

Decoherence in Phase Space

M. Murakami[†], G.W. Ford[‡], and R.F. O'Connell[†]

[†]*Department of Physics and Astronomy, Louisiana State University,*

Baton Rouge, LA 70803-4001

[‡]*Department of Physics, University of Michigan,*

Ann Arbor, MI 48109-1120

Abstract

Much of the discussion of decoherence has been in terms of a particle moving in one dimension that is placed in an initial superposition state (a Schrödinger "cat" state) corresponding to two widely separated wave packets. Decoherence refers to the destruction of the interference term in the quantum probability function. Here, we stress that a quantitative measure of decoherence depends not only on the specific system being studied but also on whether one is considering coordinate, momentum or phase space. We show that this is best illustrated by considering Wigner phase space where the measure is again different. Analytic results for the time development of the Wigner distribution function for a two-Gaussian Schrödinger "cat" state have been obtained in the high-temperature limit (where decoherence can occur even for negligible dissipation) which facilitates a simple demonstration of our remarks.

Typeset using REVTeX

I. INTRODUCTION

Decoherence refers to the destruction of a quantum interference pattern and is relevant to the many experiments that depend on achieving and maintaining entangled states. Examples of such efforts are in the areas of quantum teleportation [?], quantum information and computation [?,?], entangled states [?], Schrödinger cats [?], and the quantum-classical interface [?]. For an overview of many of the interesting experiments involving decoherence, we refer to Refs. [?] and [?].

Much of the discussion of decoherence [?,?,?,?]? has been in terms of a particle moving in one dimension that is placed in an initial superposition state (a Schrödinger "cat" state) corresponding to two widely separated wave packets, each of the same form but having their centers x_0 at $x_0 = \pm d/2$ so that the packets are separated by a distance d . Thus, in an obvious notation we write the wave function of the superposition state as

$$\psi(x, t) = N[\psi_1(x, t) + \psi_2(x, t)], \quad (1.1)$$

where ψ_1 and ψ_2 denote the packets with centers at $d/2$ and $-d/2$, respectively, and N is the normalization constant. Hence

$$P(x, t) = N^2(|\psi_1|^2 + |\psi_2|^2 + 2\text{Re}\{i_1^*\psi_2\}). \quad (1.2)$$

Thus, the probability distribution consists of three contributions, two of which correspond to the separate packets, whereas the third is an interference term.

Many investigators have considered free particle Gaussian wave packets and this has also been our choice. However, in contrast to widespread current opinion, we showed that it is possible to obtain "Decoherence without Dissipation" which is actually the title of a paper [?] in which we showed that, working solely within the framework of elementary quantum mechanics and equilibrium statistical mechanics, decoherence can in fact occur at high temperature T even for vanishingly small dissipation. More precisely, we consider an ensemble of particles in thermal equilibrium, but so weakly coupled to a heat bath that we

can neglect dissipation in the equation of motion so that we have a Maxwell distribution of initial velocities [?]. The results obtained from such a calculation are in agreement with those obtained in the appropriate limit from more sophisticated calculations within the framework of nonequilibrium statistical mechanics [?,?].

It is generally known that a quantitative measure of decoherence depends not only on the specific system being studied but also on whether one is considering coordinate or momentum space. We show that this is best illustrated by considering Wigner phase space where the measure is again different. Thus, using the techniques developed in [?], we obtain analytic results for the time development of the Wigner distribution function for a two-Gaussian Schrödinger "cat" state in the high-temperature limit which facilitates a simple demonstration of our remarks.

As a preliminary, we consider in Sec. II the case of a free particle Gaussian wave packet. Starting with the position wave function $\psi(x, t)$, we then calculate the corresponding momentum wave function $\tilde{\psi}(p, t)$ and the corresponding Wigner distribution. We then examine the effect of temperature on the various quantities. In Sec. III, we generalize to the two-Gaussian superposition state. These results enable us to obtain the rate of decay of decoherence in position, momentum and phase space, which we discuss in Sec. IV.

II. SINGLE GAUSSIAN WAVE PACKET

The solution of the free-particle Schrödinger at time t , given that the solution at $t = 0$ is a minimum uncertainty wave packet, centered at x_0 and moving with velocity v_0 , is [?]

$$\psi(x, t) = \frac{1}{\left[2\pi \left(\sigma + \frac{i\hbar t}{2m\sigma}\right)^2\right]^{1/4}} \exp \left\{ -\frac{(x - x_0 - v_0 t)^2}{4\sigma^2 + (2i\hbar t/m)} + i\frac{mv_0}{\hbar}x - i\frac{mv_0^2 t}{2\hbar} \right\}. \quad (2.1)$$

The probability distribution is

$$P(x; t) = |\psi(x, t)|^2 = [2\pi\sigma^2(t)]^{-1/2} \times \exp \left\{ -\frac{(x - x_0 - v_0 t)^2}{2\sigma^2(t)} \right\}, \quad (2.2)$$

which is a Gaussian centered at the mean position of the particle at time t with variance given by $\sigma^2(t) = \sigma^2 + (\hbar t/2m\sigma)^2$. It is convenient for later analysis in the two-Gaussian case to express these results in a more compact form, that is

$$\psi(x, t) = (2\pi\Sigma^2)^{-1/4} \exp\{i(mv^2/2\hbar)t\} \exp\left\{-\frac{x_1^2}{4\sigma\Sigma} + i\frac{mv}{\hbar}x_1\right\}, \quad (2.3)$$

and

$$P(x, t) = [2\pi\sigma^2(t)]^{-1/2} \exp\left\{-\frac{x_1^2}{2\sigma^2(t)}\right\}, \quad (2.4)$$

where

$$x_1 = x - vt \quad (2.5)$$

and

$$\begin{aligned} \Sigma &= \sigma + i\left(\frac{\hbar t}{2m\sigma}\right) \\ &= \sigma + iv_q t. \end{aligned} \quad (2.6)$$

where

$$v_q = \frac{\hbar}{2m\sigma}. \quad (2.7)$$

Also

$$\begin{aligned} \Sigma\Sigma^* &= \sigma^2 + \left(\frac{\hbar t}{2m\sigma}\right)^2 = \sigma^2 + (v_q t)^2 \\ &\equiv \sigma^2(t). \end{aligned} \quad (2.8)$$

The corresponding momentum wave function is

$$\begin{aligned} \tilde{\psi}(p, t) &= \frac{1}{(2\pi\hbar)^{1/2}} \int_{-\infty}^{\infty} dx \psi(x, t) \exp\left(-\frac{ipx}{\hbar}\right) \\ &= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/4} \exp\left\{-i\frac{(p^2 + m^2v^2)t}{2m\hbar}\right\} \exp\left\{-\frac{\sigma^2(p - mv)^2}{\hbar^2}\right\}, \end{aligned} \quad (2.9)$$

and hence the momentum probability distribution is

$$\begin{aligned}
P(p, t) &= |\tilde{\psi}(p, t)|^2 \\
&= \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{1/2} \exp\left\{-\frac{2\sigma^2(p - mv)^2}{\hbar^2}\right\}.
\end{aligned} \tag{2.10}$$

We note from (??) and (??) that the variance in coordinate space increases with increasing t whereas we see from (??) that the variance in momentum space is time independent.

Going beyond our previous investigations [?], we now turn to the determination of the Wigner distribution function $W(x, p, t)$ given by

$$\begin{aligned}
W(x, p, t) &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ipy/\hbar} \psi^* \left(x + \frac{y}{2}, t\right) \psi \left(x - \frac{y}{2}, t\right) dy \\
&= (\pi\hbar)^{-1} \exp\left\{-\frac{X^2}{2\sigma^2} - \frac{2\sigma^2 P^2}{\hbar^2}\right\},
\end{aligned} \tag{2.11}$$

where

$$X \equiv x - \frac{pt}{m}, \tag{2.12}$$

and

$$P \equiv p - mv. \tag{2.13}$$

Next we consider the case of a particle in thermal equilibrium, but so weakly coupled to the environment that we can neglect dissipation. The principles of statistical mechanics then tell us that we obtain the corresponding probability distribution by averaging the distribution (??) over a thermal distribution of velocities. The result is

$$\begin{aligned}
P_T(x, t) &= \sqrt{\frac{m}{2\pi kT}} \int_{-\infty}^{\infty} dv \exp\left\{-\frac{mv^2}{2kT}\right\} P(x, t) \\
&= (2\pi w^2)^{-1/2} \exp\left(-\frac{x^2}{2w^2}\right),
\end{aligned} \tag{2.14}$$

where

$$\begin{aligned}
w^2(t) &= \sigma^2(t) + \frac{kT}{m} t^2 \\
&= \sigma^2 + v_q^2 t^2 + \bar{v}^2 t^2,
\end{aligned} \tag{2.15}$$

and

$$\bar{v} = \sqrt{\frac{kT}{m}}. \quad (2.16)$$

The corresponding result for the thermally averaged Wigner distribution is

$$W_T(x, p, t) = \left[(\pi\hbar)^{-1} \left(\frac{v_q^2}{v_q^2 + \bar{v}^2} \right)^{1/2} \right] \exp \left\{ -\frac{x^2}{2\sigma^2} - \frac{w^2 p^2}{2m^2 \sigma^2 (v_q^2 + \bar{v}^2)} + \frac{xpt}{m\sigma^2} \right\}. \quad (2.17)$$

As a check, we note that integration of (??) over p gives (??). Furthermore, integration over x gives

$$P_T(p, t) = [2\pi m^2 (v_q^2 + \bar{v}^2)]^{-1/2} \exp \left\{ -\frac{p^2}{2m^2 (v_q^2 + \bar{v}^2)} \right\}, \quad (2.18)$$

for the momentum thermal distribution.

III. TWO-GAUSSIAN WAVE PACKET

The result has the same form as (??). For the Gaussian case, we now write it in the form

$$\begin{aligned} \psi^{(2)}(x, t) &= N \left\{ \psi \left(x - \frac{d}{2} \right) + \psi \left(x + \frac{d}{2} \right) \right\} \\ &= N \exp \left\{ i(mv^2/2\hbar)t \right\} \exp \left(i \frac{mv}{\hbar} x_1 \right) \left\{ (2\pi\Sigma^2)^{-1/4} \exp \left(-\frac{\left(x_1 - \frac{d}{2} \right)^2}{4\sigma\Sigma} \right) + (d \rightarrow -d) \right\}, \end{aligned} \quad (3.1)$$

where

$$N = [2(1 + e^{-d^2/8\sigma^2})]^{-1/2}. \quad (3.2)$$

It follows that [?]

$$\begin{aligned} P^{(2)}(x, t) &= N^2 [2\pi\sigma^2(t)]^{-1/2} \left\{ \exp \left(-\frac{\left(x_1 - \frac{d}{2} \right)^2}{2\sigma^2(t)} \right) + (d \rightarrow -d) \right. \\ &\quad \left. + 2 \exp \left(-\frac{x_1^2 + \frac{d^2}{4}}{2\sigma^2(t)} \right) \cos \frac{\hbar t d x_1}{4m\sigma^2\sigma^2(t)} \right\}, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned}
P_T^{(2)}(x, t) = N^2 [2\pi w^2(t)]^{-1/2} & \left\{ \exp\left(-\frac{(x - \frac{d}{2})^2}{2w^2}\right) + (d \rightarrow -d) \right. \\
& \left. + 2 \exp\left(-\frac{x^2}{2w^2} - \frac{\sigma^2 w^2 + (\bar{v}t)^2 (v_q t)^2}{\sigma^2 \sigma^2(t) w^2} \frac{d^2}{8}\right) \cos \frac{\hbar t d x}{4m\sigma^2 w^2} \right\}. \quad (3.4)
\end{aligned}$$

In addition, after some algebra, we find that the corresponding Wigner distribution is

$$W^{(2)}(x, p, t) = N^2 \left\{ W\left(X - \frac{d}{2}, P\right) + W\left(X + \frac{d}{2}, P\right) + 2 \cos\left(\frac{Pd}{\hbar}\right) W(X, P) \right\}, \quad (3.5)$$

where X and P are given in (??) and (??) and $W(x, p, t)$ is given in (??). Also, after thermal averaging, we obtain

$$\begin{aligned}
W_T^{(2)}(x, p, t) = N^2 & \left(W_T\left(X - \frac{d}{2}, P\right) + W_T\left(X + \frac{d}{2}, P\right) \right. \\
& \left. + W_T(x, p, t) \exp\left\{-\frac{d^2}{8\sigma^2} \frac{\bar{v}^2}{\bar{v}^2 + v_q^2}\right\} \cos\left\{\frac{pd}{\hbar} \frac{v_q^2}{\bar{v}^2 + v_q^2}\right\} \right), \quad (3.6)
\end{aligned}$$

where $W_T(x, p, t)$ is given in (??). In order to obtain the momentum distribution, we simply integrate (??) over x to obtain

$$P^{(2)}(p, t) = 2N^2 P(p, t) \left\{ 1 + \cos\left(\frac{Pd}{2\hbar}\right) \right\}, \quad (3.7)$$

where $P(p, t)$ is given by (??). Thus, as with $P(p, t)$, we note that $P^{(2)}(p, t)$ is time independent. Next, carrying out the thermal average, we obtain

$$P_T^{(2)}(p, t) = P_T(p, t) \left\{ 1 + \exp\left(-\frac{m^2 d^2 \bar{v}^2 v_q^2}{2(\bar{v}^2 + v_q^2)}\right) \cos\left(\frac{pd}{2\hbar} \frac{\bar{v}^2}{\bar{v}^2 + v_q^2}\right) \right\}, \quad (3.8)$$

which, of course, is also independent of t .

We now have all the results necessary to discuss decoherence decay rates, which will be the subject of Sec. 4.

IV. DECOHERENCE DECAY RATES

In general, it is clear from the above that, in the case of a two-Gaussian superposition state, the probability distribution consists of three contributions, two of which correspond to the separate packets, whereas the third is an interference term. The interference term

is characterized by the cosine factor. One measures the disappearance of the interference term, that is, the loss of coherence (decoherence), by defining an attenuation coefficient $a(t)$, which is the ratio of the factor multiplying the cosine to twice the geometric mean of the first two terms.

Thus, in the case of decoherence in coordinate space, one sees from an examination of (??) that

$$a(t) = \exp \left\{ -\frac{\frac{kT}{m}t^2 d^2}{8\sigma^4 + 8\sigma^2 \frac{kT}{m}t^2 + \frac{2r^2 t^2}{m^2}} \right\}. \quad (4.1)$$

For short times (characteristic of decoherence time scales), whereas the t dependent terms in the denominator are negligible, the t dependent terms in the numerator remain, and thus we obtain

$$a(t) \cong e^{-t^2/\tau_d^2}, \quad (4.2)$$

where the decoherence time is

$$\tau_d = \frac{\sqrt{8}\sigma^2}{\bar{v}d}, \quad (4.3)$$

and $\bar{v} = \sqrt{kT/m}$ is the mean thermal velocity. This is consistent with the results obtained in Refs. [?,?,?], where we found that the dominant contribution to decoherence at high temperatures ($kT \ll \hbar\gamma$, where γ is typical dissipative decay rate), is independent of dissipation.

Turning now to momentum space, it is clear that the right-side of (??) is independent of time t . Hence, there is no decoherence in momentum space, which is what we expect from physical considerations.

Decoherence in phase space is obtained from (??) leading to

$$a(t) = \exp \left\{ \frac{d^2}{8\sigma^2} \frac{v_q^2}{\bar{v}^2 + v_q^2} \text{right} \right\}. \quad (4.4)$$

Thus, similar to the case with momentum space, there is no decoherence in phase space. We conclude that, for the two-Gaussian superposition state, decoherence is manifest only in coordinate space.

REFERENCES

- [1] Zeilinger, A., 2000, *Sci. Am.*, **282**(4), 32.
- [2] Bennett, C.H., 1995, *Phys. Today*, **48**(10), 24.
- [3] *Quantum Information* [Special Issue of Physics World, 1998, **11**(3)].
- [4] Haroche, S., 1998, *Phys. Today*, **51**(7), 36.
- [5] Zeilinger, A., 2000, *Nature*, **408**, 639.
- [6] Tegmark, M. and Wheeler, J.A., 2001, *Sci. Am.*, **284**(2), 68.
- [7] Myatt, C.J., King, B.E., Turchette, Q.A., et al., 2000, *Nature* (London), **403**, 269.
- [8] Giulini, D., Joos, E., Kiefer, C., Kupisch, J., et al., 1996, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, New York).
- [9] Paz, J.P. and Zurek, W.H., 2001, *Coherent Atomic Matter Waves, Les Houches Lectures*, Kaiser, R., Westbrook, C., and David, F., eds. (Springer, Berlin).
- [10] Ford, G.W., Lewis, J.T., and O'Connell, R. F., 2001, *Phys. Rev. A*, **64**, 032101.
- [11] Ford, G.W. and O'Connell, R.F., 2001, *Phys. Lett. A*, **286**, 87.
- [12] Ford, G.W. and O'Connell, R.F., 2001, *Phys. Rev. D*, **64**, 105020.
- [13] Ford, G.W. and O'Connell, R.F., 2002, *Am. J. Phys.*, **70**, 319.
- [14] Hillery, M., O'Connell, R.F., Scully, M.O., and Wigner, E.P., 1984, *Phys. Rep.*, **106**, 121.