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Effect of an external field on decoherence: Part II

JIAN ZUO and R. F. O'CONNELL*

Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana 70803-4001, USA.
*E-mail: rfoc@rouge.phys.lsu.edu

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Abstract. A previous paper analysed the experimental results of Myatt *et al.* [*Nature*, **403**, 269 (2000)] on the decoherence of superposition states of Be ions confined in a harmonic potential by a linear Paul trap and subject to a random external electric field. Based on a rigorous derivation in the absence of an external force which had previously been obtained, a heuristic approach to the problem was presented. Whereas this approach was simple and physically appealing, a more rigorous analysis is clearly desirable, which is the content of this paper. It also makes clear that, for the case of a random force, decoherence is intrinsic to quantum mechanics and does not require a dissipative environment.

1. Introduction

Engineering decoherence is a subject of much topical interest [1], both from the viewpoint of investigations into the fundamentals of quantum mechanics and information on mechanisms that might hinder the realization of schemes to achieve quantum computation, teleportation, etc. A pioneering experiment dealing with this subject is described in recent papers from the NIST group of Wineland [2, 3].

Myatt *et al.* [2] used a linear Paul trap to confine single Be ions in a harmonic potential and then prepared various superposition states. Next, they induced decoherence by coupling the single ion to a reservoir, which they controlled in various ways. Such a reservoir gives rise to an external force $f(t)$ in the equation of motion of the system, in contrast to the usual intrinsic fluctuation force $F(t)$ that arises from interaction with an ambient thermal dissipative environment [4], which, of course, will always be present, even at $T = 0$. Reference [4] is a rigorous and detailed analysis (based on the use of non-equilibrium statistical mechanics and quantum Brownian motion theory) of decoherence effects in a general dissipative environment for arbitrary temperatures, but for the case $f(t) = 0$.

Our previous analysis of the problem with $f(t) \neq 0$ [5] was heuristic to a certain degree since it assumed that the well-established results given in [4] for the $f(t) = 0$ case could be taken over to the $f(t) \neq 0$ case by simply incorporating the effect of $f(t)$ in the result for the mean-square-displacement of the charged quantum particle, a key ingredient of [4]. Whereas this method has the merit of getting an answer quickly in a physically appealing manner, it is desirable to also carry out a more rigorous calculation. In fact, since for the case of the NIST experiment [2, 3], both the temperature $T \to 0$ and the dissipative decay rate $\gamma \to 0$, it should be clearly

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possible to carry out such a calculation within the realm of standard quantum mechanics. Our purpose here is to show that this can be achieved by making use of the Feynman–Hibbs expression [6] for the propagator and wave function of an oscillator driven by an external force $f(t)$. The results we obtain for the decoherence agree with those previously obtained in the case of a random $f(t)$. In addition, we reach the important conclusion that, for the case of a random force, decoherence is intrinsic to quantum mechanics and does not require a dissipative environment. (Decoherence without dissipation can also occur due to temperature effects [7, 8].)

In section 2, we discuss the case of the oscillator subject to a driven (external) force $f(t)$ and, using the relevant propagator, we obtain the time-dependent wave function. Next, in section 3, we obtain a very general expression for the probability distribution at time t for the case of an arbitrary initial state. Section 4 is devoted to an explicit evaluation of the probability distribution for the case of the oscillator in a single Gaussian state and we analyse the effect of $f(t)$ both for the random and non-random cases. Then, in section 5, we consider a superposition of Gaussian states, separated by a distance d, which leads to an analysis of the effect of $f(t)$ on decoherence. Conclusions are presented in section 6.

2. Wave function and propagator for a driven oscillator

In general, a wave function at time t may be written in terms of the initial ($t = 0$) wave function as

$$\psi(x, t) = \int_{-\infty}^{\infty} \mathrm{d}x' K(x, t; x', 0)\psi(x', 0), \tag{2.1}$$

where K is the propagator. In the particular case where H is time independent

$$K(x, t; x', 0) = \langle x| \exp\left(-\frac{\mathrm{i}}{\hbar}Ht\right)|x'\rangle. \tag{2.2}$$

For the driven oscillator, the Lagrangian is [6]

$$L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 + f(t)x, \tag{2.3}$$

and the action is given by

$$S = \int_0^t \mathrm{d}s\, L(s). \tag{2.4}$$

The result for the classical action is (see [6], p. 64)

$$\begin{aligned} S_{cl} = {} & \frac{m\omega}{2\sin(\omega t)}[(x_i^2 + x_f^2)\cos(\omega t) - 2x_i x_f] \\ & + \frac{x_f}{\sin(\omega t)}\int_0^t \mathrm{d}s \sin(\omega s) f(s) + \frac{x_i}{\sin(\omega t)}\int_0^t \mathrm{d}s \sin(\omega(t-s))f(s) \\ & - \frac{1}{m\omega\sin(\omega t)}\int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \sin(\omega u)\sin(\omega(t-s))f(s)f(u), \end{aligned} \tag{2.5}$$

where x_i and x_f refer to the initial and final coordinates of the path and t is the time difference. Also

$$K(x_f, t; x_i, 0) = \left\{\frac{m\omega}{2\pi \mathrm{i}\hbar\sin(\omega t)}\right\}^{1/2} \exp\left\{\frac{\mathrm{i}}{\hbar}S_{cl}\right\}. \tag{2.6}$$

It is convenient to re-write (2.5) in the form

$$\begin{aligned} S_{cl} &= S_1(x_i, x_f) + x_f I_1 + x_i I_2 - I_3 \\ &= S_1(x_i, x_f) + \frac{x_f}{\sin\omega t} J_1 + \frac{x_i}{\sin\omega t} J_2 - \frac{1}{m\omega\sin\omega t} J_3, \end{aligned} \tag{2.7}$$

where

$$S_1(x_i, x_f) = \frac{m\omega}{2\sin(\omega t)}[(x_i^2 + x_f^2)\cos(\omega t) - 2x_i x_f] \tag{2.8}$$

$$J_1 = \int_0^t \mathrm{d}s \sin(\omega s) f(s) \tag{2.9}$$

$$J_2 = \int_0^t \mathrm{d}s \sin(\omega(t-s)) f(s) \tag{2.10}$$

$$J_3 = \int_0^t \mathrm{d}s \int_0^s \mathrm{d}u \sin(\omega u)\sin(\omega(t-s)) f(s) f(u). \tag{2.11}$$

As a check, we note that when $f(t) = 0$, it is clear from (2.9) to (2.11) that the integrals $J_1 = J_2 = J_3 = 0$. Hence, $S_{cl} \to S_1$ given by (2.8) so, from (2.6), we obtain

$$K(x_f, t; x_i, 0) = \left\{\frac{m\omega}{2\pi\mathrm{i}\hbar\sin(\omega t)}\right\}^{1/2} \exp\left\{\frac{\mathrm{i}}{\hbar}\frac{m\omega}{2\sin(\omega t)}\left[\left(x_i^2 + x_f^2\right)\cos(\omega t) - 2x_i x_f\right]\right\}, \tag{2.12}$$

which is the propagator for the free oscillator [9]. Next, we define

$$J \equiv x_d(t) = \int_0^t \mathrm{d}s\, G(t-s) f(s)\,\mathrm{d}s, \tag{2.13}$$

where $x_d(t)$ denotes the coordinate displacement due to the external force and where the Green function for the oscillator is given by (making use of equations (2.6) and (2.7) of [10] in the limit of zero dissipation)

$$G(t) = \frac{\sin\omega t}{m\omega}. \tag{2.14}$$

Thus, it follows that

$$J_2 = m\omega J, \tag{2.15}$$

and

$$J = \frac{1}{m\omega}\int_0^t \mathrm{d}s \sin\omega(t-s) f(s). \tag{2.16}$$

3. Probability distribution

We now go beyond the exposition of Feynman–Hibbs to discuss the probability distribution

$$\begin{aligned} P(x;t) &= |\psi(x,t)|^2 \\ &= \int_{-\infty}^{\infty} \mathrm{d}x'' \int_{-\infty}^{\infty} \mathrm{d}x'\, K^*(x, x''; t) K(x, x'; t)\psi^*(x'', 0)\psi(x', 0). \end{aligned} \tag{3.1}$$

Thus, using (2.6), and noting from (2.5) that S_{cl} is real, we obtain

$$
\begin{aligned}
K^*(x, x''; t)K(x, x'; t) &= \left\{\frac{m\omega}{2\pi\hbar \sin \omega t}\right\} \exp\left\{\frac{i}{\hbar}[S_{cl}(x, x') - S_{cl}(x, x'')]\right\} \\
&= \left\{\frac{m\omega}{2\pi\hbar \sin \omega t}\right\} \exp\left\{\frac{i}{\hbar}([S_1(x, x') - S_1(x_1, x'')] + (x' - x'')I_2)\right\} \\
&= \left\{\frac{m\omega}{2\pi\hbar \sin \omega t}\right\} \exp\left\{\frac{i}{\hbar}\frac{m\omega}{2 \sin \omega t}\left[\left(x'^2 - x''^2\right) \cos \omega t - 2x(x' - x'')\right]\right. \\
&\quad \left. + \frac{i}{\hbar}\frac{(x' - x'')}{\sin \omega t}J_2\right\} \\
&= \left\{\frac{m\omega}{2\pi\hbar \sin \omega t}\right\} \exp\left\{a\left(x'^2 - x''^2\right) + b(x' - x'')\right\},
\end{aligned}
\tag{3.2}
$$

where

$$
a = \frac{i}{\hbar}\frac{m\omega}{2 \sin \omega t} \cos \omega t = \frac{i}{4\sigma^2}\frac{\cos \omega t}{\sin \omega t} \tag{3.3}
$$

and

$$
\begin{aligned}
b &= -\frac{i}{\hbar}\frac{m\omega x}{\sin \omega t} + \frac{i}{\hbar \sin \omega t}J_2 \\
&= \frac{i}{2\sigma^2 \sin \omega t}(J - x),
\end{aligned}
\tag{3.4}
$$

and where

$$
\sigma^2 = \frac{\hbar}{2m\omega}. \tag{3.5}
$$

4. Oscillator in a single Gaussian state

The initial wave function is given by

$$
\psi(x, 0) = (2\pi\sigma^2)^{-1/4} \exp\left\{-\frac{x^2}{4\sigma^2}\right\}, \tag{4.1}
$$

where $\sigma^2 = \hbar/2m\omega$. Hence

$$
\psi^*(x'', 0)\psi(x', 0) = (2\pi\sigma^2)^{-1/2} \exp\left\{-\frac{x'^2 + x''^2}{4\sigma^2}\right\}, \tag{4.2}
$$

Substitute (3.2) and (4.2) in (3.1) to get

$$
\begin{aligned}
P(x, t) = (2\pi\sigma^2)^{-1/2}\left\{\frac{m\omega}{2\pi\hbar \sin \omega t}\right\} &\int_{-\infty}^{\infty} dx' \exp\left\{-\left(-a + \frac{1}{4\sigma^2}\right)x'^2 + bx'\right\} \\
\times &\int_{-\infty}^{\infty} dx'' \exp\left\{-\left(a + \frac{1}{4\sigma^2}\right)x''^2 - bx''\right\}.
\end{aligned}
\tag{4.3}
$$

Integration leads to

$$P(x,t) = (2\pi\sigma^2)^{-1/2}\left\{\frac{m\omega}{2\pi\hbar\sin\omega t}\right\}\pi\left[\left(-a+\frac{1}{4\sigma^2}\right)\left(a+\frac{1}{4\sigma^2}\right)\right]^{-1/2}$$
$$\times\exp\left\{\frac{b^2\sigma^2}{-4a\sigma^2+1}+\frac{b^2\sigma^2}{4a\sigma^2+1}\right\}$$
$$= (2\pi\sigma^2)^{-1/2}\left\{\frac{m\omega}{2\hbar\sin\omega t}\right\}\frac{4\sigma^2}{[-(4a\sigma^2)^2+1]^{1/2}}\exp\left\{\frac{2b^2\sigma^2}{[-(4a\sigma^2)^2+1]}\right\}, \quad (4.4)$$

where, from (3.3) and (3.4),

$$a^2 = -\frac{m^2\omega^2}{4\hbar^2\sin^2\omega t}\cos^2\omega t = -\frac{1}{(4\sigma^2)^2}\frac{\cos^2\omega t}{\sin^2\omega t}, \quad (4.5)$$

$$b^2 = -\frac{(J-x)^2}{(2\sigma^2)^2\sin^2\omega t} \quad (4.6)$$

so that

$$\frac{1}{-(4a\sigma^2)^2+1} = \frac{\sin^2\omega t}{\cos^2\omega t+\sin^2\omega t} = \sin^2\omega t. \quad (4.7)$$

Hence

$$\frac{4\sigma^2}{[-(4a\sigma^2)^2+1]^{1/2}} = 4\sigma^2\sin\omega t, \quad (4.8)$$

and

$$\frac{2b^2\sigma^2}{[-(4a\sigma^2)^2+1]} = -\frac{(J-x)^2}{2\sigma^2}. \quad (4.9)$$

Thus, from (4.4), (4.8) and (4.9), one obtains

$$P(x,t) = (2\pi\sigma^2)^{-1/2}\exp\left\{-\frac{(x-J)^2}{2\sigma^2}\right\}$$
$$= (2\pi\sigma^2)^{-1/2}\exp\left\{-\frac{[x-x_d(t)]^2}{2\sigma^2}\right\}, \quad (4.10)$$

where $x_d(t)$ is given by (2.13). We now examine two different scenarios.

4.1. *External force not random*

We see immediately from (4.10) that if $f(t)$ is not random, the width of the wave packet remains the same but the position of the maximum of $P(x,t)$ changes in time as $f(t)$ changes in time. It is of interest to note that this is similar to the probability for a coherent state with an equilibrium position $x_d(t)$ which is continually changing in time. As a result, it will be no surprise to see that this leads (as we will see in section 5), in the two-Gaussian case, to no decoherence at zero temperature.

However, if $f(t)$ is random, then (as we will now show) the width changes in time but the position of the maximum remains the same. It also leads to a decoherence (analogous to the situation where a translational velocity does not contribute to decoherence but a random velocity does [7, 8]).

4.2. *Random external force*

Motivated by the situation for the NIST experiment [2, 3], we now assume the external force is a *classical* Gaussian process, with correlation

$$\langle f(t)f(t')\rangle = 2\zeta kT_{eff}\delta(t-t') \equiv g\delta(t-t'), \tag{4.11}$$

and

$$\langle f(t)\rangle = 0. \tag{4.12}$$

Hence, from (2.13) to (2.15),

$$\langle J\rangle = 0, \tag{4.13}$$

and

$$\begin{aligned} m^2\omega^2\langle J^2\rangle &= g\int_0^t \mathrm{d}s\,\sin^2\{\omega(t-s)\} = g\int_0^t \mathrm{d}t'\,\sin^2\omega t' \\ &= g\frac{t}{2}\left\{1-\frac{\sin 2\omega t}{2\omega t}\right\}. \end{aligned} \tag{4.14}$$

Now, we consider $P(x,t)$ given in (4.10), in order to evaluate its ensemble average. From the basic result

$$\langle \exp \mathrm{i}A\rangle = \exp\left\{-\frac{1}{2}\langle A^2\rangle\right\} \tag{4.15}$$

one obtains the following very general relationship:

$$\begin{aligned} \langle P(x,t)\rangle &= \left\langle \left[2\pi\sigma^2\right]^{-1/2}\exp\left\{-\frac{(J-x)^2}{2\sigma^2}\right\}\right\rangle \\ &= \left[2\pi(\sigma^2+\langle J^2\rangle)\right]^{-1/2}\exp\left\{-\frac{x^2}{2[\sigma^2+\langle J^2\rangle]}\right\} \\ &= \left[2\pi W^2(t)\right]^{-1/2}\exp\left\{-\frac{x^2}{2W^2(t)}\right\}, \end{aligned} \tag{4.16}$$

where

$$W^2(t) = \sigma^2 + \langle J^2\rangle = \sigma^2 + \langle x_d^2(t)\rangle. \tag{4.17}$$

Thus, we obtain the key result that, in contrast to a non-random force, a delta-correlated random force leads to a contribution to the width of a wave-packet which, as we shall see in section 5, leads to decoherence in the case of a superposition state of two separated Gaussians. Because of the importance of this result, as a check, we also employed a second method of derivation, in which the ensemble average is calculated prior to carrying out the integrations over x_1 and x_2 (see Appendix A for details) and obtained identical results.

In the case where $f(t) = 0$, it is seen that $W^2(t) \to \sigma^2$, i.e. the oscillator width then remains unchanged with time, as expected [11].

5. Oscillator in a two-Gaussian Schrödinger cat superposition state

In general, the probability $P(x, t)$ for a Schrödinger cat superposition consists of three contributions, two of which correspond to the separate packets whereas the third is an interference term. Decoherence is generally understood to refer to the disappearance of the oscillatory nature of the interference term but we should emphasize that the integrated probability associated with the interference term is actually constant in time. Moreover, whereas it is true that, for a free particle, the interference term spreads out so much that the oscillatory nature is no longer manifest, in the case of an oscillator at finite temperature but with $\gamma = 0$ and $f(t) = 0$, recurrences occur so that the oscillations in the interference term persist for all time [12]. Thus, turning to the question of how decoherence should be measured leads to the conclusion that there is no unique answer. It really depends on the system being studied and what is of interest experimentally. For example, one might focus solely on the behaviour of the interference term but, keeping in mind that the two direct terms also depend on time, another possibility (which we have adopted) is to contrast the behaviour of the interference term with these terms. Thus, as before [4, 8], as a measure of decoherence, we define an attenuation coefficient $a(t)$ as the ratio of the amplitude of the interference term to twice the geometric mean of the first two terms.

Returning to the problem at hand, the initial state is

$$\psi(x, 0) = (8\pi\sigma^2)^{-1/4}\left(1 + e^{-d^2/8\sigma^2}\right)^{-1/2}\left\{\exp\left[-\frac{(x - d/2)^2}{4\sigma^2}\right] + \exp\left[-\frac{(x + d/2)^2}{4\sigma^2}\right]\right\}. \tag{5.1}$$

Next, we calculate $\psi(x, t)$ which enables us to obtain the following possibilities.

5.1. *External field not random*

Using (5.1) in (2.1), we obtain

$$\begin{aligned} P(x, t) &= |\psi(x, t)|^2 \\ &= \frac{1}{2(1 + e^{-d^2/8\sigma^2})}\frac{1}{\sqrt{2\pi\sigma^2}}\left\{\exp\left[-\frac{(x' - \frac{d}{2}\cos\omega t)^2}{2\sigma^2}\right] + \exp\left[-\frac{(x' + \frac{d}{2}\cos\omega t)^2}{2\sigma^2}\right]\right. \\ &\quad \left. + 2a(t)\exp\left(-\frac{d^2\cos^2\omega t}{8\sigma^2}\right)\exp\left(-\frac{(x')^2}{2\sigma^2}\right)\cos\left(\frac{\mathrm{d}x'\sin\omega t}{2\sigma^2}\right)\right\} \end{aligned} \tag{5.2}$$

where

$$x' \equiv x - J(t) \tag{5.3}$$

and

$$a(t) = 1. \tag{5.4}$$

Note that our definition of $a(t)$ ensures that $a(t) = 1$ for all times in the case where $T = 0$ and $\gamma = 0$, corresponding to a pure Schrödinger cat state at all times (as discussed in detail in section VI B of [8]). What we have now shown is that a non-random field does not affect the result, i.e. it does not give rise to decoherence.

This conclusion, of course, is not surprising because in this case we are simply creating a coherent state [11] but it does provide a check on our specific calculation. We now turn to the more interesting case of a random field.

5.2. *Random external field*

In this case our starting-point is (5.2). We also make use of the result given in the result (4.16) for the single Gaussian case. Applying this result to (5.2) leads, for the ensemble average of the two-Gaussian superposition probability, to the result

$$\langle P(x,t)\rangle = \langle|\psi(x,t)|^2\rangle$$
$$= \frac{1}{2(1+e^{-d^2/8\sigma^2})}\frac{1}{\sqrt{2\pi W^2}}\left\{\exp\left[-\frac{(x-\frac{d}{2}\cos\omega t)^2}{2W^2}\right]+\exp\left[-\frac{(x+\frac{d}{2}\cos\omega t)^2}{2W^2}\right]\right.$$
$$\left. + 2a(t)\exp\left(-\frac{d^2\cos^2\omega t}{8W^2}\right)\exp\left(-\frac{x^2}{2W^2}\right)\cos\left(\frac{\mathrm{d}x\sin\omega t}{2W^2}\right)\right\}, \tag{5.5}$$

where $W^2(t)$ is given in (4.17) and now

$$a(t) = \exp\left(-\frac{d^2\langle J^2(t)\rangle}{8\sigma^2 W^2}\right). \tag{5.6}$$

From (2.13) and the discussion immediately following, we identified J as x_d, the coordinate displacement due to the external force. Thus, as in [5], we define

$$s_d(t) = \langle x_d^2(t)\rangle, \tag{5.7}$$

which is the mean-square displacement due to the external force. Hence, also using (4.17), we may write (5.6) in the form

$$a(t) = \exp\left\{-\frac{s_d(t)d^2}{8\sigma^2[\sigma^2+s_d(t)]}\right\}, \tag{5.8}$$

where we recall that σ is the initial width of the individual wave packets.

We stress again that our definition of decoherence is not unique. However, if one is interested in observing only how the interference term behaves as a function of time, it is clear from (5.5) that $a(t)$ again plays a dominant role. For further discussion of the result given in (5.8), we refer to [5].

Finally, we note that one could also obtain probabilities in momentum space or phase space [13]. In the latter case, one is either dealing with Wigner functions or density matrices but, since these are quantities which are not measurable, our view is that definitions of decoherence are best expressed in terms of the measurable coordinate probabilities.

6. Conclusions

The result given in (5.8) coincides with that given in equation (24) of [5]. Thus, we have obtained a rigorous justification that the intuitive approach given in [5] works and we have also verified that, in contrast to the case of a random external field and as expected, a non-random field does not give rise to decoherence.

Moreover, an important conclusion which follows explicitly from our present analysis, is that, for the case of a random external force, decoherence is intrinsic to quantum mechanics and does not require the presence of a dissipative environment. This is another example of 'decoherence without dissipation' which can also occur due to temperature effects [7, 8].

Appendix A: Verification of (4.16)

From (4.15), we see that

$$\left\langle \exp\left\{\frac{\mathrm{i}}{\hbar}\frac{(x'-x'')}{\sin\omega t}J_2\right\}\right\rangle = \exp\left\{-\frac{(x'-x'')^2}{2\hbar^2\sin^2\omega t}\langle J_2^2\rangle\right\}$$
$$= \exp\left\{-\frac{(x'^2+x''^2-2x'x'')}{8\sigma^4\sin^2\omega t}\langle J^2\rangle\right\}. \tag{A1}$$

Substitute (A1) in the second to last line of (3.2) to get

$$\begin{aligned}&\langle K^*(x,x'';t)K(x,x';t)\rangle\\ &= \left\{\frac{m\omega}{2\pi\hbar\sin\omega t}\right\}\exp\left\{\frac{\mathrm{i}}{4\sigma^2\sin\omega t}\left[\left(x'^2-x''^2\right)\cos\omega t-2x(x'-x'')\right].\right.\\ &\quad\left.-\frac{x'^2+x''^2-2x'x''}{8\sigma^4\sin^2\omega t}\langle J^2\rangle\right\}\\ &= \left\{\frac{m\omega}{2\pi\hbar\sin\omega t}\right\}\exp\left\{A_1x'^2-A_2x''^2+B(x'-x'')+Dx'x''\right\},\end{aligned} \tag{A2}$$

where

$$A_1 = \frac{\mathrm{i}}{4\sigma^2}\frac{\cos\omega t}{\sin\omega t} - \frac{\langle J^2\rangle}{8\sigma^4\sin^2\omega t}, \tag{A3}$$

$$A_2 = -A_1^* = \frac{\mathrm{i}}{4\sigma^2}\frac{\cos\omega t}{\sin\omega t} + \frac{\langle J^2\rangle}{8\sigma^4\sin^2\omega t}, \tag{A4}$$

$$B = -\frac{\mathrm{i}}{2\sigma^2}\frac{x}{\sin\omega t}, \tag{A5}$$

and

$$D = \frac{\langle J^2\rangle}{4\sigma^4\sin^2\omega t} = -2\,\mathrm{Re}\,A_1. \tag{A6}$$

Once again, we consider a single Gaussian state where, as before,

$$\psi^*(x'',0)\psi(x',0) = (2\pi\sigma^2)^{-1/2}\exp\left\{-\frac{x'^2+x''^2}{4\sigma^2}\right\}. \tag{A7}$$

Thus, substituting (A2) and (A7) in (3.1), we get

$$
\begin{aligned}
\langle P(x,t)\rangle &= (2\pi\sigma^2)^{-1/2}\left\{\frac{m\omega}{2\pi\hbar\sin\omega t}\right\}\int_{-\infty}^{\infty}\mathrm{d}x'\int_{-\infty}^{\infty}\mathrm{d}x'' \\
&\quad\times\exp\left\{-\left(\frac{1}{4\sigma^2}-A_1\right)x'^2-\left(\frac{1}{4\sigma^2}+A_2\right)x''^2+B(x'-x'')+Dx'x''\right\} \\
&\equiv (2\pi\sigma^2)^{-1/2}\left\{\frac{m\omega}{2\pi\hbar\sin\omega t}\right\}I. \qquad \text{(A8)}
\end{aligned}
$$

Now, in general, [14]

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\mathrm{d}x_1\int_{-\infty}^{\infty}\mathrm{d}x_2\exp\left\{-\frac{1}{2}x_iA_{ij}x_j+B_ix_i\right\},\quad i=1,2 \\
&=\int_{-\infty}^{\infty}\mathrm{d}x_1\mathrm{d}x_2\exp\left\{-\frac{1}{2}\left[A_{11}x_1^2+A_{22}x_2^2+2A_{12}x_1x_2\right]+B_1x_1+B_2x_2\right\} \\
&=\frac{2\pi}{\sqrt{\det A}}\exp\left\{\frac{1}{2}B_i(A^{-1})_{ij}B_j\right\} \\
&=\frac{2\pi}{\sqrt{\det A}}\exp\left\{\frac{1}{2}\left[B_1^2(A^{-1})_{11}+B_2^2(A^{-1})_{22}\right]+B_1B_2(A^{-1})_{12}\right\}. \qquad \text{(A9)}
\end{aligned}
$$

Thus, in the particular case of the integral, I, defined in (A8),

$$A_{11}=\left(\frac{1}{2\sigma^2}-2A_1\right) \qquad \text{(A10)}$$

$$A_{22}=\left(\frac{1}{2\sigma^2}+2A_2\right)=\left(\frac{1}{2\sigma^2}-2A_1^*\right) \qquad \text{(A11)}$$

$$A_{12}=A_{21}=-D=-\frac{\langle J^2\rangle}{4\sigma^4\sin^2\omega t} \qquad \text{(A12)}$$

$$B_1=B \qquad \text{(A13)}$$

$$B_2=-B. \qquad \text{(A14)}$$

Also

$$
\begin{aligned}
\det A &= A_{11}A_{22}-A_{12}^2 \\
&=\left(\frac{1}{2\sigma^2}-2A_1\right)\left(\frac{1}{2\sigma^2}+2A_2\right)-D^2 \\
&=\frac{1}{(2\sigma^2)^2}-4A_1A_2+\frac{1}{\sigma^2}(A_2-A_1)-D^2 \\
&=\frac{1}{(2\sigma^2)^2}+4|A_1|^2-\frac{2}{\sigma^2}\operatorname{Re}A_1-4(\operatorname{Re}A_1)^2 \\
&=\frac{1}{(2\sigma^2)^2}+4(\operatorname{Im}A_1)^2-\frac{2}{\sigma^2}\operatorname{Re}A_1 \\
&=\frac{1}{(2\sigma^2)^2}+\frac{1}{(2\sigma^2)^2}\frac{\cos^2\omega t}{\sin^2\omega t}+\frac{\langle J^2\rangle}{4\sigma^6\sin^2\omega t} \\
&=\frac{1}{4\sigma^6\sin^2\omega t}\left\{\sigma^2+\langle J^2\rangle\right\} \\
&=\frac{1}{4\sigma^6\sin^2\omega t}W^2(t), \qquad \text{(A15)}
\end{aligned}
$$

where

$$W^2(t) = \sigma^2 + \langle J^2 \rangle. \tag{A16}$$

In addition

$$\left(A^{-1}\right)_{ij} = \frac{c_{ji}}{\det A}, \tag{A17}$$

where c_{ji} is the co-factor, so that

$$\left(A^{-1}\right)_{11} = A_{22}/\det A \tag{A18}$$

$$\left(A^{-1}\right)_{22} = A_{11}/\det A \tag{A19}$$

$$\left(A^{-1}\right)_{12} = \left(A^{-1}\right)_{21}/\det A = -A_{12}/\det A. \tag{A20}$$

Hence

$$\begin{aligned}
I &= \frac{2\pi}{\sqrt{\det A}} \exp\left\{\frac{1}{2}\left[B_1^2(A^{-1})_{11} + B_2^2(A^{-1})_{22}\right] + B_1 B_2 (A^{-1})_{12}\right\} \\
&= \frac{2\pi}{\sqrt{\det A}} \exp\left(\left\{\frac{1}{2}\left[B_1^2 A_{22} + B_2^2 A_{11}\right] - B_1 B_2 A_{12}\right\} \Big/ \det A\right) \\
&= \frac{2\pi}{\sqrt{\det A}} \exp\left(\left\{\frac{1}{2}(A_{11} + A_{22})B^2 + B^2 A_{12}\right\} \Big/ \det A\right) \\
&= \frac{2\pi}{\sqrt{\det A}} \exp\left(\left\{\left[\frac{1}{2}(A_{11} + A_{22}) + A_{12}\right] B^2\right\} \Big/ \det A\right) \\
&= \frac{2\pi}{\sqrt{\det A}} \exp\left(\left\{\left[\left(\frac{1}{2\sigma^2} - 2\mathrm{Re}\,A_1\right) - D\right] B^2\right\} \Big/ \det A\right) \\
&= \frac{2\pi}{\sqrt{\det A}} \exp\left\{\frac{B^2}{2\sigma^2} \frac{1}{\det A}\right\} \\
&= \frac{2\pi}{\sqrt{\det A}} \exp\left\{-\frac{1}{8\sigma^6 \sin^2 \omega t} \frac{x^2}{\det A}\right\}.
\end{aligned} \tag{A21}$$

Thus, using (A21) and (A15) in (4.13), we obtain

$$\langle P(x, t) \rangle = N \exp\left\{-\frac{x^2}{2W^2(t)}\right\}, \tag{A22}$$

where

$$N = (2\pi\sigma^2)^{-1/2} \frac{m\omega}{\hbar \sin \omega t} \frac{1}{\sqrt{\det A}}, \tag{A23}$$

and, from (A15),

$$\begin{aligned}
W^2(t) &= \sigma^2 \left(\frac{\hbar \sin \omega t}{m\omega}\right)^2 \det A \\
&= \frac{1}{2\pi N^2}.
\end{aligned} \tag{A24}$$

Hence

$$\langle P(x,t)\rangle = \left[2\pi W^2(t)\right]^{-1/2}\exp\left\{-\frac{x^2}{2W^2(t)}\right\}, \tag{A25}$$

where

$$W^2(t) = \sigma^2 + \langle J^2\rangle, \tag{A26}$$

in agreement with (4.16).

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