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Quantum Noise Effects in Strongly Driven Systems

G. W. Ford* and R. F. O'Connell**

* *Department of Physics, University of Michigan, Ann Arbor, MI, 48109-1120 USA*
** *Department of Physics and Astronomy, Louisiana State University, Baton Rouge, LA, 70803-4001 USA*
e-mail: rfoc@rouge.phys.lsu.edu

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Abstract—For strongly driven systems where the coupling is weak and for frequencies near resonance, the Lax formula has proved to be an invaluable tool for calculating two-time correlation functions, as we have recently discussed in detail [1]. Nevertheless, we also stressed that this does not imply a quantum generalization of the Onsager hypothesis but in fact that the correct quantum generalization of this hypothesis is the fluctuation dissipation theorem. Here we discuss the fluctuation-dissipation theorem in more detail (by expressing a general correlation function in terms of both a relaxation function and a response function) and the conditions under which it leads to the Onsager hypothesis.

1. INTRODUCTION

In earlier publications [1] we gave a simple rigorous demonstration that the (classical) regression hypothesis of Onsager fails in the quantum case. We also emphasized there that the correct quantum generalization of the regression hypothesis is the fluctuation-dissipation theorem of Callen and Welton [2]. Our purpose here is to discuss this theorem in more detail (by expressing a general correlation function in terms of both a relaxation function and a response function) and the conditions under which it leads to the Onsager hypothesis. At the same time we shall see more in detail how the Onsager hypothesis fails in the quantum case.

2. THE FLUCTUATION–DISSIPATION THEOREM

The fluctuation–dissipation theorem can be stated in two equivalent forms. The first is a linear relation expressing the correlation function $C_{jk}(t)$ in terms of the relaxation function $A_{jk}(t)$. The second is a similar relation between the correlation function and the response function $G_{jk}(t)$. In the Appendix we give a formal general definition of these quantities. The theorem takes the following simple form relating the Fourier transforms (denoted by a tilde):

$$\tilde{C}_{jk}(\omega) = u(\omega)\tilde{A}_{jk}(\omega). \tag{2.1}$$

Here $u(\omega)$ is the *universal* spectrum of quantum noise,

$$u(\omega) = \frac{\hbar\omega}{2}\coth\left(\frac{\hbar\omega}{2kT}\right). \tag{2.2}$$

In applications of the theorem, the relaxation function is a solution of a macroscopic equation. The theorem thus states that the spectrum of correlations, $\tilde{C}_{jk}(\omega)$, is that of quantum noise as seen through a filter, $\tilde{A}_{jk}(\omega)$, corresponding to the relaxation function.

In the classical limit, the noise spectrum is the flat spectrum of white noise,

$$\lim_{\hbar\to 0} u(\omega) = kT. \tag{2.3}$$

The correlation function is then a simple constant multiplying the relaxation function. This is the Onsager hypothesis: the correlation function satisfies the same macroscopic equation as does the relaxation function. But, in the quantum case, the noise spectrum is never flat and the regression hypothesis never holds.

The failure of the hypothesis is seen most forcefully in the case of zero temperature, where the noise spectrum is that of quantum zero point fluctuations,

$$\lim_{T\to 0} u(\omega) = \frac{1}{2}\hbar|\omega|. \tag{2.4}$$

The fluctuation–dissipation can then be put in the time-dependent form,

$$C_{jk}(t) = \frac{\hbar}{2\pi}\frac{d}{dt}P\int_{-\infty}^{\infty} dt'\frac{A_{jk}(t')}{t-t'}, \tag{2.5}$$

where P stands for principal value. Clearly, $C_{jk}(t)$ will never satisfy the same equation as $A_{jk}(t)$.

In the case of finite temperature, one can write [4]

$$C_{jk}(t) = kTA_{jk}(t) - \frac{\pi(kT)^2}{2\hbar}\int_{-\infty}^{\infty} dt'\frac{A_{jk}(t')}{\sinh^2[\pi kT(t-t')/\hbar]}. \tag{2.6}$$

Here the first term is the classical correlation, the added term is quantum mechanical. The singular nature of the zero temperature limit (2.5) is a typical quantum phenomenon.

In summary, we have shown that in the classical limit and for macroscopic variables corresponding to a dynamical variable of the system, the fluctuation–dissipation theorem is a proof of the Onsager regression hypothesis. On the other hand, in the quantum case, where $\hbar$ is not zero, the factor $u(\omega)$ is a nontrivial function of frequency, and this leads to the necessity of simply utilizing the fluctuation–dissipation theorem directly [5] since the Onsager hypothesis does not apply to the quantum case. However, we should also emphasize that the fluctuation–dissipation theorem assumes that the applied forces are weak (see Appendix A) and the question arises as to what technique to use for strongly driven systems. The answer is the Lax formula [1, 6], if we are considering systems where the coupling is weak and the frequencies of interest are near resonance. However, there are situations where off-resonance frequencies play an important role, as we have recently discussed [1]. Thus, we are left with an outstanding and important problem viz. how can the fluctuation-dissipation theorem be generalized to apply to strongly driven systems over the whole range of frequencies?

APPENDIX A: THE FLUCTUATION–DISSIPATION THEOREM

The theorem deals with a system with Hamiltonian H in equilibrium at temperature T. The expectation of an operator $\mathbb{C}$ in the Hilbert space of H is defined to be

$$\langle \mathbb{C} \rangle \equiv \frac{\operatorname{Tr}\left\{\mathbb{C}\exp\left(-\frac{H}{kT}\right)\right\}}{\operatorname{Tr}\left\{\exp\left(-\frac{H}{kT}\right)\right\}}. \quad \text{(A1)}$$

The (Heisenberg) operator, $\mathbb{C}(t)$, that evolves in time t from the operator $\mathbb{C}$ is defined to be

$$\mathbb{C}(t) \equiv \exp(iHt/\hbar)\mathbb{C}\exp(-iHt/\hbar). \quad \text{(A2)}$$

Consider now a set of dynamical variables (self-adjoint operators) $y_1, y_2, \ldots$ in the Hilbert space of H. For convenience it is assumed, and it can always be arranged, that the expectation of each is zero,

$$\langle y_j \rangle = 0, \quad j = 1, 2, \ldots. \quad \text{(A3)}$$

The correlation functions are defined to be

$$C_{jk}(t-t') = \frac{1}{2}\langle y_j(t)y_k(t') + y_k(t')y_j(t)\rangle. \quad \text{(A4)}$$

The relaxation functions are defined in terms of the perturbed Hamiltonian,

$$H_f = H - \sum_j f_j y_j, \quad \text{(A5)}$$

where f_j, $j = 1, 2, \ldots$, are real c-number constants (generalized forces). One imagines that these forces are fixed in the distant past and the system (with Hamiltonian H_f) is allowed to come to equilibrium at temperature T. Then at $t = 0$, the forces are removed and the system evolves in time according to the Hamiltonian H. The expectation value of y_j is then

$$y_j^{(a)}(t) \equiv \frac{\operatorname{Tr}\left\{y_j(t)\exp\left(-\frac{H_f}{kT}\right)\right\}}{\operatorname{Tr}\left\{\exp\left(-\frac{H_f}{kT}\right)\right\}}. \quad \text{(A6)}$$

Here in the numerator $y_j(t)$, given by the definition (A2), is the Heisenberg operator evolving in time under the unperturbed Hamiltonian H. Finally, one assumes that f is small and expands,

$$y_j^{(a)}(t) = \sum_j A_{jk}(t) f_k + \ldots. \quad \text{(A7)}$$

The functions $A_{jk}(t)$ thus defined are the relaxation functions (sometimes also called the after-effect functions). Note that, according to this definition, the relaxation functions are defined for both positive and negative times. This is adopted as a convention.

One now introduces the Fourier transform of these functions,

$$\tilde{A}_{jk}(\omega) = \int_{-\infty}^{\infty} dt e^{i\omega t} A_{jk}(t), \quad \text{(A8)}$$

and similarly for $\tilde{C}_{jk}(\omega)$. The first form of the fluctuation–dissipation theorem [2] is then given by the relation (2.1).

The response function is defined in terms of the perturbed time-dependent Hamiltonian,

$$H_f(t) = H - \sum_j f_j(t) y_j, \quad \text{(A9)}$$

where $f_j(t)$, $j = 1, 2, \ldots$, are c-number time-dependent applied generalized forces. It is assumed that these vanish in the distant past,

$$\lim_{t \to -\infty} f_j(t) = 0. \quad \text{(A10)}$$

In the distant past the system will therefore be in equilibrium under the unperturbed Hamiltonian H and one consider the subsequent development in the presence of the applied forces, with Hamiltonian (A9). This time development corresponds to the unitary operator $U(t)$ that is the solution of the equation

$$i\hbar\frac{\partial U}{\partial t} = H_f(t)U, \qquad \text{(A11)}$$

with the condition that in the distant past,

$$\lim_{t\to-\infty} U(t)\exp(iHt/\hbar) = 1. \qquad \text{(A12)}$$

One now introduces the expectation value of y_j,

$$y_j^{(r)}(t) = \langle U^\dagger(t)y_jU(t)\rangle, \qquad \text{(A13)}$$

which is the thermal expectation of the operator y_j developing in time under the action of the applied forces. Finally, one assumes that the forces are small and expands,

$$y_j^{(r)}(t) = \int_{-\infty}^{t} dt'\sum_j G_{jk}(t-t')f_k(t') + \dots \qquad \text{(A14)}$$

The function $G_{jk}(t)$ thus defined is the response function (also called the Green function or the generalized susceptibility). As a convention the response function vanishes for negative times:

$$G_{jk}(t) = 0, \quad t<0. \qquad \text{(A15)}$$

This is sometimes called the causality condition. Indeed, one can use time-dependent perturbation theory to expand the unitary operator $U(t)$ and obtain the explicit formula

$$G_{jk}(t) = \frac{1}{i\hbar}\langle[y_k, y_j(t)]\rangle\theta(t), \qquad \text{(A16)}$$

where $\theta(t)$ is the Heaviside function, corresponding to the causality condition.

The Fourier transform of the response function is frequently denoted by $\alpha_{jk}(\omega)$ and is defined to be

$$\alpha_{jk}(z) = \int_0^\infty dt e^{izt}G_{jk}(t), \quad \operatorname{Im} z>0. \qquad \text{(A17)}$$

This is clearly an analytic function in the upper half z-plane, corresponding to the causality condition (A15). The Fourier transform of the relaxation function is related to that of the response function through the formula,

$$i\omega\tilde{A}_{jk}(\omega) = \alpha_{jk}(\omega+i0^+) - \alpha_{kj}(\omega+i0^+)^*. \qquad \text{(A18)}$$

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