INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 4183–4186 PII: S0305-4470(02)34814-5

Note on the derivative of the hyperbolic cotangent

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Received 13 March 2002
Published 26 April 2002
Online at stacks.iop.org/JPhysA/35/4183

Abstract

In a letter to *Nature* (Ford G W and O'Connell R F 1996 *Nature* **380** 113) we presented a formula for the derivative of the hyperbolic cotangent that differs from the standard one in the literature by an additional term proportional to the Dirac delta function. Since our letter was necessarily brief, shortly after its appearance we prepared a more extensive unpublished note giving a detailed explanation of our argument. Since this note has been referenced in a recent article (Estrada R and Fulling S A 2002 *J. Phys. A: Math. Gen.* **35** 3079) we think it appropriate that it now appear in print. We have made no alteration to the original note.

PACS numbers: 02.30.Nw, 02.50.−r

In [1] we published the formula

$$\frac{\mathrm{d}}{\mathrm{d}x}\coth(x) = -\mathrm{csch}^2(x) + 2\delta(x), \tag{1}$$

and gave an argument showing that it is correct. In this note we give some additional detail on the derivation of the formula. First, however, it might be useful to point out that the function $\coth(x)$ increases by +2 as x goes from $-\infty$ to $+\infty$, yet its derivative is everywhere negative, except at $x = 0$. How can a function that is everywhere decreasing still increase? We shall see how the answer is given by this formula.

We should emphasize that, as should be obvious from the appearance of the Dirac delta function, this is a formula of *distributions*. As a function, $\coth(x)$ and its derivative are undefined at $x = 0$, but as distributions they can be given meaning for all real x and it is for these distributions that the formula is correct. In general, a distribution is the limit of a sequence of good functions [2], where a good function and all its derivatives are continuous and bounded for all x. In the following we give an explicit example of such a sequence for the various terms in the formula.

We can define $\coth(x)$ as a distribution as the limit as $\epsilon \to 0$ of the good function

$$\coth(x,\epsilon) \equiv \mathrm{Re}\,\{\coth(x + \mathrm{i}\epsilon)\} = \frac{\sinh(2x)}{\cosh(2x) - \cos(2\epsilon)}. \tag{2}$$

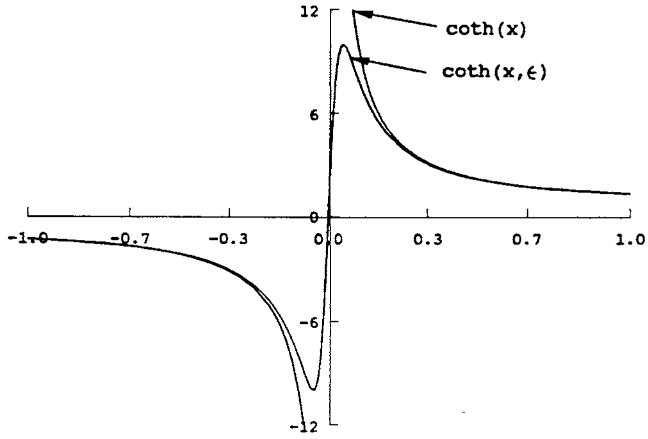

Figure 1. The function $\coth(x)$ and its smooth approximation $\coth(x, \varepsilon)$.

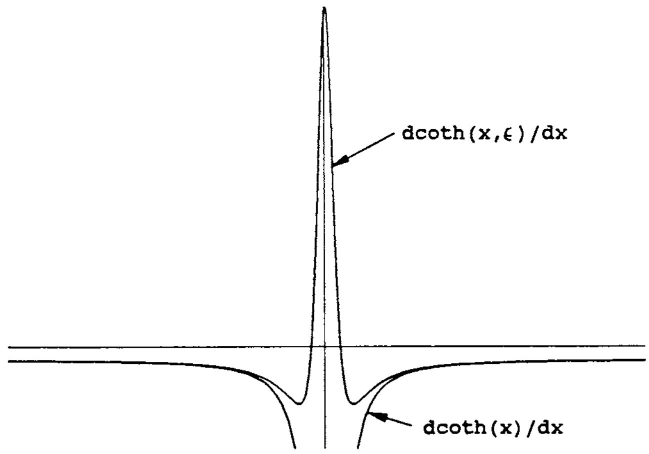

Figure 2. The derivative of $\coth(x)$ and its smooth approximation $\coth(x, \varepsilon)$.

For small ϵ, this function is very close to $\coth(x)$ except in a narrow range about $x = 0$ where, instead of diverging, it turns over and smoothly connects through the origin. This is shown in figure 1 for $\epsilon = 0.05$.

The derivative of this function is also a good function:

$$\frac{\mathrm{d}\coth(x, \epsilon)}{\mathrm{d}x} = 2\frac{1 - \cos(2\epsilon)\cosh(2x)}{[\cosh(2x) - \cos(2\epsilon)]^2}. \tag{3}$$

If we plot this function, we see that, for ϵ small, it will be very close to $-\mathrm{csch}^2(x)$, except for a narrow range of width of order ϵ about $x = 0$, where there is a large positive peak. This is shown in figure 2. The area under the central peak must exceed the (negative) area under the wings by exactly +2, since that is the net change of $\coth(x, \epsilon)$ as x is carried from $-\infty$ to $+\infty$. This is exactly accounted for by the delta function in formula (1). Thus, the term $-\mathrm{csch}^2(x)$ in that equation is to be considered as a distribution with area zero[3].

[3] This justifies our assertion that, in the physical application described in [1], this term vanishes in the classical limit.

As a more explicit and detailed example of this separation, we write the right-hand side of (2) as the sum of two good functions, the first of which has zero net change as x is carried from $-\infty$ to $+\infty$, while the second will have a net change of two and its derivative will approximate the delta function. Thus, we can write (2) in the form

$$\coth(x,\epsilon) = F(x,\epsilon) + G(x,\epsilon), \tag{4}$$

where F and G are good functions given by

$$\begin{aligned} F(x,\epsilon) &= \frac{\sinh(2x)}{\cosh(2x)-\cos(2\epsilon)} - \frac{1}{\pi/2-\epsilon}\arcsin\left(\frac{\cos(\epsilon)\sinh(x)}{\sqrt{\sinh^2(x)+\sin^2(\epsilon)}}\right), \\ G(x,\epsilon) &= \frac{1}{\pi/2-\epsilon}\arcsin\left(\frac{\cos(\epsilon)\sinh(x)}{\sqrt{\sinh^2(x)+\sin^2(\epsilon)}}\right). \end{aligned} \tag{5}$$

For fixed non-zero ϵ, each of these is a good function of x. The derivatives are given by

$$\begin{aligned} \frac{\mathrm{d}F(x,\epsilon)}{\mathrm{d}x} &= 2\frac{1-\cos(2\epsilon)\cosh(2x)}{[\cosh(2x)-\cos(2\epsilon)]^2} - \frac{\sin(2\epsilon)}{(\pi/2-\epsilon)[\cosh(2x)-\cos(2\epsilon)]}, \\ \frac{\mathrm{d}G(x,\epsilon)}{\mathrm{d}x} &= \frac{\sin(2\epsilon)}{(\pi/2-\epsilon)[\cosh(2x)-\cos(2\epsilon)]}. \end{aligned} \tag{6}$$

In the limit as $\epsilon \to 0$,

$$F(x,\epsilon) \to \frac{x}{|x|}\frac{2}{\mathrm{e}^{2|x|}-1}, \qquad G(x,\epsilon) \to \frac{x}{|x|}. \tag{7}$$

Note that, when these limiting values are put in (4), we get exactly the separation given in equation (2) of [1]. What we have done here is to show explicitly that each term in the separation corresponds to the limit of a good function.

If we consider the derivatives in this limit, we see that

$$\frac{\mathrm{d}F(x,\epsilon)}{\mathrm{d}x} \to -\mathrm{csch}^2(x), \qquad \frac{\mathrm{d}G(x,\epsilon)}{\mathrm{d}x} \to 2\delta(x). \tag{8}$$

Hence, $\mathrm{d}F/\mathrm{d}x$ is a good function that goes to $-\mathrm{csch}^2(x)$ for any finite x and which has the property that its integral from $-\infty$ to $+\infty$ is zero. This last follows since $F(x,\epsilon)$ vanishes for $x \to \pm\infty$.

Formula (1) is surprising, since the delta function at the origin arises, so to speak, from the behaviour at infinity rather than that at the origin. In this connection it is perhaps worthwhile to make the comparison with the well known distribution of the principal value of x^{-1}, which can be defined as the limit of the good function

$$P(x,\epsilon) = \frac{x}{x^2+\epsilon^2}. \tag{9}$$

In the limit as $\epsilon \to 0$,

$$P(x,\epsilon) \to P\frac{1}{x}, \tag{10}$$

where here P denotes the principal value. Like our smooth approximation to $\coth(x)$, $P(x,\epsilon)$ is, for small ϵ, very close to $1/x$ except in a narrow range about $x = 0$, where it turns over and smoothly connects through the origin. Also, as with our smooth approximation to $\coth(x)$, the derivative of $P(x,\epsilon)$ is very close to $-x^{-2}$ except for a narrow range of width of order ϵ about $x = 0$, where there is a large positive peak. However, in this case the area under the central peak equals that in the wings, since the net change of $P(x,\epsilon)$ as x is carried from $-\infty$ to $+\infty$ is zero. Thus, no delta function appears in the derivative.

Perhaps still more surprising is what we see if we form the difference of the two functions: $\coth(z) - z^{-1}$, where we have denoted the variable as z to emphasize that here we are talking about functions and not distributions. This difference function is continuous and bounded for all real z. The same is true of its derivative, so there is no delta function! What has happened? The answer is that the difference of two functions is not necessarily the same as the difference of the corresponding distributions. In this instance, one must take into account the definition of the distributions at $x = 0$, where the functions are undefined. Recall in particular that $\text{csch}^2(x)$ in formula (1) is defined, like the derivative of Px^{-1}, to be a distribution with area zero, so the delta function must appear.

References

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