EFFECT OF THE EARTH'S QUADRUPOLE MOMENT ON THE PRECESSION OF A GYROSCOPE

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Abstract. It is shown that there is a contribution to the precession of a gyroscope due to the quadrupole moment of the Earth. The magnitude of this additional precession is a maximum for a gyroscope in a satellite in equatorial orbit around the Earth at a moderate altitude, and exceeds the experimental error for observations which cover a span of about one year or greater.

An exciting new possibility for testing Einstein's general theory of relativity is the gyroscope experiment which was proposed by SCHIFF (1960, 1964). This test of the theory, in contrast to the classical tests, is also sensitive to off-diagonal terms in the metric (THIRRING and LENSE, 1918). In addition, as the present author recently pointed out (O'CONNELL, 1968a) it has the further merit of offering what is perhaps the best possibility for testing the Brans-Dicke theory (BRANS and DICKE, 1961). Precession of a gyroscope in the linear theories of Birkhoff and of Belinfante and Swihart has also been considered (PUSTOVOĬT and BAUTIN, 1964). In all of these calculations it is assumed that the motion of the gyroscope takes place in the field of a rotating *spherically-symmetric Earth*. However, the Earth is not spherical and it is our purpose in this communication to show that the Earth's quadrupole moment gives an additional contribution to the gyroscope precession. The contribution is small but for observation times of a year or more it can be greater than the experimental error.

The Newtonian potential ϕ at a point due to a body of mass M and quadrupole moment Q is

$$\phi = \frac{GM}{r} + Q\frac{GM}{r^3}(1 - 3\cos^2\theta), \tag{1}$$

where r is the distance from the center of M to the point and θ is the azimuthal angle from the polar axis. In Einstein's theory of general relativity, the metric which represents this static distribution of matter, correct to lowest order in ϕ (in our units $c=1$), may be written (MCVITTIE, 1965)

$$ds^2 = e^{\mu}\,dt^2 - e^{-\mu}(dr^2 + r^2\,d\theta^2 + r^2\sin^2\theta\,d\phi^2), \tag{2}$$

where

$$e^{\mu} = (1 - 2\phi). \tag{3}$$

To obtain the maximum contribution due to the quadrupole moment we take $\theta=\pi/2$ (equatorial orbit). As with SCHIFF (1960) we make use of the equations of motion of spinning test particles (PAPAPETROU, 1951; CORINALDESI and PAPAPETROU, 1951) and then make coordinate and Lorentz transformations to obtain the precession of a gyroscope as measured by a co-moving observer travelling with the gyroscope. Denoting the spin angular momentum vector in the co-moving system by $\mathbf{S}_0$ we find that

$$\frac{\mathrm{d}\mathbf{S}_0}{\mathrm{d}t} \propto \frac{\mathrm{d}\mu}{\mathrm{d}r} = \frac{2GM}{r^2}\left(1 + \frac{3Q}{r^2}\right) e^{-\mu} \tag{4}$$

which, to lowest order in ϕ and for $Q=0$ agrees with the results of SCHIFF (1960) which in turn lead to the result that the angular velocity of precession, $\mathbf{\Omega}_\mathrm{E}$ say, is

$$\mathbf{\Omega}_\mathrm{E} = \mathbf{\Omega}_\mathrm{T} + \mathbf{\Omega}_\mathrm{DS} + \mathbf{\Omega}_\mathrm{LT}, \tag{5}$$

where $\mathbf{\Omega}_\mathrm{T}$, $\mathbf{\Omega}_\mathrm{DS}$ and $\mathbf{\Omega}_\mathrm{LT}$ are the so-called (SCHIFF, 1960; O'CONNELL, 1968a) Thomas, De Sitter, and Lense-Thirring contributions, respectively. In the case of $Q \neq 0$ we find that there is an additional contribution, $\mathbf{\Omega}_Q$ say, to the RHS of Equation (5). It is not difficult to show that

$$\mathbf{\Omega}_Q = \frac{3Q}{r^2}\,\mathbf{\Omega}_\mathrm{DS} = J\,(R/r)^2\,\mathbf{\Omega}_\mathrm{D}{}^\mathrm{S}, \tag{6}$$

where R is the radius of the earth and J is equal to 1.6375×10^{-3} (JEFFREYS, 1962). Thus, for an orbit in the Earth's equatorial plane, we find

$$\mathbf{\Omega}_Q = 1.70 \times 10^{-2}\,(R/r)^3\,\boldsymbol{\omega}_\mathrm{o}, \tag{7}$$

where $\boldsymbol{\omega}_\mathrm{o}$ is the instantaneous orbital angular velocity of the gyroscope. For a circular orbit, we get the convenient expression

$$\mathbf{\Omega}_Q = 1.40 \times 10^{-2}\,(R/r)^{9/2}\,\hat{\omega}_\mathrm{o}\ \text{arc sec/year}. \tag{8}$$

It now appears possible (EVERITT and FAIRBANK, 1966) that the gyroscope spin axis can be read to an accuracy of 1.0×10^{-2} arc sec. Thus, for a satellite in equatorial orbit around the Earth at a moderate altitude, the magnitude of the precession per year due to the quadrupole moment of the Earth may be larger than the experimental error. Moreover, it is important to note that *this is a secular effect*; so that the greater the length of time over which measurements are taken, the greater will be the contribution of Ω_Q. The expression for Ω_Q corresponding to all possible orientations of the orbit with respect to the Earth's equator is much more laborious to obtain and will be considered in detail elsewhere (O'CONNELL, 1968b).

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ABSTRACTS OF FORTHCOMING PAPERS

Ya. B. Zeldovich and R. A. Sunyaev: **The Interaction of Matter and Radiation in a Hot-Model Universe.** In Russian and in English. (Received 30 December, 1968.)

In this paper we continue the investigation initiated by Weymann as to the reason why the spectrum of the residual radiation deviates from a Planck curve. We shall consider the distortions of the spectrum resulting from radiation during the recombination of a primeval plasma. Analytical expressions are obtained for the derivation from an equilibrium spectrum due to a Compton scattering by hot electrons. On the basis of the observational data it is concluded that a period of neutral hydrogen in the evolution of the universe is unavoidable. It is shown that any injection of energy at $t > 10^9$ sec (redshift $z < 10^5$) leads to deviation from an equilibrium spectrum.

Zdeněk Kopal: **The Precession and Nutation of Deformable Bodies, II.** (Received 18 January, 1969.)

In a previous paper of this series (Kopal, 1968a) the Eulerian equations have been set up which govern the precession and nutation of selfgravitating bodies of viscous fluid in inertial coordinates which are at rest in space. In order to facilitate their solution, in the present investigation we shall transform these equations to the rotating body-axes; and shall explicitly evaluate all their coefficients arising as a result of second-harmonic dynamical tides.

Following the introductory Section 1 which contains a mathematical statement of the problem, the requisite transformation of coordinates will be outlined in Section 3, and applied to the equations of motion in Section 5. The corresponding moments and products of inertia appropriate for self-gravitating configurations of arbitrary internal structure will be formulated in Section 5; while the deformation terms arising from second-harmonic dynamical tides raised on centrally-condensed configurations will be evaluated in sections 3 and 6. The concluding Section 6 will then contain a specification of the components of the disturbing force.

The next stage of our investigation – namely, a construction of actual solutions of the equations governing precession and nutation of fluid bodies in different cases of astrophysical interest – has been postponed for a separate paper.

P. Brosche: **Das Geschwindigkeitsellipsoid der HII Regionen in M33.** (Received 27 January, 1969.)

The peculiar radial velocities of HII-regions in M33 varies with position angle in a manner that can be explained by an ellipsoidal distribution with minor axis in the direction of rotation. The amplitude of the variations is, however, too great as compared with theory or experience in the galaxy.

A. E. Roy: **Luni-Solar Perturbations of an Earth Satellite.** (Received 28 January, 1969.)

Luni-solar perturbations of the orbit of an artificial Earth satellite are given by modifying the analytical theory of an artificial lunar satellite derived by the author in recent papers. Expressions for the first-order changes, both secular and periodic, in the elements of the geocentric Keplerian orbit of the earth satellite are given, the Moon's geocentric orbit, including solar perturbations in it, being found by using Brown's lunar theory.

The effects of Sun and Moon on the satellite orbit are described to a high order of accuracy so that the theory may be used for distant earth satellites.

Petr Vaníček: **Approximate Spectral Analysis by Least-Square Fit (Successive Spectral Analysis).** (Received 28 January, 1969.)

An approximate method of spectral analysis called 'successive spectral analysis' based upon the mean-quadratic approximation of an empirical function by generalized trigonometric polynomial with both unknown frequencies and coefficients is developed. A few quotations describing some properties of the method as well as one of the possible methods for numerical solution are given.

L. Rosino, G. Chincarini, and A. Mammano: **Nova Vulpeculae 1968 No. 2 (Kohoutek).** (Received 3 February, 1969.)

Nova Vul 1968 No. 2 (Kohoutek) has been studied on Asiago material obtained before and after the announcement of discovery. The nova, fainter than 20 magnitude at minimum, has brighened on July 16. The maximum (9.25 *B*) has been reached on July 19. From the light curve the star can be classified as a normal fast nova. Objective prism spectra taken near maximum display the presence of absorption systems with radial velocities from −550 to −2200 km/sec. On slit spectrograms obtained at the end of October, the nova has been found in the nebular stage with wide emission bands of H, He I, He II, N II, N III and forbidden lines of O III, N II, O I. The degree of excitation is slowly increasing.

The nova is strongly reddened by interstellar absorption.

J. E. Dyson: **The Interpretation of Hydrogen Radio Recombination Lines.** (Received 6 February, 1969.)

Computations of the high level populations of hydrogen in gaseous nebulae are used to compare observations of radio recombination lines with theoretical predictions based on possible line enhancement. Attempts to confirm the existence of maser action from electron temperatures derived on the assumption of thermodynamic equilibrium are inconclusive. There is evidence that most of the low ($\lesssim$ 5000 K) derived temperatures can be increased by at most a few per cent allowing for line enhancement. Measured ratios of the peak temperatures of lines of the same frequencies originating from different upper quantum levels, indicate maser action if Stark broadening is not taken into account. The inclusion of Stark broadening produces confirmation of maser action only in the central regions of the Orion Nebula in the case of the $137\beta/109\alpha$ ratio, and in Orion, and, possibly, IC 1795 and M17, in the case of the $197\beta/156\alpha$ ratio.

A. Kovetz: **Isentropic Stars in General Relativity.** (Received 14 February, 1969.)

In an investigation of the evolution of homogeneous, isentropic stars through stages of diminishing entropy, Rakavy and Shaviv have recently (1968) found that stars of mass less than M_c (Chandrasekhar's limiting mass for white dwarfs) evolve into white dwarfs, while stars of mass greater than M_c approach a (singular) state of minimum entropy. An elementary explanation of these results is given and qualitative effects of general relativity are discussed. It is found that stars which are lighter than the Oppenheimer and Volkoff (1939) limit become white dwarfs, while heavier stars must become dynamically unstable at a finite stage in their evolution.

C. J. Waddington: **The Fragmentation of Cosmic-Ray Nuclei in Interstellar Hydrogen.** (Received 17 February, 1969.)

In order to calculate the effects of traversal of interstellar matter on the charge spectrum of the cosmic radiation it is necessary to have values for the fragmentation parameters of nuclei of each element into all lighter elements. Most of these values have not been experimentally determined. As a consequence, they have been calcultated from a semi-empirical mass spallation relation designed to fit the available partial cross-sections obtained from radio chemical determinations. This calculation has attempted to take into account the conditions that are peculiar to the cosmic ray problem. Values of the parameters are given for three characteristic energies and a comparison is made with the space experimental data. The effects of using these parameters in a calculation of the extrapolation of the charge spectrum through interstellar space are shown for some representative cases.

ZDENĚK KOPAL: **The Precession and Nutation of Deformable Bodies, III.** (Received 19 February, 1969.)

In preceding papers of this series (Kopal, 1968; 1969) the Eulerian equations have been set up which govern the precession and nutation of self-gravitating fluid globes of arbitrary structures in inertial coordinates (space-axes) as well as with respect to the rotating body axes; with due account being taken of the effects arising from equilibrium as well as dynamical tides.

In Section 1 of the paper, the explicit form of these equations is recapitulated for subsequent solutions. Section 2 contains then a detailed discussion of the coplanar case (in which the equation of the rotating configuration and the plane of its orbit coincide with the invariable plane of the system); and small fluctuations in the angular velocity of axial rotation arising from the 'tidal breathing' in eccentric binary systems are investigated.

In Section 3, we consider the angular velocity of rotation about the Z'-axis to be constant, but allow for finite inclination of the equator to the orbital plane. The differential equations governing such a problem are set up exactly in terms of the time-dependent Eulerian angles θ and ϕ, and their coefficients averaged over a cycle. In Section 4, these equations are linearized by the assumption that the inclinations of the equator and the orbit to the invariable plane of the system are small enough for their squares to be negligible; and the equations of motion reduced to their canonical form.

The solution of these equations – giving the periods of precession and nutation of rotating components of close binary systems, as well as the rate of nodal regression which is synchronised with precession – are expressed in terms of the physical properties of the respective system and of its constituent components; while the concluding Section 6 contains a discussion of the results, in which the differences between the precession and nutation of rigid and fluid bodies are pointed out.

JEFFREY D. ROSENDHAL: **The Effect of Electron Scattering on Curves of Growth.** (Received 19 February, 1969.)

A new exact solution of the macroscopic line transfer equation including electron scattering terms has been obtained for a Milne-Eddington Model atmosphere, and curves of growth based on this solution have been calcultated. The results indicate that for lines formed by scattering there is a systematic change in the appropriate theoretical curve of growth as electron scattering becomes an increasingly important source of continuous opacity. In a case where electron scattering is the dominant opacity source, the abundance necessary to produce a given line strength may be decreased by a factor of 2–5 and the derived velocity parameter decreased by 20–30% due to the shifts in the theoretical curves.