

Attractive and repulsive quantum forces from dimensionality of space

I Białynicki-Birula^{1,2}, M A Cirone¹, J P Dahl^{1,3}, R F O'Connell⁴
and W P Schleich¹

¹ Abteilung für Quantenphysik, Universität Ulm, D-89069 Ulm, Germany

² Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotnikow, 32/46, 02-668, Warsaw, Poland

³ Chemical Physics, Department of Chemistry, Technical University of Denmark, DTU 207, DK-2800 Lyngby, Denmark

⁴ Department of Physics, Louisiana State University, Baton Rouge, LA 70803-4001, USA

Received 19 January 2002

Published 29 July 2002

Online at stacks.iop.org/JOptB/4/S393

Abstract

Two particles of identical mass attract and repel each other even when there exist no classical external forces and their average relative momentum vanishes. This quantum force depends crucially on the number of dimensions of space.

Keywords: Quantum mechanics, quantum force, dimensionality of space, entanglement

1. Introduction

The concept of a force acting on a particle has undergone many changes in the history of physics. In Newton's mechanics the force, that is a change of the momentum in time, is due to a change of a potential as a function of space [1]. The theory of action-at-a-distance [2] eliminates the potential altogether in favour of forces only. Unfortunately in this approach retarded as well as advanced forces appear, that is the particle feels forces not only from its past but also from its future. Finally, in general relativity [3] a particle is always in free fall, that is it does not feel any gravitational forces but follows along geodesics. Viewed from flat spacetime it acts as if forces would influence its course. A new concept of force arises in quantum mechanics [4]. Pauli's exclusion principle postulates that fermions can never assume the same quantum state. This feature can be interpreted as another representation of a force. Moreover, in quantum field theory [5] the interaction between two particles is achieved by the exchange of particles such as the photon, the W boson or the graviton.

In the present paper we introduce yet another different notion of force: two free particles of identical mass attract or repel each other even when there exist no classical external forces and their average relative momentum vanishes. This quantum force depends crucially on the number of space dimensions.

We use non-relativistic quantum mechanics to study the relative motion of two entangled particles of identical mass in the absence of any external force. We show that, when the two

particles are constrained to one dimension, that is when they can only move along a line, they first attract and then repel each other. This effect occurs for appropriately chosen initial wavefunctions. However, when they move in two and three dimensions and start from an initial wavefunction of the same mathematical form, they never attract but only repel each other.

Our paper is organized as follows: in section 2 we express the average separation between two particles in terms of hyperspherical coordinates. We show that this separation is proportional to the hyperradius. Section 3 is dedicated to the discussion of the time dependence of the average separation. Here we show that the number of dimensions of space can induce attraction or repulsion of otherwise free particles. This phenomenon is based on analytical results for the distance between the two particles which is made possible by a special choice of the initial wavefunction. In section 4 we connect this phenomenon with the quantum centrifugal potential. Section 5 presents our conclusions.

2. Separation between two particles

The main goal of the present paper is to study the time evolution of the separation between two free particles in its dependence on the number d of dimensions of the space they live in. In this case the configuration space has $D = 2d$ dimensions. Our space-dimension-induced effect of attractive and repulsive forces results from the entanglement between the two particles. Throughout this paper we focus on wavefunctions which

depend on the $D = 2d$ space coordinates only through the hyperradius r but not the angles. In this hyperspace of D dimensions such a wavefunction corresponds to an s-wave, that is a state of vanishing angular momentum.

In the present section we first define the average separation $s^{(d)}$ of the two particles living in a d -dimensional space. We then express $s^{(d)}$ in hyperspherical coordinates [6] and show that it is proportional to the average hyperradius.

2.1. Formulation of problem

We first define the average separation

$$s^{(d)} \equiv \langle |\vec{r}_1 - \vec{r}_2| \rangle \equiv \int d^d \vec{r}_1 \int d^d \vec{r}_2 W(\vec{r}_1, \vec{r}_2; t) |\vec{r}_1 - \vec{r}_2| \quad (1)$$

between the two particles described by their position variables \vec{r}_1 and \vec{r}_2 in d -dimensional space.

The time dependence of the probability density $W \equiv |\Psi^{(2d)}(\vec{r}_1, \vec{r}_2; t)|^2$ follows from the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi^{(2d)}(\vec{r}_1, \vec{r}_2; t) \equiv -\frac{\hbar^2}{2M} \Delta^{(2d)} \Psi^{(2d)}(\vec{r}_1, \vec{r}_2; t) \quad (2)$$

for the two-particle wavefunction $\Psi^{(2d)}(\vec{r}_1, \vec{r}_2; t)$ which lives in the D -dimensional configuration space and

$$\Delta^{(2d)} \equiv \Delta_{\vec{r}_1}^{(d)} + \Delta_{\vec{r}_2}^{(d)} \quad (3)$$

denotes the Laplacian in hyperspace.

We choose an initial wavefunction

$$\Psi_0^{(2d)}(\vec{r}_1, \vec{r}_2; t = 0) = \Psi_0^{(2d)}(r; t = 0) \equiv \Psi_0^{(D)}(r; t = 0) \quad (4)$$

which depends on the hyperradius

$$r \equiv \sqrt{\vec{r}_1^2 + \vec{r}_2^2} \quad (5)$$

only.

This wavefunction corresponds to an s-wave in configuration space and enjoys a vanishing angular momentum. The subsequent time evolution

$$i\hbar \frac{\partial}{\partial t} \Psi_0^{(D)}(r; t) = -\frac{\hbar^2}{2M} \left[\frac{\partial^2}{\partial r^2} + \frac{D-1}{r} \frac{\partial}{\partial r} \right] \Psi_0^{(D)}(r; t) \quad (6)$$

preserves this property. At any instant of time the wavefunction depends on the hyperradius only.

2.2. Distance

We now express the separation

$$|\vec{r}_1 - \vec{r}_2| = \sqrt{\vec{r}_1^2 + \vec{r}_2^2 - 2\vec{r}_1 \cdot \vec{r}_2} = \sqrt{r^2 - 2\vec{r}_1 \cdot \vec{r}_2} \quad (7)$$

between the two particles in hyperspherical coordinates. In the last step we have already made use of the definition equation (5) of the hyperradius r .

We recall the definition [6]

$$(\vec{r}_1)_j = r f_j(\theta, \varphi_1, \dots, \varphi_{2d-2}) \quad (8)$$

of the j th component of the position vector \vec{r}_1 in terms of hyperspherical coordinates $r, \theta, \varphi_1, \dots, \varphi_{2d-2}$. Here the functions f_j depend solely on the angles $\theta, \varphi_1, \dots, \varphi_{2d-2}$.

Similarly, the components

$$(\vec{r}_2)_k = r f_{d+k}(\theta, \varphi_1, \dots, \varphi_{2d-2}) \quad \text{for } k = 1, \dots, d \quad (9)$$

of the second particle involve the angular functions f_{d+k} .

Hence, the separation

$$\begin{aligned} |\vec{r}_1 - \vec{r}_2| &= r \left[1 - 2 \sum_{j=1}^d f_j \cdot f_{d+j} \right]^{1/2} \\ &\equiv r \cdot f(\theta, \varphi_1, \dots, \varphi_{2d-2}) \end{aligned} \quad (10)$$

between the two particles is the product of the hyperradius r and a function f that solely depends on the angles.

In hyperspherical coordinates the integration over the coordinates \vec{r}_1 and \vec{r}_2 of the two particles takes the form

$$\int d^d \vec{r}_1 \int d^d \vec{r}_2 = \int_0^\infty dr r^{D-1} \int d\Omega^{(D)} \quad (11)$$

where $d\Omega^{(D)}$ denotes the solid angle in the D -dimensional hyperspace.

At this point it is advantageous to introduce the radial wavefunction $u_0^{(D)} = u_0^{(D)}(r; t)$ in hyperspace corresponding to vanishing angular momentum. It is connected to the wavefunction $\Psi_0^{(D)}$ via the relation

$$\Psi_0^{(D)}(r; t) = \frac{1}{\sqrt{S^{(D)}}} \frac{u_0^{(D)}(r; t)}{r^{(D-1)/2}} \quad (12)$$

where $S^{(D)} \equiv \int d\Omega^{(D)}$ denotes the surface of the D -dimensional unit sphere.

With the help of the representation (10) of the separation, the formula (11) for the integration over the coordinates and the radial wavefunction $u_0^{(D)}$ defined by equation (12) we arrive at the expression

$$s^{(d)}(t) = \langle r^{(2d)} \rangle(t) \cdot C^{(2d)} \quad (13)$$

for the average separation of the two particles. Here

$$\langle r^{(2d)} \rangle(t) \equiv \int_0^\infty dr r |u_0^{(2d)}(r; t)|^2 \quad (14)$$

denotes the average hyperradius and

$$\begin{aligned} C^{(2d)} &\equiv \frac{1}{S^{(2d)}} \int d\Omega^{(2d)} f \\ &= \frac{1}{S^{(2d)}} \int d\Omega^{(2d)} \left[1 - 2 \sum_{j=1}^d f_j f_{d+j} \right]^{1/2} \end{aligned} \quad (15)$$

is a geometrical factor expressing the ratio between the average of f and the surface of the unit surface.

Since we are not interested in the absolute numbers but only in the time dependence of $s^{(d)}$ there is no need to evaluate the geometric factor $C^{(2d)}$. According to equation (13) the time dependence of $s^{(d)}$ arises solely from the time dependence of the average hyperradius $\langle r^{(2d)} \rangle$.

3. Attraction and repulsion without classical forces

So far we have not specified the initial wavefunction except that it corresponds to an s -wave, that is the wavefunction depends on the hyperradius only. In the present section we choose the initial wavefunction in a way that allows us to derive analytical expressions for the time dependence of the distance between the two particles.

In [7] we have focused on the time evolution of the initial wavefunction

$$\begin{aligned}\Phi_0(\vec{r}_1, \vec{r}_2) &\equiv \Psi_0^{(2d)}(\vec{r}_1, \vec{r}_2; t = 0) \\ &\equiv \mathcal{N}(\vec{r}_1^2 + \vec{r}_2^2) \exp\left[-\frac{1}{2} \frac{\vec{r}_1^2 + \vec{r}_2^2}{\delta r^2}\right]\end{aligned}\quad (16)$$

which depends solely on the hyperradius, that is

$$\Phi_0(\vec{r}_1, \vec{r}_2) = \mathcal{N} r^2 \exp\left(-\frac{1}{2} \frac{r^2}{\delta r^2}\right).\quad (17)$$

Here \mathcal{N} denotes the normalization constant and δr is a measure for the width of the wavefunction in hyperradius.

For this wavefunction we have derived [7] the formula

$$\langle r^{(2d)} \rangle(t) = \frac{1 + (t/\tau)^2/a^{(2d)}}{\sqrt{1 + (t/\tau)^2}} r_0^{(2d)}\quad (18)$$

for the time dependence of the average hyperradius. In order to achieve a compact expression we have introduced the characteristic time $\tau \equiv \delta r^2 M/\hbar$ and the parameter

$$a^{(2d)} \equiv 1 + \frac{8d}{4d^2 + 3}\quad (19)$$

together with the initial hyperradius

$$r_0^{(2d)} \equiv \frac{\Gamma(d + \frac{5}{2})}{\Gamma(d + 2)} \delta r,\quad (20)$$

determined by the ratio of gamma functions.

When we substitute equation (18) into formula (13) for the average separation we arrive at the explicit expression

$$s^{(d)}(t) = \frac{1 + (t/\tau)^2/a^{(2d)}}{\sqrt{1 + (t/\tau)^2}} s_0^{(d)}\quad (21)$$

for the time dependence of $s^{(d)}$. Here the initial separation $s_0^{(d)} \equiv r_0^{(2d)} \mathcal{C}^{(2d)}$ is the product of the initial average hyperradius and the average of the function f over the unit sphere.

For short times, that is for $t \ll \tau$, we can expand the square root which yields

$$s^{(d)}(t) \cong \left[1 + \left(\frac{1}{a^{(2d)}} - \frac{1}{2}\right) \left(\frac{t}{\tau}\right)^2\right] s_0^{(d)}.\quad (22)$$

Hence, for $a^{(2d)} > 2$ the separation of the two particles first decreases. However, for $a^{(2d)} < 2$ the separation increases.

With the help of the definition (19) of $a^{(2d)}$ we find for two particles moving along a line, that is $d = 1$,

$$a^{(2)} = 1 + \frac{8}{7} = 2 + \frac{1}{7} > 2.\quad (23)$$

Hence, they initially approach each other which indicates an attractive force.

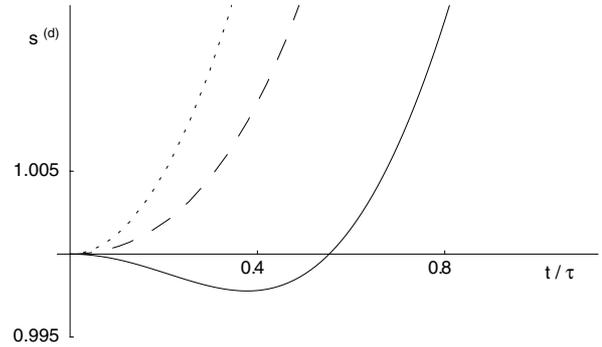


Figure 1. Average separation between two particles moving freely along a line (full curve), in a plane (broken curve) or in three-dimensional space (dotted curve). The initial wavefunction is the non-Gaussian given by equation (17). Here we have scaled the separation in terms of the initial separation $s_0^{(d)}$.

However, when they move in two or three dimensions, that is for $d = 2$ or 3, we find

$$a^{(4)} = 1 + \frac{16}{19} < 2\quad (24)$$

and

$$a^{(6)} = 1 + \frac{8}{13} < 2.\quad (25)$$

Consequently, in both cases the particles separate, which suggests that a repulsive force is at work.

The long time behaviour, $t \gg \tau$, follows from equation (21) by neglecting the term of unity compared to $(t/\tau)^2$ which yields

$$s^{(d)}(t) \cong \frac{1}{a^{(2d)}} \frac{t}{\tau} s_0^{(d)}.\quad (26)$$

The average separation of the two particles increases linearly in time for all three cases, that is $d = 1-3$.

Figure 1 shows this dimension-dependent phenomenon of attraction and repulsion in more detail. For this purpose we display the time dependence of the separation of the two particles given by equation (21). We observe the transient attraction for two particles along a line but a repulsion in the two other cases.

4. Quantum fictitious force

This phenomenon of attraction and repulsion of two particles despite the absence of a classical force stands out most clearly in the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} u_0^{(D)}(r; t) = \left[-\frac{\hbar^2}{2M} \frac{\partial^2}{\partial r^2} + V^{(D)}(r)\right] u_0^{(D)}(r; t)\quad (27)$$

for the radial wavefunction $u_0^{(D)}(r; t)$ of vanishing angular momentum in D -dimensional configuration space.

This equation follows immediately, when we substitute the ansatz equation (12) for the wavefunction into the Schrödinger equation (6). Here we have introduced the quantum fictitious potential [8, 9]

$$V^{(D)}(r) \equiv \frac{\hbar^2}{2M} \frac{(D-1)(D-3)}{4r^2} \equiv \alpha \frac{\beta(D)}{r^2}\quad (28)$$

with the abbreviation $\alpha \equiv \hbar^2/(8M)$ and the strength $\beta(D) \equiv (D-3)(D-1)$.

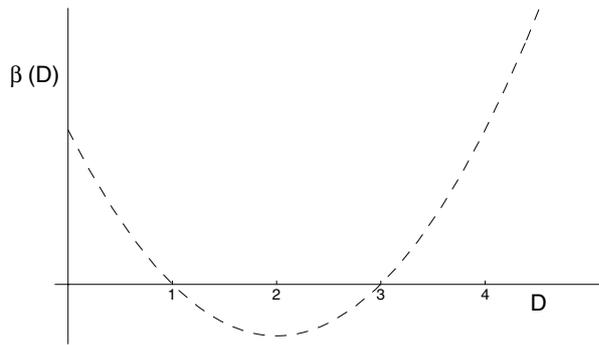


Figure 2. Strength $\beta(D) \equiv (D-1)(D-3)$ of the quantum fictitious potential $V^{(D)}(r) \equiv \alpha\beta(D)/r^2$ as a function of dimension D of configuration space where $\alpha \equiv \hbar^2/(8M)$. For $D = 1$ and 3 the potential vanishes. For $D = 2$ it is negative and attractive. However, for $D \geq 4$ it is always positive and repulsive. In order to guide the eye we have connected the discrete dimensions by a broken curve.

In figure 2 we display β as a function of D . We note that for $D = 1$ and 3 the potential vanishes. However, for $D = 2$ the potential

$$V^{(2)}(r) \equiv V_Q(r) \equiv -\frac{\hbar^2}{2M} \frac{1}{4r^2} \quad (29)$$

is negative and corresponds to an attractive force. In contrast, for $D \geq 4$ we always have positive potentials which correspond to repulsive forces. The strength of the potential, that is the strength of the repulsion, increases quadratically with the number of dimensions.

Hence, for two particles in d dimensions, that is $D = 2d$, we find for $d = 1$, that is for two particles on a line, the attractive potential $V^{(2)}$. In contrast, in two and three dimensions, that is $D = 2 \times 2 = 4$ and $2 \times 3 = 6$, the potentials $V^{(4)}$ and $V^{(6)}$ are always repulsive, leading to a repulsion of the two particles, in complete agreement with the behaviour displayed in figure 1.

5. Conclusions

We conclude by summarizing our main results. Even in the absence of classical forces two particles can feel an attractive or repulsive quantum force, depending on the number of dimensions of the space they move in and the initial wavefunction. Three features make this effect possible:

- (i) the wave nature of matter,
- (ii) the form of the Laplacian in different dimensions, and
- (iii) entanglement between the two particles.

We emphasize that our analysis is not restricted to the particular choice (17) of our initial wavefunction. A crucial

condition is that the wavefunction is not a Gaussian. For a Gaussian there is no entanglement between the two particles and the individual wavefunctions factorize.

So far we have focused on the case of two particles. However, the quantum fictitious potential (28) suggests that the case of more than two particles is also interesting. Indeed, for N particles in three space dimensions we have effectively $D = 3N$ dimensions. For $1 \ll N$ the repulsive quantum centrifugal potential

$$V^{(3N)} \cong \frac{\hbar^2}{2M} \frac{9N^2}{4r^2} \quad (30)$$

gives rise to a strong repulsion between the particles culminating in a rapid explosion. This explosion is rather dramatic since the potential depends quadratically on the number of particles. However, this phenomenon does not occur for the initial wavefunction (17); a more sophisticated state is needed as shown in [10].

Acknowledgments

We thank M V Berry, M Fedorov, R Glauber, D Greenberger, M J W Hall, D Kobe, G Süßmann and S Varro for many fruitful discussions. Two of us (IBB and JPD) gratefully acknowledge the support of the Humboldt Foundation. The work of WPS is partially supported by DFG. WPS and MAC gratefully acknowledge financial support from the network ‘QUEST’, HPRN-CT-2000-00121, of the IHP program of the European Union.

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