Noise in gravitational wave detector suspension systems: A universal model

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In a recent review of gravitational wave detectors, Ricci and Brillet discussed models for noise in the various suspension systems and concluded that ''. . . there is probably no universal model . . . .'' Here we present such a model which is based on work carried out by Ford, Lewis, and the present author [Phys. Rev. A **37**, 4419 (1988)]; the latter work presents a very general dissipative model (which has been applied already to many areas of physics) with the additional merit of being based on a microscopic Hamiltonian. In particular, we show that all existing models fall within this framework. Also, our model demonstrates (a) the advantages of using the Fourier transform of the memory function to parametrize the data from interferometric detectors such as the Laser Interferometric Gravitational Wave Observatory (rather than the presently-used Zener function) and (b) the fact that a normal-mode analysis is generally not adequate, consistent with a conclusion reached by Levin [Phys. Rev. D **57**, 659 (1998)].

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A key element in the detection of gravitational waves is the mechanical system which is used to measure the relative displacements of suspended mirrors [interferometric experiments, such as the Laser Interferometric Gravitational Wave Observatory (LIGO)] or the displacement of a resonant mass antenna (bar experiments). First, one has to take account of noise in interpreting the results and, secondly, one has to understand in detail how the mechanical motion is read out (such as optical detection systems involving photocurrents in the case of interferometric detectors and either capacitive transducers or superconducting quantum interference devices in the case of the bar detectors). Here, we present a general framework for describing noise effects in detectors systems. Our results apply to a very general dissipative environment. However, after outlining the very general framework which we have developed, for definiteness we will concentrate mostly on the LIGO-like experiments [1,2] and, more specifically, the test mass suspensions.

Many experimental investigations of LIGO suspension systems have been carried out already, notably by Weiss [3], Saulson [4], and co-workers [5–7]. There are many different kinds of noise sources, etc. (thermal; quantum; seismic; residual gas; internal friction, flexing, and creep; metrological;...) which affect the displacements of the suspension. Is there a general model which incorporates all such sources? The answer is that Ford, Lewis and O'Connell (FLO) have developed such a model [2] [which we refer to as the independent-oscillator (IO) model] for other purposes. However, it is clear that this model embraces nearly all situations presently discussed in the gravitational wave detection literature and our goal is to demonstrate its usefulness in specific cases and to show, in particular, how it can be utilized to gain information about noise sources.

Turning to specifics, we consider a topic which has received much attention in the recent literature [5–8] viz. the effect of thermal noise on an extended mass suspended by an anelastic wire. The approach used by all these authors is *phenomenological* and goes back to the work of Zener [9–11], who used such an approach to analyze experiments carried out as early as 1936 [12]. In one variation of this approach [4,5,7], internal damping in materials is treated by consideration of a complex spring ''constant'' (in momentum space), i.e., $k\tilde{x}(\omega)$ is replaced by $k[1-i\phi(\omega)]\tilde{x}(\omega) \equiv k_{\text{eff}}\tilde{x}(\omega)$, where $\phi(\omega)$ is the Zener phenomenological function and where in general a superposed tilde denotes Fourier transform and we have used $-i$ (instead of the oft-used $+i$) to conform to our definition of Fourier transform [2]. In another variation [6,7], a frequency-dependent complex Young's modulus is employed. It was also realized that, in general, $\phi(\omega)$ must have both real and imaginary parts which are related by a Kramers-Kronig (KK) relation. However, there is a *serious problem* arising from the fact that, in general, KK relations often have subtraction terms associated with them (see, for example, the case of the KK relations between the real and imaginary parts of the dielectric constant for a medium with a static electrical conductivity [13]). Thus, whereas there is a definite relationship between the real and imaginary parts of $\phi(\omega)$, there is no way of knowing precisely what it is from a phenomenological approach.

Despite the fact that the use of $\phi(\omega)$ is ubiquitous in the literature, there appears to be no attempt to relate it to a simple microscopic model other than one particular analysis by Zener based on thermoelastic damping, in contrast to our approach which displays the more general framework into which this parametrization fits. Furthermore, our approach is based on a well-defined Hamiltonian, which leads to what we dubbed the independent oscillator (IO) model. This model is the basis of work carried out by Ford, Lewis, and the present author on dissipative and fluctuation phenomena in quantum mechanics [2], a review of which appears in Ref. [14]. In particular, we will point out that referring to the spring as having a complex spring constant can be misleading since, as we shall show explicitly, the imaginary part of k_{eff} does not depend in any way on the properties of the spring per se (apart from an overall nonessential factor of m where m is the mass of the spring) but, instead, it depends on the nature of the dissipative environment (primarily the suspension wire in the case of LIGO).

To put this work and our general approach in perspective, we will first present some historical background. The study of fluctuation phenomena in science began in essence in 1827 with the observations of the Scottish botanist, Robert Brown. It is interesting to note that these early observations are still a source of great interest and controversy [15]. An explanation of these results was first provided by Einstein [16] using a discrete time assumption. Langevin later presented an entirely new approach [17] in the form of a stochastic differential equation. For a survey of this early work we refer to the review by Chandrasekhar [18]. It soon became apparent that a Langevin-type equation provides the framework for discussing fluctuation and dissipative phenomena over a wide spectrum of physical phenomena.

In general, there is an intimate connection between fluctuations and dissipation, which is referred to as the fluctuation-dissipation (FD) theorem. For example, Nyquist [19] showed that the random fluctuations in voltage across a resistor measured by Johnson [20] are determined by its impedance. A general quantum formulation of the FD theorem appears in the celebrated paper of Callen and Welton [21]. This theorem is a key ingredient of the pioneering work of Kubo [22,23] on linear response theory in nonequilibrium statistical mechanics. Correlations of the type discussed below are widely used in the work of Kubo and others. Another major advance is contained in the work of Mori [24], who showed that a microscopic equation of motion can generally be transformed into the form of a generalized quantum Langevin equation (GLE).

In recent years, there has been widespread interest in dissipative problems arising in a variety of areas in physics. As it turns out, solutions of many of these problems are encompassed by a generalization of Langevin's equation to encompass quantum, memory, and non-Markovian effects, as well as arbitrary temperature and the presence of an external potential $V(x)$. As in Ref. [2], we refer to this as the generalized quantum Langevin equation (GLE):

$$m\ddot{x}+\int_{-\infty}^{t}dt'\,\mu(t-t')\dot{x}(t')+V'(x)=F(t)+f(t), \quad (1)$$

where $V'(x)=dV(x)/dx$ is the negative of the time-independent external force and $\mu(t)$ is the so-called memory function. $F(t)$ is the random (fluctuation or noise) force and $f(t)$ is a c-number external force (due to a gravitational wave, for instance). In addition (keeping in mind that measurements of Δx generally involve a variety of readout systems involving electrical measurements), it should be strongly emphasized that ''--the description is more general than the language--'' [2] in that $x(t)$ can be a generalized displacement operator (so that, for instance, Δx could represent a voltage change).

A detailed discussion of Eq. (1) appears in Ref. [2]. In particular, it was pointed out the GLE corresponds to a macroscopic description of a quantum system interacting with a quantum-mechanical heat-bath and that this description can be precisely formulated, using such general principles as causality and the second law of thermodynamics. We also stressed that this is a model-independent description. However, the most general GLE can be realized with a simple and convenient model, viz., the independent-oscillator model. The Hamiltonian of the IO system is

$$H=\frac{p^2}{2m}+V(x)+\sum_j\left(\frac{p_j^2}{2m_j}+\frac{1}{2}m_j\omega_j^2(q_j-x)^2\right)-xf(t). \quad (2)$$

Here m is the mass of the quantum particle while m_j and ω_j refer to the mass and frequency of heat-bath oscillator j. In addition, x and p are the coordinate and momentum operators for the quantum particle and q_j and p_j are the corresponding quantities for the heat-bath oscillators.

The infinity of choices for the m_j and ω_j give this model its great generality. In particular, it can describe nonrelativistic quantum electrodynamics, the Schwabl-Thirring model, the Ford-Kac-Mazur (FKM) model, and the Lamb model [2].

In this context, it should be noted that, whereas H in Eq. (2) has been put into a form in which all the heat-bath oscillators interact with the central oscillator of interest (the detector) but not with each other, we have shown that this is the most general H one can write down to describe most types of dissipation encountered in the literature; in particular, it is unitarily equivalent to the FKM model in which all the oscillators are coupled (as discussed in Sec. V E of Ref. [2]).

Use of the Heisenberg equations of motion leads to the GLE (1) describing the time development of the particle motion, with

$$\mu(t)=\sum_j m_j\omega_j^2\cos(\omega_j t)\,\theta(t), \quad (3)$$

where $\theta(t)$ is the Heaviside step function. Also

$$F(t)=\sum_j m_j\omega_j^2 q_j^h(t), \quad (4)$$

where $q^h(t)$ denotes the general solution of the homogeneous equation for the heat-bath oscillators (corresponding to no interaction). These results were used to obtain the (model-independent) result for the (symmetric) autocorrelation of $F(t)$, viz.,

$$\frac{1}{2}\langle F(t)F(t')+F(t')F(t)\rangle$$
$$=\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Re}[\tilde{\mu}(\omega+iO^+)]\hbar\omega\coth(\hbar\omega/2kT)$$
$$\times\cos[\omega(t-t')], \quad (5)$$

where $\tilde{\mu}(\omega)$ is the Fourier transform of the memory function $\mu(t)$. This type of equation is referred to by Kubo [22] as the second fluctuation-dissipation theorem and we note that it can be written down explicitly once the GLE is obtained. Also, its evaluation requires only knowledge of Re $\tilde{\mu}(\omega)$. On the other hand, the first fluctuation-dissipation theorem is an equation involving the autocorrelation of $x(t)$ and its explicit evaluation requires a knowledge of the generalized susceptibility $\alpha(\omega)$ (to be defined below) which is equivalent to knowing the solution to the GLE and also requires knowledge of both Re $\tilde{\mu}(\omega)$ and Im $\tilde{\mu}(\omega)$. This solution is readily obtained when $V(x)=0$, corresponding to the original Brownian motion problem [14]. As shown by FLO [25], a solution is also possible in the case of an oscillator. Taking $V(x)=\frac{1}{2}m\omega_0^2x^2$, these authors obtained [see Eqs. (1)–(3) of Ref. [25]]

$$\tilde{x}(\omega)=\alpha(\omega)\{\tilde{F}(\omega)+\tilde{f}(\omega)\}, \tag{6}$$

where

$$\alpha(\omega)=[-m\omega^2+m\omega_0^2-i\omega\tilde{\mu}(\omega)]^{-1}, \tag{7}$$

and the superposed tilde is used to denote the Fourier transform. Thus, $\tilde{x}(\omega)$ is the Fourier transform of the operator $x(t)$:

$$\tilde{\chi}(\omega)=\int_{-\infty}^{\infty}dt x(t)e^{i\omega t}. \tag{8}$$

Also, since Eq. (3) implies that $\mu(t)$ is 0 for negative t, we have

$$\tilde{\mu}(\omega)=\int_0^{\infty}dt\mu(t)e^{i\omega t}, \mathrm{Im}\,\omega>0. \tag{9}$$

Thus $\tilde{\mu}(\omega)$ is analytic in the upper half-plane, Im $\omega>0$.

We have now all the tools we need to calculate various correlation functions which represent, in essence, observable quantities. Before doing so, it is convenient to rewrite Eq. (5) in the form

$$C_{FF}(\tau)\equiv\frac{1}{2}\langle F(t)F(t')+F(t')F(t)\rangle$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\tilde{C}_{FF}(\omega)e^{-i\omega\tau}, \tag{10}$$

where $\tau=t-t'$ and where

$$\tilde{C}_{FF}(\omega)=\mathrm{Re}[\tilde{\mu}(\omega+i0^+)]\hbar\omega\coth(\hbar\omega/2kT). \tag{11}$$

In deriving this result we have used the fact that the integrand on the right side of Eq. (5) is an even function of ω. Next, using Eqs. (6) and (10), it is straightforward to prove that the noise contribution [obtained by setting $f(t)=0$] to the coordinate autocorrelation is given by [26]

$$C_{xx}(\tau)\equiv\frac{1}{2}\langle x(t)x(t')+x(t')x(t)\rangle$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\tilde{C}_{xx}(\omega)e^{-i\omega\tau}, \tag{12}$$

where

$$\tilde{C}_{xx}(\omega)=|\alpha(\omega)|^2\tilde{C}_{FF}(\omega)=\hbar\,\mathrm{Im}\,\alpha(\omega)\coth(\hbar\omega/2kT), \tag{13}$$

and where the second equality in Eq. (13) follows from use of the relation

$$\mathrm{Im}\,\alpha(\omega)=\omega|\alpha(\omega)|^2\,\mathrm{Re}\,\tilde{\mu}(\omega), \tag{14}$$

which, in turn, follows directly from Eq. (7). We note that (12) and (13) are nothing more than the fluctuation-dissipation theorem of the first kind [22] except that our results are more explicit [see Eq. (31) below in this context] being related to a specific model.

In a similar manner, we obtain, for the ensemble average of the product of the displacement and random force

$$C_{XF}(\tau)=\frac{1}{2}\langle x(t)F(t')+F(t')x(t)\rangle$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\tilde{C}_{XF}(\omega)e^{-i\omega\tau}, \tag{15}$$

where

$$\tilde{C}_{XF}(\omega)=\alpha(\omega)\tilde{C}_{FF}(\omega)$$
$$=\alpha(\omega)\mathrm{Re}\,\tilde{\mu}(\omega)\hbar\omega\coth(\hbar\omega/2kT). \tag{16}$$

We now have all the tools we need at our disposal for the calculation of observable quantities. For example, taking $\tau=0$ in Eq. (12), and using Eq. (14), gives the ensemble average of the square of the displacement due to noise

$$\langle x^2(t)\rangle=\frac{\hbar}{2\pi}\int_{-\infty}^{\infty}d\omega\omega|\alpha(\omega)|^2\,\mathrm{Re}\,\tilde{\mu}(\omega)\coth(\hbar\omega/2kT)$$
$$=\frac{\hbar}{\pi}\int_0^{\infty}d\omega\omega|\alpha(\omega)|^2\,\mathrm{Re}\,\tilde{\mu}(\omega)\coth(\hbar\omega/2kT)$$
$$\equiv\int_0^{\infty}P(\omega)d\omega, \tag{17}$$

where

$$P(\omega)=\frac{\hbar}{\pi}\omega|\alpha(\omega)|^2\,\mathrm{Re}\,\tilde{\mu}(\omega)\coth(\hbar\omega/2kT), \tag{18}$$

is the power spectrum of the coordinate fluctuations.
In a similar manner, from Eqs. (10) and (11), we have

$$\langle F^2(t)\rangle\equiv\int_0^{\infty}P_F(\omega)d\omega, \tag{19}$$

where

$$P_F(\omega)=\frac{\hbar}{\pi}\mathrm{Re}\,\tilde{\mu}(\omega)\coth(\hbar\omega/2kT). \tag{20}$$

It is clear from Eqs. (18) and (20) that

$$P(\omega)=|\alpha(\omega)|^2P_F(\omega). \tag{21}$$

In particular, we note the key role played by $\tilde{\mu}(\omega)$, given by Eqs. (9) and (3), in all of these results. Once more, we stress their generality, which follows from the infinity of choices for the memory function displayed in Eq. (3).

In the case of resonant bar detectors, $\tilde{\mu}(\omega)$ is taken to be $m\gamma$, where γ is a constant. Thus, in particular, if we take $\omega_0 \ll \gamma$ and substitute Eq. (7) into Eq. (17), we immediately find the well-known high-temperature result $\langle x^2(t)\rangle = kT/m\omega_0^2$ and the zero-temperature result $\langle x^2(t)\rangle = (\hbar/2m\omega_0)$. However, in the case of nonresonant LIGO detectors, which are responsive to a range of frequencies, the frequency dependence of $\tilde{\mu}(\omega)$ is essential.

Next, in order to make contact with the approaches which use the Zener function, we write $k = m\omega_0^2$, so that the generalized susceptibility (7) may be written as

$$\alpha(\omega) = \{-m\omega^2 + k_{\rm eff}\}^{-1}, \tag{22}$$

where

$$k_{\rm eff} = k\left\{1 - \frac{i}{k}\omega\tilde{\mu}(\omega)\right\} \equiv k\{1 - i\phi(\omega)\}. \tag{23}$$

In other words, the quantity $\phi(\omega)$ appearing in phenomenological theories is simply given by

$$\phi(\omega) = \frac{1}{k}\omega\tilde{\mu}(\omega). \tag{24}$$

However, from our perspective, it is not helpful to discuss the results in terms of an effective complex spring constant $k_{\rm eff}$ since from Eq. (3) we see that $\mu(t)$ depends only on the parameters of the heat-bath. In fact, as emphasized in Ref. [2], the memory function is independent of the external potential, the particle mass and the temperature T. In other words, the imaginary part of $k_{\rm eff}$ does not depend in any way on the properties of the spring (apart from an overall nonessential factor of m) but, instead, it depends on the nature of the dissipative environment (primarily the suspension wire in the case of LIGO).

There are several key reasons why it is better to fit the experimental results by using $\tilde{\mu}(\omega)$ instead of $\phi(\omega)$ [apart from the fact that Eq. (23) could be misleading since it tends to obscure the fact that $\phi(\omega)$ itself can also have an imaginary part]. These stem from the fact that we know a lot about the properties of $\tilde{\mu}(\omega)$, regardless of the nature of the heat bath. Most important, as discussed in Ref. [2], $\tilde{\mu}(z)$ is not only analytic in the upper half-plane, ${\rm Im}\, z>0$, but it is what is referred to as a *positive real function* with the consequence that the relation between its real and imaginary parts is given by the Stieltjes inversion theorem, i.e., a Kramers-Kronig relation with at most one subtraction term, which can be absorbed into the particle mass term. Thus, in essence, ${\rm Re}\,\tilde{\mu}(\omega)$ characterizes the function $\tilde{\mu}(\omega)$ and, as can be seen from Eqs. (18) and (20), this is the key ingredient in determining observable results. Another useful characteristic of $\tilde{\mu}(\omega)$ is the reality condition [2], viz.,

$$\tilde{\mu}(\omega + iO^+) = \tilde{\mu}(-\omega + i0^+)^*. \tag{25}$$

Also, $\tilde{\mu}(\omega)$ must have the asymptotic form [26]

$$\tilde{\mu}(\omega) \approx -ic_1\omega + c_2 + ic_3/\omega. \tag{26}$$

where c_1, c_2, and c_3 are positive constants, at least one of which must be nonzero.

Next, we examine the question of where resonances occur in our expression (17) for $\langle x^2(t)\rangle$. Substituting Eq. (7) in Eq. (18) we obtain

$$P(\omega) = \frac{\hbar}{\pi}\frac{\omega\,{\rm Re}\,\tilde{\mu}(\omega)\coth(\hbar\omega/2kT)}{\left\{m^2\left(\omega^2 - \omega_0^2 - \frac{\omega}{m}{\rm Im}\,\tilde{\mu}(\omega)\right)^2 + [\omega\,{\rm Re}\,\tilde{\mu}(\omega)]^2\right\}}. \tag{27}$$

From henceforth, we will assume weak coupling since this is the situation for gravity wave detectors; this corresponds to the conditions

$$\omega_0 \gg {\rm Re}\,\tilde{\mu}(\omega), {\rm Im}\,\tilde{\mu}(\omega). \tag{28}$$

In this case, the resonance frequency is shifted by an amount [27]

$$\Delta\omega_0 = \frac{1}{2m}{\rm Im}\,\tilde{\mu}(\omega_0) \ll \omega_0. \tag{29}$$

We will now examine Eq. (27) in two cases of interest.

(a) Resonance at $\omega \approx \omega_0$. This corresponds to the situation for bar detectors. In this case, we may have $\omega^2 - \omega_0^2 \approx 2\omega(\omega - \omega_0)$ and hence

$$\begin{aligned} P(\omega) &= \frac{\hbar}{4\pi m^2}\frac{{\rm Re}\,\tilde{\mu}(\omega)\coth(\hbar\omega/2kT)}{\omega^2\left[\{\omega - (\omega_0 + \Delta\omega_0)\}^2 + \left\{\frac{1}{2m}{\rm Re}\,\tilde{\mu}(\omega)\right\}^2\right]} \\ &= \frac{\hbar}{2m\omega^2}\coth(\hbar\omega/2kT)\,\delta[\omega - (\omega_0 + \Delta\omega_0)] \\ &\rightarrow \frac{kT}{m\omega_0^2}\delta[\omega - (\omega_0 + \Delta\omega_0)] \quad \text{for} \quad kT \gg \hbar\omega_0, \end{aligned} \tag{30}$$

which is the usual weak-coupling high-temperature resonance result. Also, of course, if $\tilde{\mu}(\omega) = m\gamma = {\rm const}$, we see from Eq. (29) that $\Delta\omega_0 = 0$, i.e., there is no resonance shift.

(b) Resonance at $\omega \approx \omega_j$. These corresponds to the case of the LIGO detectors and arise because Eq. (17) also contains the factor ${\rm Re}\,\tilde{\mu}(\omega)$ which, in terms of the heat-bath parameters, is given by [2]

$${\rm Re}[\tilde{\mu}(\omega + i0^+)] = \frac{\pi}{2}\sum_j m_j\omega_j^2[\delta(\omega - \omega_j) + \delta(\omega + \omega_j)], \tag{31}$$

and thus we see explicitly that resonances also occur at the normal-mode frequencies of the heat-bath. Thus, in the case of LIGO, this will give information on the nature of the dissipative effect of the suspension wires. All of these properties should be a guide to the experiment in choosing suitable parameters to fit the data. In particular, we note that, because of the presence of the ${\rm Re}\,\tilde{\mu}(\omega)$ in both the numerator and denominator of Eq. (27), the contribution of the individual normal modes to $P(\omega)$ is not additive, i.e., the so-called expansion theorem [28] is not valid. Our conclusion that the normal-mode decomposition used by other authors [4] is not correct is consistent with the work of Levin [29]. In fact, we regard Levin's approach and our approach to be complementary as far as an analysis of the LIGO experiment is concerned (and in this context he goes beyond our analysis in one respect by considering the spatial distribution of the dissipation but our results can easily be generalized to this situation simply by integration over the detector surface) but whereas Levin's analysis is solely confined to the LIGO case and high temperatures, our analysis is applicable to all gravitational wave detector systems as well as to a plethora of dissipative systems discussed in the literature.

However, for high frequencies and weak coupling [$\omega \gg \omega_0 \gg \mathrm{Re}\,\mu(\omega), \mathrm{Im}\,\tilde{\mu}(\omega)$],

$$P(\omega) \to \frac{\hbar}{\pi m^2 \omega^2} \mathrm{Re}\,\tilde{\mu}(\omega) \coth(\hbar\omega/2kT)$$

$$\to \frac{2kT}{\pi m^2 \omega^4} \mathrm{Re}\,\tilde{\mu}(\omega) \quad \text{for} \quad kT \ll \hbar\omega \tag{32}$$

so that the expansion theorem is valid in this limit. Experimentally, there seems to be some evidence for $\tilde{\mu}(\omega)$ being proportional to ω (the so-called ''structural damping model,'' corresponding to a frequency-independent $\phi(\omega)$ [1,4,30,31]), which corresponds to a power spectrum $P(\omega) \sim \omega^{-3}$ which falls rapidly at higher frequencies. It is also of interest to note that our general framework clearly has wider applications, in particular, it is relevant to the measurement of the Newtonian constant G by use of a torsion balance using a time-of-swing method where it was found that the ''spring constant'' appears to increase with frequency [32].

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