rate can be substantial, differing only slightly with the degree of polarization. For any polarization P, we observe that the transition rate increases linearly with H for $H/H_c>2.7$.

An interesting observation is that the total transition rate depends upon the polarization of the neutron source. This is not the case for free-field transitions. The reason for this effect is connected with the inability of electrons to exist in the $n=0$ spin-up Landau state.

We conclude that for an unpolarized source $w>w_0$ for all values of the magnetic field. For a polarized source $w>w_0$ except for $H\ll H_c$. The astrophysical implications of this phenomena will be discussed elsewhere.[18]

ACKNOWLEDGMENT

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[18] R. F. O'Connell and J. J. Matese, Nature (to be published).