

# Decoherence in Quantum Systems

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**Abstract**—We discuss various definitions of decoherence and how it can be measured. We compare and contrast decoherence in quantum systems with an infinite number of eigenstates (such as the free particle and the oscillator) and spin systems. In the former case, we point out the essential difference between assuming “entanglement at all times” and entanglement with the reservoir occurring at some initial time. We also discuss optimum calculational techniques in both arenas.

**Index Terms**—Decoherence, dissipation, quantum information, Wigner distribution.

## I. INTRODUCTION

THERE IS presently intense interest in the fundamentals of quantum theory and applications. In particular, the superposition principle (and related work on Schrödinger cats), entanglement, and the quantum–classical interface are at the cutting edge of topical research, especially in respect to their relevance to quantum computing, quantum information processing, quantum teleportation, and quantum encryption. Since superposition states are very sensitive to decoherence, reservoir theory has attracted much recent interest.

Decoherence is the physical process which is responsible for the emergence of the classical world from the quantum world [1]–[8]. In order to understand the phenomenon in depth, much attention has been devoted to the case of a free particle interacting with a reservoir. Most of these investigations have assumed that entanglement with the reservoir occurs at some initial time, leading to the conclusion that a dissipative environment is necessary to achieve decoherence. However, we have shown that, in the case of “entanglement with the reservoir at all times,” decoherence can occur simply due to temperature effects without requiring dissipation [8]–[10], [29], [30]. In addition, we pointed out that there are many definitions of decoherence [11], and we argued that the preferred definition involves probabilities in coordinate space since they can be measured (in contrast to the off-diagonal components of the density matrix, as considered often in the literature). Moreover, the fact that the free particle wave packet spreads even in the absence of a reservoir necessitates a careful consideration of how decoherence should be defined.

By contrast, an oscillator wave packet does not spread in the absence of a reservoir, so its investigation was clearly worthy of investigation [12], [23]. This brought to light another interesting feature viz. the fact that revivals of coherence can occur. A new dimension was added to the problem by consideration of

an external field [13], [14], [31], of interest because of the recent experiments of the Wineland group [15], [16].

Thus, in Sections II and III, we present a detailed discussion of decoherence for both a free particle and an oscillator, respectively, in an arbitrary reservoir, including remarks on what we consider to be the optimal tools for such calculations. Section IV is devoted to a discussion of decoherence in spin systems and the significant differences that arise compared to systems with an infinite number of eigenstates. In Section V, we summarize and discuss our conclusions.

## II. DECOHERENCE FOR A FREE PARTICLE

Decoherence refers to the destruction of a quantum interference pattern and is relevant to the many experiments that depend on achieving and maintaining entangled states. Examples of such efforts are in the areas of quantum teleportation, quantum information and computation, entangled states, Schrödinger cats, and the quantum–classical interface. Much of the discussion of decoherence has been in terms of a particle moving in one dimension that is placed in an initial superposition state (Schrödinger “cat” state) corresponding to two widely separated Gaussian wave packets, each with variance  $\sigma^2$  and separated by a distance  $d$ . We consider the particle to be coupled to an arbitrary reservoir such that in the distant past the complete system is in thermal equilibrium at temperature  $T$ . For such a state, the probability distribution at time  $t$  can be shown to be of the form [8]

$$P(x, t) = \frac{1}{2 \left(1 + e^{-\frac{d^2}{8\sigma^2}}\right)} \times \left\{ P_0 \left(x - \frac{d}{2}, t\right) + P_0 \left(x + \frac{d}{2}, t\right) + 2e^{-\frac{d^2}{8\omega^2(t)}} a(t) P_0(x, t) \cos \frac{[x(0), x(t)] x d}{4i\sigma^2 \omega^2(t)} \right\} \equiv P_1 + P_2 + 2P_I \cos \theta(t) \quad (1)$$

where  $P_0$  is the probability distribution for a single wavepacket, given by

$$P_0(x, t) = \frac{1}{\sqrt{2\pi\omega^2(t)}} \exp \left\{ -\frac{x^2}{2\omega^2(t)} \right\}. \quad (2)$$

Here and in (1),  $\omega^2(t)$  is the variance of a single wave packet, which in general is given by

$$\omega^2(t) = \sigma^2 - \frac{[x(0), x(t)]^2}{4\sigma^2} + s(t) \quad (3)$$

where  $\sigma^2$  is the initial variance,  $[x(0), x(t)]$  is the commutator, and

$$s(t) = \left\langle \{x(t) - x(0)\}^2 \right\rangle \quad (4)$$

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is the mean square displacement. The temperature dependence enters only in  $s(t)$ . Also,  $a(t)$ , which can be defined as the ratio of the factor multiplying the cosine in the interference term to twice the geometric mean of the first two terms [8], is given by the following exact general formula:

$$a(t) = \exp \left\{ -\frac{s(t)d^2}{8\sigma^2 w^2(t)} \right\}. \quad (5)$$

The question now arises as how to define decoherence. It is often described as the disappearance of the interference term with time. However, this is not strictly correct since the integrated probability of each of the three terms in (1) is constant in time. In fact, if we define the common normalization factor in (1) as

$$N = \left[ 2 \left( 1 + e^{-\frac{d^2}{8\sigma^2}} \right) \right]^{-\frac{1}{2}} \quad (6)$$

then we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} dx P(x, t) &= N^2 \left\{ 1 + 1 + 2 \exp \left( -\frac{d^2}{8\sigma^2} \right) \right\} \\ &= 2N^2 \left\{ 1 + \exp \left( -\frac{d^2}{8\sigma^2} \right) \right\} \\ &= 1. \end{aligned} \quad (7)$$

It is clear that the integrated probability of each of the three terms is constant in time. Also, the integrated probability of the interference term is smaller than the integrated probability of either of the direct terms by a factor  $\exp(-d^2/8\sigma^2)$  which, for macroscopic  $d$  and microscopic  $\sigma$ , is very small. Thus, for example, when  $(d/\sigma) = 5$ , we see that  $\exp(-d^2/8\sigma^2) \approx 4.4 \times 10^{-2}$ , which is a very small number compared to unity.

In fact, the effect of decoherence is to cause the interference wave packet to spread in time, with a concomitant decrease in its amplitude, consistent with the fact that its integrated probability is constant in time. Moreover, we note the ubiquitous cosine term which is a feature of the interference contribution. Thus, a measure of decoherence is to focus on the interference term by calculating and measuring the time dependence of  $P_I$ . However, if decoherence is taken to mean the effects of a reservoir (not necessarily dissipative), then this definition has a problem since it gives a nonzero value for the rate of decoherence in the absence of a reservoir. The problem stems from the fact that the wave packet for a free particle spreads naturally in time, a standard result in quantum mechanics [17]. In fact, this result immediately follows from (3) since, for a free particle without dissipation and with zero temperature,  $s(t) = 0$  and  $[x(0), x(t)] = i\hbar t/m$ , so that

$$w^2(t) \rightarrow \sigma^2 + \frac{\hbar^2 t^2}{4m^2 \sigma^2}. \quad (8)$$

As a result, all of the three wave packets in (1) spread in time. Thus, in order to correct for this dynamical spreading, we were led to define [8]

$$a(t) = \frac{P_I}{(P_1 P_2)^{\frac{1}{2}}} \quad (9)$$

as a measure of decoherence. It has the virtue of reducing to unity in the absence of a reservoir [10], [29], [30].

In the case of Ohmic dissipation, high temperature ( $kT \gg \hbar\gamma$ ), small times ( $\gamma t \ll 1$ ), and  $d \gg \lambda_{\text{th}}$ ,  $\sigma$ , we obtained [8], [9]  $s(t) = (kT/m)t^2$  and  $w^2(t) \approx \sigma^2$ . Hence, (5) reduces to

$$a(t) \rightarrow \exp \left\{ -\left( \frac{t}{\tau_d} \right)^2 \right\} \quad (10)$$

where

$$\tau_d = \frac{\sqrt{8}\sigma^2}{\sqrt{\frac{kT}{m}}d} \quad (11)$$

exhibiting ‘‘decoherence without dissipation.’’ Here,  $\gamma$  is the Ohmic decay rate, and  $\lambda_{\text{th}} = \hbar/\sqrt{mkT}$  is the deBroglie wavelength. Hence,

$$\lambda_{\text{th}}^2 = (5 \cdot 2 \times 10^{-21} \text{ cm})^2 \left( \frac{1 \text{ gm}}{m} \right) \left( \frac{300 \text{ K}}{T} \right). \quad (12)$$

For example, the choice  $m = 1 \text{ gm}$ ,  $d = 1 \text{ cm}$ , and  $T = 300 \text{ K}$  leads to the very large ratio  $d/\lambda_{\text{th}} = 2 \times 10^{20}$ . In addition, we note that

$$\frac{kT}{\hbar\gamma} = \frac{T(K)}{\gamma(10^{11} \text{ s}^{-1})}. \quad (13)$$

On the other hand, if we consider a low-temperature environment ( $kT \ll \hbar\gamma$ ) [18], then, using the quantum Langevin equation [19] to evaluate  $w^2(t)$ , we obtained

$$a(t) = \exp \left\{ \left( \frac{t}{\tau_0} \right)^2 \left[ \log \frac{\zeta t}{m} + \gamma_E - \frac{3}{2} \right] \right\}, \quad \tau \ll t \ll \left( \frac{\zeta}{m} \right)^{-1} \quad (14)$$

with

$$\tau_0 \equiv \frac{m\sigma^2}{d} \sqrt{\frac{8\pi}{\hbar\zeta}} \quad (15)$$

and  $\zeta = \gamma m$  and where  $\gamma_E = 0.577215665$  is Euler’s constant. Thus, in the low-temperature regime, decoherence requires a dissipative environment.

We emphasize again that all of the above results correspond to the case of the free particle being entangled with the environment at all times. Initially (or in the distant past), the complete system is in equilibrium at temperature  $T$  [8]. At time  $t = 0$ , say, a first measurement is carried out which prepares the system of two widely separated wave packets. Then, after a time  $t$ , a second measurement is carried out which probes the system.

Most other investigations have assumed that the quantum system is initially at temperature zero and uncoupled from the reservoir, which is at temperature  $T$ . When the quantum system is brought into contact with the reservoir at  $t = 0$ , it takes a time  $\sim \gamma^{-1}$  in order for thermal equilibrium to be achieved, which is generally much larger than the decoherence decay time so that decoherence occurs before thermal equilibrium is achieved. Such a calculation generally involves the use of density matrix equations. Thus, in order to make a careful comparison with our results corresponding to ‘‘entanglement at all times,’’ we solved the exact Hu–Paz–Zhang (HPZ) master equation [7] and obtained results for an arbitrary reservoir and arbitrary

temperatures. In the particular case of high temperature, we obtained

$$a(t) \cong \exp \left\{ -\frac{\zeta k T d^2 t^3}{12 m^2 \sigma^4 + 3 \hbar^2 t^2} \right\}, \quad t \ll \frac{m}{\zeta} \equiv \gamma^{-1}. \quad (16)$$

If we suppose that the slit width is negligibly small, we find  $a(t) \cong \exp\{-t/\tau_d\}$ , where  $\tau_d = 3\hbar^2/\zeta k T d^2$ . This is similar to the decoherence time that often appears in the literature. However, as we have mentioned above, this result corresponds to a particle in an initial state that is effectively at temperature zero, which is suddenly coupled to a heat bath at high temperature. The result is therefore unphysical in the sense that the initial states does not correspond to that envisioned when we speak of a system at temperature  $T$ . Thus, with this scenario, we get no decoherence, i.e.,  $a(t) = 1$ , when  $\gamma = 0$ . On the other hand, we found that we could repair this unphysical difficulty by choosing the initial temperature to be the same as the reservoir, in which case the result given in (10) is again obtained.

We conclude this section by commenting on the techniques used. The ‘‘entanglement at all times’’ calculation utilized quantum probability distributions (which are related to Wigner distributions which are probabilities in quantum phase space [20]–[22], [12], [32], [33]) in conjunction with results obtained by use of the stationary solution to the generalized quantum Langevin equation [19].

For the case where the oscillator and reservoir are initially decoupled, we formulated the problem in terms of the Langevin equation for the initial value problem. In fact, a master equation was not required, but, in order to make contact with other investigators, we actually derived the HPZ exact master equation from the Langevin equation approach. Moreover, the Langevin approach enabled us to obtain explicit results for the time-dependent coefficients in the HPZ equation. Our strategy was based on use of the Wigner distribution, eventually leading to explicit and very general results for the Wigner distribution for the reduced system of an oscillator in an arbitrary state and in an arbitrary heat bath. These results were then used to obtain coordinate probabilities.

The Wigner distribution is, of course, the Fourier transform of the density matrix [20], so that results obtained by use of the former are equivalent to results obtained by use of the latter. However, Wigner functions are much easier to use in practice because, as distinct from density matrices, they are not operators and they are always real. They describe, in essence, quantum phase space. However, just as we object to using the density matrix as a quantitative way to describe decoherence, we also would have the same objections to the use of Wigner distributions for that purpose [11], but, instead, we regard them as a wonderful tool to calculate probability distributions, which are measurable quantities. A detailed review of the use of Wigner distributions in this context appears in [21] and [32].

### III. DECOHERENCE FOR AN OSCILLATOR

A free oscillator wave packet does not spread in time (i.e., its width does not depend on time) but it oscillates back and forth with a frequency  $\omega$ , where  $\omega$  is the oscillator frequency [22], [33]. In other words, its shape does not change but the peak of

the wave packet has the time dependence  $x_0 \cos \omega t$ . Thus, in contrast to the free particle, there is no dynamical spreading for an oscillator wave packet; it is a coherent state.

In the case of a Schrödinger cat superposition of two coherent states separated at a distance  $d$  and in thermal equilibrium in a nondissipative reservoir, we obtained a result for  $P(x, t)$  which is similar in structure to the result given in (1), the difference being that the interference term, in common with the other terms, is coherent in the sense that the wave packet shape does not change but, instead, it oscillates in time and persists for all time. Explicitly, the corresponding result for  $a(t)$  is [23]

$$a(t) = \exp \left\{ -\frac{m \omega d^2 \cos^2 \omega t}{2 \hbar \sinh \left( \frac{\hbar \omega}{k T} \right)} \right\}. \quad (17)$$

It is interesting to note that, for small times ( $\omega t \ll 1$ ) after the times for which the attenuation factor has its maximum value of unity ( $\cos \omega t = 0$ ), and if we recall that the initial width  $\sigma$  of the oscillator wave packet is given by  $\sigma^2 = (\hbar/2m\omega)$ , then, for  $\omega \rightarrow 0$ , we find that  $a(t)$  reduces to the result given in (10) and (11) for the free-particle case. However, in contrast to the latter case, for later times, there is a revival of  $a(t)$  toward its maximum value. We believe that this revival of coherence in a nondissipative thermal reservoir is a common feature, the only exception being the free particle. For the case of  $\gamma \neq 0$ , we refer to [7].

Finally, we remark that a nonrandom external field  $f(t)$  does not cause decoherence, in contrast to the case of a random  $f(t)$  [13], [14], [31], a result of relevance for the analysis of the experimental results obtained by the Wineland group [15], [16].

### IV. DECOHERENCE IN SPIN SYSTEMS

Whereas the study of decoherence for a free particle and an oscillator is very relevant for the study of the classical–quantum correspondence, it is now generally believed that spin systems (where the emphasis is on the spin and associated magnetic moment of quantum particles) are more relevant to quantum computing, teleportation and information processing. There already exists extensive studies of such systems [24]–[26], so we confine ourselves to some general remarks.

The main advantage of spin qubits are that they interact weakly with their environment. In general, as distinct from the study of free particle and oscillator systems interacting with a reservoir, we do not have to worry to the same extent with the question of when entanglement with the environment occurs, since the system is generally controlled by external fields. Thus, density matrix techniques will invariably constitute the tool of choice. This brings up the question as to whether or not there is an optimum way to solve these operator equations. We already saw in Sections II and III that Wigner distribution functions proved to be a wonderful calculational tool for the study of a system with an infinite spectrum of states. However, for spin systems, there is also what we consider a very useful and physically appealing tool viz. the use of the spin polarization vector. This has found application for particles of arbitrary spin in a magnetic field but not interacting with a reservoir [27] but, strangely enough, it has not usually been adopted for the more challenging situations where a reservoir is present. Ford and O’Connell actually used this

technique in order to calculate the spectrum of resonance fluorescence for a driven two-level atom [28], instead of using the corresponding but more unwieldy master equation. Thus, in the Appendix, we use the spin-polarization vector to study an even simpler system. This is the well-known two-level system (describing either a spin 1/2 system or a two-level atom) interacting with a reservoir, and we feel that our calculation demonstrates the simplicity and physically appealing nature of using the spin-polarization vector.

## V. DISCUSSION

We have pointed out that, whereas it is easy to describe qualitatively what is meant by decoherence, difference of opinions may arise as how best to define its signature in a quantitative manner. Different results may ensue depending on the following:

- 1) whether one is dealing with coordinate, momentum, or quantum phase space probabilities (and we favor the former since they are measurable);
- 2) how natural dynamical spreading of a wave packet in the case of a free particle is separated from spreading due to environmental effects;
- 3) whether one considers entanglement with a reservoir as existing for all times or simply taking place at some initial time;
- 4) whether external forces come into play;
- 5) the system under consideration since quantum systems with an infinite number of eigenstates (such as the free particle and the oscillator) generally exhibit decoherence decay times much smaller than relaxation times, whereas, for spin systems, decoherence decay times are often comparable to relaxation times (which is the reason they are preferred for applications).

In addition, we pointed out the great calculational advantages of using either Wigner functions (for the free particle and the oscillator) or spin polarization vectors (for spin systems).

## APPENDIX A

### SOLUTION OF THE MASTER EQUATION FOR A TWO-LEVEL SYSTEM INTERACTING WITH A RESERVOIR, USING THE SPIN-POLARIZATION VECTOR

The master equation for the reduced density matrix  $\rho$  may be written in the form

$$\begin{aligned} \frac{d\rho}{dt} &= -\frac{\gamma}{2}(1+\bar{n})\{[\sigma_+, \sigma_-\rho] + [\rho\sigma_+, \sigma_-]\} \\ &\quad -\frac{\gamma}{2}\bar{n}\{[\sigma_-, \sigma_+\rho] + [\rho\sigma_-, \sigma_+]\} \\ &= \frac{\gamma}{2}(1+\bar{n})(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-) \\ &\quad + \frac{\gamma}{2}\bar{n}(2\sigma_+\rho\sigma_- - \sigma_-\sigma_+\rho - \rho\sigma_-\sigma_+) \end{aligned} \quad (A1)$$

where  $\bar{n}$  is the thermal average boson number

$$\bar{n} = \left[ \exp \left\{ \frac{\hbar\omega}{kT} \right\} - 1 \right]^{-1}. \quad (A2)$$

Also

$$\sigma_{\pm} = \frac{1}{2}(\sigma_x \pm i\sigma_y) \quad (A3)$$

where the  $\sigma_i$ 's are the usual Pauli spin matrices.

Now  $\rho$  can be expanded in terms of the complete set  $(I, \sigma_i)$  with real coefficients as follows:

$$\rho = \frac{1}{2}(I + \vec{P} \cdot \vec{\sigma}) \quad (A4)$$

where  $I$  is the identity matrix and  $\vec{P}$  is the polarization vector. This leads in the usual manner to

$$\vec{P} = \langle \vec{\sigma} \rangle = Tr(\vec{\sigma}\rho) \quad (A5)$$

i.e., the polarization vector is the ensemble average of the spin vector. Thus,

$$\frac{d\vec{P}}{dt} = Tr \left( \vec{\sigma} \frac{d\rho}{dt} \right). \quad (A6)$$

Next, we note that

$$\begin{aligned} Tr \{ \vec{\sigma}(\sigma_+\sigma_-\rho + \rho\sigma_+\sigma_- - 2\sigma_-\rho\sigma_+) \} \\ &= \langle \vec{\sigma}\sigma_+\sigma_- + \sigma_+\sigma_-\vec{\sigma} - 2\sigma_+\vec{\sigma}\sigma_- \rangle \\ &= \left\langle \frac{1}{2}\vec{\sigma}(1+\sigma_z) + \frac{1}{2}(1+\sigma_z)\vec{\sigma} + (1+\sigma_z)\hat{z} \right\rangle \\ &= \langle \vec{\sigma} + \hat{z} + \hat{z} + \sigma_z\hat{z} \rangle \\ &= \vec{P} + \hat{z} \cdot \vec{P}\hat{z} + 2\hat{z} \end{aligned} \quad (A7)$$

and

$$\begin{aligned} Tr \{ \vec{\sigma}(\sigma_-\sigma_+\rho + \rho\sigma_-\sigma_+ - 2\sigma_+\rho\sigma_-) \} \\ &= \langle \vec{\sigma}\sigma_-\sigma_+ + \sigma_-\sigma_+\vec{\sigma} - 2\sigma_-\vec{\sigma}\sigma_+ \rangle \\ &= \left\langle \frac{1}{2}\vec{\sigma}(1-\sigma_z) + \frac{1}{2}(1-\sigma_z)\vec{\sigma} - (1-\sigma_z)\hat{z} \right\rangle \\ &= \langle \vec{\sigma} - 2\hat{z} + \sigma_z\hat{z} \rangle \\ &= \vec{P} + \hat{z} \cdot \vec{P}\hat{z} - 2\hat{z}. \end{aligned} \quad (A8)$$

Hence, using (A1) and (A6)–(A8), we have

$$\begin{aligned} \frac{d\vec{P}}{dt} &= -\frac{\gamma}{2}(1+\bar{n})(\vec{P} + P_z\hat{z} + 2\hat{z}) - \frac{\gamma}{2}\bar{n}(\vec{P} + P_z\hat{z} - 2\hat{z}) \\ &= -\frac{\gamma}{2}(2\bar{n}+1)(\vec{P} + P_z\hat{z}) - \gamma\hat{z}. \end{aligned} \quad (A9)$$

At equilibrium (where  $P = P_0$ )

$$\frac{d\rho}{dt} = 0 \quad (A10)$$

so that

$$\frac{dP}{dt} = 0. \quad (A11)$$

Hence, the right-hand side of (A9) is zero, which implies

$$\vec{P} + P_z\hat{z} = -\frac{2\hat{z}}{2\bar{n}+1}. \quad (A12)$$

It is thus clear that, in thermal equilibrium at temperature  $T$ ,  $\vec{P}$  is along  $\hat{z}$ , i.e.,

$$\vec{P} = P_z \hat{z} \equiv P_0 \hat{z}. \quad (\text{A13})$$

Thus, from (A12), we obtain

$$P_0 = -\frac{1}{2\bar{n} + 1} = -\tanh\left(\frac{\hbar\omega}{2kT}\right). \quad (\text{A14})$$

For  $T = 0$  ( $\bar{n} = 0$ ), we see that  $P_0 = -1$  corresponding to all of the spins being down. On the other hand, for  $T \rightarrow \infty$  ( $\bar{n} \rightarrow \infty$ ), we obtain  $P_0 \rightarrow 0$ , corresponding to the equal number of up and down spins. Hence, (A9) becomes

$$\frac{d\vec{P}}{dt} = \frac{\gamma}{2P_0}(\vec{P} + P_z \hat{z}) - \gamma \hat{z}. \quad (\text{A15})$$

Hence,

$$\begin{aligned} \frac{dP_z}{dt} &= \frac{\gamma}{P_0} P_z - \gamma \\ &= \frac{P_0 - P_z}{T_1} \end{aligned} \quad (\text{A16})$$

where

$$\frac{1}{T_1} \equiv -\frac{\gamma}{P_0} = \gamma(2\bar{n} + 1) = \gamma \coth\left(\frac{\hbar\omega}{2kT}\right). \quad (\text{A17})$$

Thus, for  $T \rightarrow 0$ ,  $T_1 \rightarrow \gamma^{-1}$  whereas for  $T \rightarrow \infty$ ,  $T_1 \rightarrow 0$ . The solution of (A16) is

$$P_z = P_0 \left\{ 1 - \exp\left(-\frac{t}{T_1}\right) \right\}. \quad (\text{A18})$$

In addition, (A15) may be written as

$$\frac{d\vec{P}}{dt} = -\frac{1}{2T_1}(\vec{P} + P_z \hat{z}) - \gamma \hat{z}. \quad (\text{A19})$$

Thus, defining  $P_\perp$  to be the component of  $\vec{P}$  perpendicular to  $\hat{z}$ , we obtain

$$\frac{dP_\perp}{dt} = -\frac{P_\perp}{2T_1} \quad (\text{A20})$$

whose solution is

$$P_\perp = P_\perp(0) \exp\left(-\frac{t}{T_2}\right) \quad (\text{A21})$$

where

$$T_2 \equiv 2T_1. \quad (\text{A22})$$

Since the magnetic moment  $\vec{M}$  of a nuclear ion with spin  $\vec{S} = (\hbar/2)\vec{\sigma}$  is given by

$$\vec{M} = g_n \frac{e}{2mc} \vec{S} = \frac{e\hbar}{2mc} \frac{g_n}{2} \vec{\sigma} = \mu_0 \frac{g_n}{2} \vec{\sigma} \quad (\text{A23})$$

where  $m$  is the nuclear mass,  $e$  is its charge,  $g_n$  is the nuclear ion  $g$  factor, and

$$\mu_0 = \frac{e\hbar}{2mc} \quad (\text{A24})$$

is the nuclear Bohr magneton, it follows that

$$\langle \vec{M} \rangle = -\mu_0 \frac{g_n}{2} \langle \vec{\sigma} \rangle = -\mu_0 \frac{g_n}{2} \vec{P}. \quad (\text{A25})$$

Hence, using (A18) and (A21), we obtain

$$\langle M_z \rangle = M_o \left\{ 1 - \exp\left(-\frac{t}{T_1}\right) \right\} \quad (\text{A26})$$

and

$$\langle M_\perp \rangle = M_\perp(0) \exp\left(-\frac{t}{T_2}\right). \quad (\text{A27})$$

We note that  $T_1$  is the so-called spin-lattice (longitudinal) relaxation time which describes the approach to thermal equilibrium whereas  $T_2$  is the dephasing (transverse) relaxation time, describing a system of spin 1/2 particles in a magnetic field.

It is also of interest to rewrite (A4) in the form

$$\rho = \frac{1}{2} \{ 1 + (\sigma_+ P_- + \sigma_- P_+) + \sigma_z P_z \} \quad (\text{A28})$$

where

$$P_\pm = (P_x \pm iP_y) \quad (\text{A29})$$

so that, as is also obvious from (A19)–(A21)

$$\frac{dP_\pm}{dt} = -\frac{P_\pm}{2T_1} \quad (\text{A30})$$

and

$$P_\pm = P_\pm(0) \exp\left(-\frac{t}{T_2}\right). \quad (\text{A31})$$

Moreover, if we write

$$\rho = \begin{pmatrix} \rho_{++} & \rho_{+-} \\ \rho_{-+} & \rho_{--} \end{pmatrix} \quad (\text{A32})$$

and use the fact that

$$\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (\text{A33})$$

it readily follows from (A28) that

$$\rho_{++} = \frac{1}{2}(1 + P_z) \quad (\text{A34})$$

$$\rho_{--} = \frac{1}{2}(1 - P_z) \quad (\text{A35})$$

$$\rho_{+-} = \frac{1}{2}(1 + P_-) \quad (\text{A36})$$

and

$$\rho_{-+} = \frac{1}{2}(1 + P_+). \quad (\text{A37})$$

Thus, using (A18) and (A31), it is clear that we obtain the familiar result that the rate of decay of the diagonal elements of the density matrix ( $T_1^{-1}$ ) is twice that of the rate of decay of the off-diagonal elements ( $T_2^{-1}$ ). However, we suggest that it is physically more appealing to say that we conclude [from (A18) and either (A21) or (A31)] that the expectation value of the spin component in the  $z$  direction (direction of the magnetic field) decays faster than the corresponding values perpendicular to the  $z$  direction.

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#### REFERENCES

- [1] H. Zeh, *Found. Phys.*, vol. 1, p. 69, 1970.
- [2] A. J. Leggett, *Suppl. Prog. Theor. Phys.*, vol. 69, p. 80, 1980.
- [3] E. P. Wigner, *Quantum Optics, Experimental Gravity and Measurement Theory*, P. Meystre and M. O. Scully, Eds. New York: Plenum, 1983, p. 43.
- [4] D. F. Walls and G. J. Milburn, *Phys. Rev. A, Gen. Phys.*, vol. 31, p. 2403, 1985.
- [5] E. Joos, H. D. Zeh, C. Kiefer, D. Giulini, J. Kupsch, and I.-O. Stamatescu, *Decoherence and the Appearance of a Classical World in Quantum Theory*, 2nd ed. Berlin, Germany: Springer-Verlag, 2003.
- [6] W. H. Zurek, *Rev. Mod. Phys.*, vol. 75, p. 715, 2003.
- [7] G. W. Ford and R. F. O'Connell, *Phys. Rev. D, Part. Fields*, vol. 64, p. 105 020, 2001.
- [8] G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A, Gen. Phys.*, vol. 64, p. 032 101, 2001.
- [9] G. W. Ford and R. F. O'Connell, *Phys. Lett. A*, vol. 286, p. 87, 2001.
- [10] —, *Amer. J. Phys.*, vol. 70, p. 319, 2002.
- [11] M. Murakami, G. W. Ford, and R. F. O'Connell, *Laser Phys.*, vol. 13, p. 180, 2002.
- [12] G. W. Ford and R. F. O'Connell, *Proc. Wigner Centennial Conf.*, Pecs, Hungary, Jul. 2002.
- [13] R. F. O'Connell and J. Zuo, *Phys. Rev. A, Gen. Phys.*, vol. 67, p. 062 107, 2003.
- [14] J. Zuo and R. F. O'Connell, *J. Mod. Opt.*, vol. 51, p. 821, 2004.
- [15] C. J. Myatt *et al.*, *Nature*, vol. 403, p. 269, 2000.
- [16] Q. A. Turchette *et al.*, *Phys. Rev. A, Gen. Phys.*, vol. 62, p. 053 807, 2000.
- [17] E. Merzbacher, *Quantum Mechanics*, 3rd ed. New York: Wiley, 1998.
- [18] G. W. Ford and R. F. O'Connell, *J. Opt.*, vol. 5, p. S609, 2003.
- [19] G. W. Ford, J. T. Lewis, and R. F. O'Connell, *Phys. Rev. A, Gen. Phys.*, vol. 37, p. 4419, 1988.
- [20] M. Hillery, R. F. O'Connell, M. O. Scully, and E. P. Wigner, *Phys. Rep.*, vol. 106, p. 121, 1984.
- [21] R. F. O'Connell, *J. Opt. B*, vol. 5, p. 5349, 2004.
- [22] M. O. Scully and M. S. Zubairy, *Quantum Optics*. Cambridge: Cambridge Univ. Press, 1997, p. 53.
- [23] G. W. Ford and R. F. O'Connell, *Acta Phys. Hung. B*, vol. 20, no. 1–2, p. 91, 2004.
- [24] S. A. Wolf *et al.*, *Science*, vol. 294, p. 1488, 2001.
- [25] D. W. Awschalom, M. E. Flatte, and N. Samarth *et al.*, *Sci. Amer.*, vol. 286, no. 6, p. 67, 2002.
- [26] J. Hogan, *Nature*, vol. 424, p. 484, 2003.
- [27] L. I. Schiff, *Quantum Mechanics*, 3rd ed. New York: McGraw-Hill, 1968, pp. 381–384.
- [28] G. W. Ford and R. F. O'Connell, *Opt. Commun.*, vol. 179, p. 451, 2000.
- [29] —, *Virtual J. Nanoscale Sci. Technol.*, vol. 5, no. 8, Feb. 2002.
- [30] —, *Virtual J. Quantum Inform.*, vol. 2, no. 3, Mar. 2002.
- [31] R. F. O'Connell and J. Zuo, *Virtual J. Quantum Inform.*, vol. 3, no. 7, Jul. 2003.
- [32] R. F. O'Connell, *J. Opt. B*, vol. 5, p. S349, 2003.
- [33] W. P. Schleich, *Quantum Optics in Phase Space*. New York: Wiley, 2001, p. 114.

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