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Driven systems and the Lax formula

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Abstract

One of the most widely used tools in quantum optics, the Lax formula for two-time correlations, is studied. The formula is then applied to two driven quantum systems: the oscillator and the two-level atom. Through a comparison with exact results for the oscillator, it is shown explicitly that the Lax formula is a weak coupling approximation that does not include quantum corrections to the Onsager regression hypothesis of classical statistical physics. For the two-level atom, the calculation of the well known Burshtein–Mollow spectrum of resonance fluorescence is extended to the case of non-zero temperature.

1. Introduction

In the field of non-equilibrium statistical mechanics there is an extensive literature on fluctuations of a system about its equilibrium state. Important results there are Onsager's regression hypothesis [1], and its quantum mechanical extension in the fluctuation-dissipation theorem of Callen and Welton [2]. Much less is known about fluctuations about the steady state of a driven system. An early paper is that of Bernard and Callen [3], who used perturbation methods to obtain general results for a weakly driven system. More important was a paper written more than a quarter century ago by Lax [4], which has been of central importance in the field of quantum optics, and where he obtained a formula for correlations in a driven quantum system that has come to be called the quantum regression theorem. While, within his stated assumptions, Lax's formula is correct, there is no quantum regression theorem [5]. That is, the quantum correlation *always* deviates from the form obtained using the regression hypothesis. Rather, the correct quantum generalization of the classical Onsager hypothesis [1] is the fluctuation-dissipation theorem of Callen and Welton [2]. But the so-called quantum regression theorem appears in every modern textbook exposition of quantum optics and, so far as we know, there are no flagrant errors in its application. How can it be that a nonexistent theorem gives correct results? Our purpose here is to elucidate this apparent contradiction by looking carefully at the approximations inherent in the Lax formula. To do this we begin in Section 2 with a brief derivation of the Lax formula. There we can point out that, while there is a formally exact formula, it depends for its implementation upon the existence of

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a master equation, which is necessarily approximate. More strongly, we assert that this approximation corresponds to weak coupling. To make this assertion explicit, in the following sections we apply the Lax formula to calculate correlations in the two systems for which there is a master equation: the oscillator-dipole and the two-level atom. We begin in Section 3 with the calculation of the correlation of fluctuations of an oscillator-dipole about the equilibrium state, i.e., we first consider the special case of the Lax formula for the undriven oscillator. We can then compare the result with the exact result obtained from the fluctuation-dissipation theorem. There we see that the expression for the correlation obtained from the Lax formula is correct only in the weak coupling approximation and then only in the neighborhood of the resonance. While it is generally recognized, at least implicitly, that the Lax formula is a weak coupling result, the restriction to resonant frequencies is not so well recognized. It is exactly this restriction that leads to the failure of the quantum regression theorem. However, it should be emphasized that applications of the Lax formula have always been to the resonant case and therefore have led to correct results [6,7]. Then in Section 4 we extend the discussion to the driven oscillator, again an exactly soluble problem. There we see that the correlation is the sum of that due to the driven motion and that calculated in the previous section. The driven motion is essentially classical, the quantum fluctuations are those contained in the fluctuations about the equilibrium state. In Section 5 we apply the Lax formula to the calculation of the correlations for a driven two-level atom. Our motives there are (i) to present a calculation that is simpler and more direct than that which is standard in the literature, (ii) to extend the Burshtein–Mollow results for resonance fluorescence to non-zero temperature, and (iii) to compare the results with those for the oscillator obtained in the previous two sections, with the view to better understand the nature of the approximations involved. Finally, in Section 6 we make some brief concluding remarks.

2. Formal derivation of the Lax formula

We present here a brief derivation of the Lax formula. While a similar discussion appears in a number of places (see, e.g., [6,7]) our emphasis is somewhat different. Our aim will be to set down the basic formulas, fixing the notation, and, most important, to point out exactly where an approximation is made.

We consider a system consisting of a finite subsystem, which we shall call 'the atom', coupled to an infinite heat bath, which we shall call 'the radiation field'. We imagine that at a time t_0 in the distant past the radiation field is in equilibrium at a temperature T. The atom, which we assume is driven by a periodic c-number force with period τ, is then coupled to the field. In the course of time the initial transients decay and the system comes to a steady state, periodic with the period of the applied force but independent of the initial state of the atom. This state corresponds to a density matrix, $\rho_S(t)$, which is periodic with period τ, $\rho_S(t+\tau)=\rho_S(t)$. For this system, we consider the correlation

$$\langle A(t_1)B(t_2)C(t_1)\rangle = \mathrm{Tr}\{U(t_2,t_1)C\rho_S(t_1)AU(t_1,t_2)B\}, \tag{2.1}$$

where A, B, and C are atomic operators and $U(t_1,t_2)$ is the unitary operator describing the evolution of the system from time t_2 to time t_1. Next we introduce

$$\rho(t_2)=U(t_2,t_1)C\rho_S(t_1)AU(t_1,t_2). \tag{2.2}$$

This is the density matrix at time t_2 which at time t_1 has the form

$$\rho(t_1)=C\rho_S(t_1)A. \tag{2.3}$$

With this, we can write

$$\langle A(t_1)B(t_2)C(t_1)\rangle=\mathrm{Tr}\{\rho(t_2)B\}. \tag{2.4}$$

That is, the correlation we seek is the expectation of the operator B with respect to the density operator $\rho(t_2)$. We should remark that, while $\rho_S(t_1)$ is a proper density matrix (i.e., a positive-definite trace-class operator), the operator ρ may not be, since the operators A and C are arbitrary. This does not effect our argument.

The quantities in the above discussion, the operators and the trace, are in the Hilbert space of the infinite system of atom coupled to the radiation field.

We would like to express these formulas in terms of a reduced description, involving only the finite atomic system. The way this is done is to introduce the partial trace over the heat bath variables,

$$\bar{\rho}(t) \equiv \mathrm{Tr}_{\mathrm{R}}\{\rho(t)\}. \tag{2.5}$$

The idea here is that we imagine the Hilbert space as a direct product of a radiation field sector, in which the operators of the radiation field act, and an atomic sector, in which the atomic operators act. The above trace is with respect to the Hilbert space of the radiation field. While this separation can be made at any instant of time, and the resulting density matrix $\bar{\rho}(t)$ is an operator in the atomic sector of Hilbert space, the evolution in time of this operator will span the whole Hilbert space. We come back to this point below. The atomic variables C and A being operators in the atomic sector of Hilbert space, this partial trace of (2.3) at time t_1 is given by

$$\bar{\rho}(t_1) = C\mathrm{Tr}_{\mathrm{R}}\{\rho_{\mathrm{S}}(t_1)\}A = C\bar{\rho}_{\mathrm{S}}(t_1)A. \tag{2.6}$$

We therefore have the general formula,

$$\langle A(t_1)B(t_2)C(t_1)\rangle = \mathrm{Tr}_{\mathrm{A}}\{\bar{\rho}(t_2)B\}, \tag{2.7}$$

where the trace is in the Hilbert space of the atomic variables and $\bar{\rho}(t_1)$ is given by (2.6).

The formula (2.7) is formally exact. (For an infinite system there is some question about the definition of the partial trace, but that need not concern us here.) Rather there is an approximation implicit in the requirement that there be a macroscopic prescription that will allow one to determine the steady state $\bar{\rho}_{\mathrm{S}}(t_1)$ and to determine the time evolution of $\bar{\rho}(t_2)$ from $\bar{\rho}(t_1)$. In practise, this prescription is always afforded by the well known master equation. This, then, is the Lax formula: the correlation is given by (2.7) with $\bar{\rho}(t_2)$ the solution of the master equation with $\bar{\rho}(t_1)$ given by (2.6) in which $\bar{\rho}_{\mathrm{S}}(t_1)$ is the steady solution of this same master equation.

The Lax formula therefore involves the approximation inherent in the derivation of the master equation. This is usually stated as the assumption that, during the time evolution, the density matrix for the system can be approximately factored: $\rho(t) \simeq \bar{\rho}(t)\rho_{\mathrm{R}}$, where ρ_{R} is the equilibrium density matrix for the radiation field. More generally, the approximation assumes that the dynamical variables for the atomic system remain in the atomic sector of Hilbert space during the evolution. Since the spread of the atomic operators into the radiation sector is due to the interaction (or coupling) term in the system Hamiltonian, it is clear that this is a weak coupling approximation. This statement of the approximation involved in the Lax formula is formal and it is perhaps not clear exactly what are its physical consequences. We therefore in the next sections apply the Lax formula to two systems: the driven oscillator and the driven two-level atom. The first of these is an exactly soluble problem, so we can compare the results, enabling us to state precisely what approximation is involved in the Lax formula. There is no exact result for the two-level atom, but we obtain results on the temperature dependence and for fluctuations about the equilibrium state which, due to the analogy with the oscillator calculation, allow us to assess the nature of the approximation in that case as well.

3. The oscillator autocorrelation function

As a means of gaining an understanding of the nature of the approximations involved in the Lax formula, we consider first fluctuations of the linear oscillator about its equilibrium state. We shall come back to the problem of the driven oscillator in the next section. Our aim therefore is to calculate the autocorrelation

$$C_0(t_2 - t_1) = \tfrac{1}{2}\langle x(t_2)x(t_1) + x(t_1)x(t_2)\rangle, \tag{3.1}$$

where the subscript '0' is there to emphasize that we are considering the undriven oscillator. From the discussion in the previous section, we see that in the Lax formula we must choose $A = B = x$ and $C = 1$ in the first term and $B = C = x$ and $A = 1$ in the second term. That is, we must calculate

$$C_0(t) = \mathrm{Tr}\{\bar{\rho}(t)x\}, \tag{3.2}$$

where $\bar{\rho}(t)$ is the solution of the master equation with

$$\bar{\rho}(0) = \tfrac{1}{2}(\bar{\rho}_{\mathrm{S}}x + x\bar{\rho}_{\mathrm{S}}). \tag{3.3}$$

The oscillator Hamiltonian is

$$H_{\mathrm{O}} = \frac{p^2}{2m} + \tfrac{1}{2}Kx^2\,, \tag{3.4}$$

where K is the force constant. It will save writing to introduce the familiar oscillator lowering and raising operators,

$$a = \frac{m\omega_0 x + \mathrm{i}p}{\sqrt{2m\hbar\omega_0}}\,, \quad a^\dagger = \frac{m\omega_0 x - \mathrm{i}p}{\sqrt{2m\hbar\omega_0}}\,, \tag{3.5}$$

where $\omega_0 = \sqrt{K/m}$ is the natural frequency of the oscillator. In terms of these operators, the master equation for the oscillator has the form [8,9]

$$\frac{\partial\bar{\rho}}{\partial t} = \frac{1}{\mathrm{i}\hbar}\left[H_{\mathrm{O}},\bar{\rho}\right] - N\frac{\gamma}{2}\Big(\left[a,a^\dagger\bar{\rho}\right] + \left[\bar{\rho}a,a^\dagger\right]\Big) - (N+1)\frac{\gamma}{2}\Big(\left[a^\dagger,a\bar{\rho}\right] + \left[\bar{\rho}a^\dagger,a\right]\Big), \tag{3.6}$$

where the characteristics of the heat bath are incorporated in the decay constant γ and

$$N = N(\omega_0) = \frac{1}{\mathrm{e}^{\hbar\omega_0/kT} - 1} \tag{3.7}$$

is the Planck expression for the mean quantum number of the oscillator at temperature T. In the present case the steady state corresponds to the equilibrium state That is, $\bar{\rho}_{\mathrm{S}} = \bar{\rho}_{\mathrm{eq}}$, where

$$\bar{\rho}_{\mathrm{eq}} = \frac{1}{N+1}\left(\frac{N}{N+1}\right)^{a^\dagger a}. \tag{3.8}$$

To proceed we note first of all that from the master equation, multiplying first by x and forming the trace and then the same with p, we obtain a simple pair of coupled equations for the mean quantities

$$\langle x(t)\rangle = \mathrm{Tr}\{\bar{\rho}(t)\,x\}\,, \quad \langle p(t)\rangle = \mathrm{Tr}\{\bar{\rho}(t)\,p\}\,. \tag{3.9}$$

These are

$$\frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} = \frac{\langle p\rangle}{m} - \frac{\gamma}{2}\langle x\rangle,$$
$$\frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} = -K\langle x\rangle - \frac{\gamma}{2}\langle p\rangle. \tag{3.10}$$

The solution of these equations is

$$\langle x(t)\rangle = \mathrm{e}^{-\gamma t/2}\left\{\langle x(0)\rangle\cos\omega_0 t + \frac{\langle p(0)\rangle}{m\omega_0}\sin\omega_0 t\right\},$$
$$\langle p(t)\rangle = \mathrm{e}^{-\gamma t/2}\left\{-m\omega_0\langle x(0)\rangle\sin\omega_0 t + \langle p(0)\rangle\cos\omega_0 t\right\}. \tag{3.11}$$

Now all we need do is remark, comparing (3.2) and (3.9), that

$$C_0(t) = \langle x(t)\rangle, \tag{3.12}$$

where $\langle x(0)\rangle$ and $\langle p(0)\rangle$ are to be calculated using the initial density matrix (3.3) with $\bar{\rho}_{\mathrm{S}} = \bar{\rho}_{\mathrm{eq}}$. That is,

$$\begin{aligned}\langle x(0)\rangle &= \mathrm{Tr}\{\bar{\rho}(0)\,x\}\\ &= \mathrm{Tr}\left\{\tfrac{1}{2}\left(\bar{\rho}_{\mathrm{eq}}\,x + x\bar{\rho}_{\mathrm{eq}}\right)x\right\}\\ &= \mathrm{Tr}\left\{\bar{\rho}_{\mathrm{eq}}\,x^2\right\}\\ &= \frac{\hbar}{2m\omega_0}(2N+1)\,,\\ \langle p(0)\rangle &= \mathrm{Tr}\{\bar{\rho}(0)\,p\}\\ &= \mathrm{Tr}\left\{\tfrac{1}{2}\left(\bar{\rho}_{\mathrm{eq}}\,x + x\bar{\rho}_{\mathrm{eq}}\right)p\right\}\\ &= \mathrm{Tr}\left\{\bar{\rho}_{\mathrm{eq}}\tfrac{1}{2}(xp+px)\right\} = 0\,.\end{aligned} \tag{3.13}$$

Therefore, we have the result

$$C_0(t) = \frac{\hbar}{2m\omega_0}\coth\frac{\hbar\omega_0}{2kT}\,\mathrm{e}^{-\frac{1}{2}\gamma|t|}\cos(\omega_0 t)\,. \tag{3.14}$$

Note that there is no quantum effect in the time dependence of this correlation.

The comparison of this result of the Lax formula with the exact results is best done in the language of Fourier transforms. The Fourier transform of (3.14) is

$$\tilde{C}_0(\omega) = \frac{\hbar}{2m\omega_0}\coth\frac{\hbar\omega_0}{2kT} \times \frac{\gamma\left(\omega_0^2 + \frac{1}{4}\gamma^2 + \omega^2\right)}{\left(\omega_0^2 + \frac{1}{4}\gamma^2 - \omega^2\right)^2 + \omega^2\gamma^2}\,. \tag{3.15}$$

We contrast this result of the Lax formula with the exact result, which from the fluctuation-dissipation theorem [2,10] is

$$\tilde{C}_0(\omega) = \hbar \coth\frac{\hbar\omega}{2kT}\,\mathrm{Im}\{\alpha(\omega)\}\,, \tag{3.16}$$

where $\alpha(\omega)$ is the oscillator susceptibility. In general the susceptibility has the form

$$\alpha(\omega) = \frac{1}{-m\omega^2 + K - \mathrm{i}\omega\tilde{\mu}(\omega)}\,, \tag{3.17}$$

where $\tilde{\mu}(\omega)$ is the Fourier transform of the memory function. In the Ohmic case ($\tilde{\mu}(\omega) = m\gamma$, $K = m\omega_0^2$) we get

$$\tilde{C}_0(\omega) = \frac{\hbar\omega}{m}\coth\left(\frac{\hbar\omega}{2kT}\right)\frac{\gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}\,. \tag{3.18}$$

In the QED case ($m = 0$, $\tilde{\mu}(\omega) = -\mathrm{i}\omega M/(1 - \mathrm{i}\omega\tau_e)$, $K = M\omega_0^2$, $\gamma = \omega_0^2\tau_e$, with M the renormalized mass and $\tau_e = 2\mathrm{e}^2/3Mc^3$) we get

$$\tilde{C}_0(\omega) = \frac{\hbar\omega}{M}\coth\left(\frac{\hbar\omega}{2kT}\right)\frac{\omega^2\tau_e}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\gamma^2}\,. \tag{3.19}$$

In the weak coupling limit, $\gamma \ll \omega_0$, these expressions are sharply peaked at the oscillator frequency ω_0 and at frequencies near ω_0 one can replace $\omega \simeq \omega_0$ in the slowly varying prefactor. Then all three expressions are the same. However, at high and low frequencies there are large deviations, even for weak coupling. In particular at high frequencies in the QED case there is a significant contribution which leads to a logarithmic divergence of $C_0(t)$ for short times. This is a typical quantum phenomenon that is missed by the master equation.

In summary, we see that the result (3.14) obtained using the Lax formula does not exhibit the quantum effects in the regression that are contained in the exact result [11]. The quantum effects are exactly those contained in the frequency dependence of the prefactor $\hbar\omega\coth(\hbar\omega/2kT)$ in the exact results. These effects are missed when in the weak coupling limit one replaces $\omega \simeq \omega_0$. We conclude that the Lax formula gives useful results in the weak coupling limit near a resonance, but it is not a quantum generalization of the Onsager hypothesis. Finally, we remark that this weak coupling result for the oscillator can be obtained simply and directly from the exact results.

4. The driven oscillator

It should be obvious from the linearity of the oscillator that the motion of the driven oscillator will be a superposition of a driven mean motion and a motion about the mean that is identical with the motion about the equilibrium state. That is, the correlation for the driven oscillator should be of the form

$$\begin{aligned} C_{\mathrm{driven}}(t_2,t_1) &\equiv \tfrac{1}{2}\langle x(t_2)\,x(t_1) + x(t_1)\,x(t_2)\rangle \\ &= C_0(t_2 - t_1) + \langle x(t_2)\rangle_{\mathrm{S}}\langle x(t_1)\rangle_{\mathrm{S}}\,, \end{aligned} \tag{4.1}$$

where $C_0(t)$ is the correlation about the equilibrium state discussed in Section 3 and $\langle x(t)\rangle_{\mathrm{S}}$ is the steady mean of the driven motion. In this section we show how this comes about from the master equation.

The master equation for the driven oscillator is the same as that given in (3.4) excepting that the Hamiltonian is to be replaced with that of the driven oscillator, $H_{\mathrm{O}} \to H_{\mathrm{driven}}$. That is,

$$\begin{aligned} \frac{\partial\bar{\rho}}{\partial t} &= \frac{1}{\mathrm{i}\hbar}\left[H_{\mathrm{driven}},\bar{\rho}\right] - N\frac{\gamma}{2}\left(\left[a,a^\dagger\bar{\rho}\right] + \left[\bar{\rho}a,a^\dagger\right]\right) \\ &\quad -(N+1)\frac{\gamma}{2}\left(\left[a^\dagger,a\bar{\rho}\right] + \left[\bar{\rho}a^\dagger,a\right]\right), \end{aligned} \tag{4.2}$$

where

$$H_{\mathrm{driven}} = \frac{p^2}{2m} + \tfrac{1}{2}Kx^2 - xf(t)\,. \tag{4.3}$$

Here $f(t)$ is the driving force which, in accord with the discussion in Section 2, is periodic with period τ: $f(t+\tau) = f(t)$. From this equation we can determine the equations of mean motion, which now take the form

$$\begin{aligned} \frac{\mathrm{d}\langle x\rangle}{\mathrm{d}t} &= \frac{\langle p\rangle}{m} - \frac{\gamma}{2}\langle x\rangle, \\ \frac{\mathrm{d}\langle p\rangle}{\mathrm{d}t} &= -K\langle x\rangle - \frac{\gamma}{2}\langle p\rangle + f(t)\,. \end{aligned} \tag{4.4}$$

The long time solution of these equations of mean motion corresponds to the steady mean motion. That is,

$$\langle x(t)\rangle_{\rm S} = \frac{1}{m\omega_0}\int_{-\infty}^{t} {\rm d}t' {\rm e}^{-\gamma(t-t')/2} \times \sin\omega_0(t-t')\, f(t')\,,$$

$$\langle p(t)\rangle_{\rm S} = \int_{-\infty}^{t} {\rm d}t' {\rm e}^{-\gamma(t-t')/2}\cos\omega_0(t-t')\, f(t')\,. \tag{4.5}$$

Note that for a periodic force, $f(t+\tau)=f(t)$, it follows that $\langle x(t)\rangle_{\rm S}$ and $\langle p(t)\rangle_{\rm S}$ are also periodic. The transformation to a frame moving with this steady mean motion corresponds to the unitary operator,

$$U(t) = \exp\left(\frac{\langle x(t)\rangle_{\rm S} p - \langle p(t)\rangle_{\rm S} x}{{\rm i}\hbar}\right). \tag{4.6}$$

It is a simple matter to show that

$$U^{\dagger}(t)\, xU(t) = x + \langle x(t)\rangle_{\rm S}\,,$$
$$U^{\dagger}(t)\, pU(t) = p + \langle p(t)\rangle_{\rm S}\,, \tag{4.7}$$

That is, $U(t)$ corresponds to a translation with the steady mean motion.

The point of introducing this transformation is that the master equation in the moving frame has exactly the form (3.6) of the master equation for the undriven oscillator. To show this, we begin by writing

$$\bar\rho(t) = U(t)\,\bar\rho'(t)U^{\dagger}(t)\,, \tag{4.8}$$

where $\bar\rho' = U^{\dagger}\bar\rho U$ is the density matrix in the moving frame. To determine the master equation for this density matrix we first note that

$$\frac{\partial\bar\rho'}{\partial t} = \frac{\partial U^{\dagger}}{\partial t}\bar\rho U + U^{\dagger}\bar\rho\frac{\partial U}{\partial t} + U^{\dagger}\frac{\partial\bar\rho}{\partial t}U = \frac{\partial U^{\dagger}}{\partial t}U\bar\rho' + \bar\rho' U^{\dagger}\frac{\partial U}{\partial t} + U^{\dagger}\frac{\partial\bar\rho}{\partial t}U. \tag{4.9}$$

Next, using the Baker–Campbell–Hausdorff theorem, ${\rm e}^{A+B} = {\rm e}^{A}{\rm e}^{B}{\rm e}^{-[A,B]/2}$, we can write (4.6) in the form

$$U(t) = {\rm e}^{\langle x\rangle_{\rm S} p/{\rm i}\hbar}{\rm e}^{-\langle p\rangle_{\rm S} x/{\rm i}\hbar}{\rm e}^{-\langle x\rangle_{\rm S}\langle p\rangle_{\rm S}/2{\rm i}\hbar}. \tag{4.10}$$

Forming the time derivative, we see that

$$\frac{\partial U}{\partial t} = \frac{1}{{\rm i}\hbar}\left\{\frac{{\rm d}\langle x\rangle_{\rm S}}{{\rm d}t}pU - \frac{{\rm d}\langle p\rangle_{\rm S}}{{\rm d}t}Ux - \frac{1}{2}\left(\langle x\rangle_{\rm S}\frac{{\rm d}\langle p\rangle_{\rm S}}{{\rm d}t} + \langle p\rangle_{\rm S}\frac{{\rm d}\langle x\rangle_{\rm S}}{{\rm d}t}\right)U\right\}. \tag{4.11}$$

Multiplying from the left by $U^{\dagger}$ and using (4.7) we see that

$$U^{\dagger}\frac{\partial U}{\partial t} = \frac{1}{{\rm i}\hbar}\left\{\frac{{\rm d}\langle x\rangle_{\rm S}}{{\rm d}t}p - \frac{{\rm d}\langle p\rangle_{\rm S}}{{\rm d}t}x + \frac{1}{2}\left(\langle x\rangle_{\rm S}\frac{{\rm d}\langle p\rangle_{\rm S}}{{\rm d}t} - \langle p\rangle_{\rm S}\frac{{\rm d}\langle x\rangle_{\rm S}}{{\rm d}t}\right)\right\}. \tag{4.12}$$

We use this and its Hermitian conjugate in the first two terms of (4.8), while in the third term we use the master Eq. (4.2) and the formulas (4.7). After some rearrangement, we find

$$\frac{\partial\bar\rho'}{\partial t} = \frac{1}{{\rm i}\hbar}\left[H_{\rm O},\bar\rho'\right] - N\frac{\gamma}{2}\left(\left[a,a^{\dagger}\bar\rho\right] + \left[\bar\rho a,a^{\dagger}\right]\right) - (N+1)\frac{\gamma}{2}\left(\left[a^{\dagger},a\bar\rho\right] + \left[\bar\rho a^{\dagger},a\right]\right) + \frac{1}{{\rm i}\hbar}\left(\frac{{\rm d}\langle p\rangle_{\rm S}}{{\rm d}t} + K\langle x\rangle_{\rm S} + \frac{\gamma}{2}\langle p\rangle_{\rm S} - f\right)\left[x,\bar\rho'\right] - \frac{1}{{\rm i}\hbar}\left(\frac{{\rm d}\langle x\rangle_{\rm S}}{{\rm d}t} - \frac{\langle p\rangle_{\rm S}}{m} + \frac{\gamma}{2}\langle x\rangle_{\rm S}\right)\left[p,\bar\rho'\right]. \tag{4.13}$$

But, using the equations of mean motion (4.4) we see that the last two terms vanish identically and, therefore, that $\bar\rho'$ satisfies exactly the master Eq. (3.6) for the undriven oscillator. In particular, the long-time steady solution is the equilibrium density matrix (3.8). Therefore, we have the result that the steady solution of the master equation for the driven oscillator is given by

$$\bar\rho_{\rm S}(t) = U^{\dagger}(t)\,\bar\rho_{\rm eq}U(t)\,. \tag{4.14}$$

We are now in a position to calculate the correlation (4.1). According to the Lax formula, this correlation is given by

$$C_{\rm driven}(t_2,t_1) = {\rm Tr}\{\bar\rho(t_2)\,x\}, \tag{4.15}$$

where $\bar{\rho}(t_2)$ is the solution of the master equation with the initial condition

$$\bar{\rho}(t_1) = \tfrac{1}{2}\{\bar{\rho}_S(t_1)\,x + x\bar{\rho}_S(t_1)\}. \tag{4.16}$$

In the moving frame, using (4.8), (4.14) and the translation formulas (4.7), this becomes

$$\begin{aligned} C_{\text{driven}}(t_2,t_1) &= \mathrm{Tr}\{\bar{\rho}'(t_2)(x + \langle x(t_2)\rangle_S)\} \\ &= \mathrm{Tr}\{\bar{\rho}'(t_2)\,x\} + \langle x(t_2)\rangle_S \mathrm{Tr}\{\bar{\rho}'(t_2)\}, \end{aligned} \tag{4.17}$$

where $\bar{\rho}'$ is the solution of the master Eq. (3.6) for the undriven oscillator with the initial condition

$$\begin{aligned} \bar{\rho}'(t_1) &= \tfrac{1}{2}\{\bar{\rho}_{eq}(x + \langle x(t_1)\rangle_S) \\ &\quad + (x + \langle x(t_1)\rangle_S)\bar{\rho}_{eq}\} \\ &= \tfrac{1}{2}(\bar{\rho}_{eq}\,x + x\bar{\rho}_{eq}) + \langle x(t_1)\rangle_S \bar{\rho}_{eq}. \end{aligned} \tag{4.18}$$

Since the master equation preserves the trace, in the second term of (4.17),

$$\mathrm{Tr}\{\bar{\rho}'(t_2)\} = \mathrm{Tr}\{\bar{\rho}'(t_1)\} = \langle x(t_1)\rangle_S, \tag{4.19}$$

while in the first term we repeat the arguments beginning with (3.9) and leading to the expression (3.14) for the correlation in the undriven case. The result (4.1) follows.

In connection with this result, we should point out that while the *form* (4.1) of the correlation is exact, the equations of mean motion (4.4) obtained from the master equation are not [12]. Rather, like the master equation itself, these are weak coupling results, valid only in that limit.

5. The Burshtein–Mollow spectrum

We consider now correlations for a driven two-level atom. In particular we shall use the Lax approximation to calculate the correlation

$$g(t_2 - t_1) = \langle \sigma_+(t_1)\,\sigma_-(t_2)\rangle. \tag{5.1}$$

This correlation is of special interest in the calculation of the spectrum of resonance fluorescence, since the radiated power is proportional to its Fourier transform,

$$\frac{dP}{d\omega} \propto \tilde{g}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega t} g(t). \tag{5.2}$$

This spectrum was first obtained by Burshtein [13] and then independently later by Mollow [14]. Both these calculations used implicitly the Lax approximation. The calculation is now standard in the literature (see, e.g., Scully and Zubairy [6])and our purpose in presenting a succinct repetition here is to compare with the corresponding calculation for the oscillator, allowing us to point out the nature of the approximation involved. In addition, we wish to extend existing results to the case of finite temperature, which can be of interest for the case of level spacings small compared with kT (high Rydberg states, for example)

We begin with the two-level atomic Hamiltonian, which for the atom driven in the rotating wave approximation (RWA) takes the form

$$H_A = \frac{\hbar\omega_0}{2}\sigma_z + \frac{\hbar\omega_R}{2}(\sigma_+ e^{-i\omega_P t} + \sigma_- e^{i\omega_P t}), \tag{5.3}$$

where $\hbar\omega_0$ is the level spacing, ω_R is the Rabi frequency and ω_P is the driving (or pump) frequency. Here we should perhaps remind ourselves that $\boldsymbol{\sigma} = \sigma_x\hat{\boldsymbol{x}} + \sigma_y\hat{\boldsymbol{y}} + \sigma_z\hat{\boldsymbol{z}}$ is a vector in a fictitious three dimensional space, $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$, and that

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.4}$$

are the Pauli matrices. The raising and lowering operators are

$$\sigma_+ = \tfrac{1}{2}(\sigma_x + i\sigma_y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

$$\sigma_- = \tfrac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.5}$$

The system consists of this atom coupled to the radiation field. The master equation for this system can be written in the form

$$\frac{\partial\bar{\rho}}{\partial t}=\frac{1}{\mathrm{i}\hbar}\left[H_A,\bar{\rho}\right]+N\frac{\gamma}{2}\left(\left[\sigma_+,\sigma_-\bar{\rho}\right]+\left[\bar{\rho}\sigma_+,\sigma_-\right]\right)+(N+1)\frac{\gamma}{2}\left(\left[\sigma_-,\sigma_+\bar{\rho}\right]+\left[\bar{\rho}\sigma_-,\sigma_+\right]\right). \tag{5.6}$$

where N is again the Planck number (3.7). To use the Lax formula to calculate the correlation (5.1), we choose in (2.4) $A=\sigma_+$, $B=\sigma_-$, and $C=1$. Then

$$g(t_2-t_1)=\mathrm{Tr}\left\{\bar{\rho}(t_2)\sigma_-\right\}, \tag{5.7}$$

where $\bar{\rho}$ is the solution of the master equation with the initial condition

$$\bar{\rho}(t_1)=\bar{\rho}_{\mathrm{S}}(t_1)\sigma_+ \ . \tag{5.8}$$

As a first step in the solution of the master equation, we introduce the transformation to the rotating frame, writing

$$\bar{\rho}(t)=U(t)\bar{\rho}'(t)U^{\dagger}(t), \tag{5.9}$$

where $U(t)$ is the unitary transformation corresponding to a rotation about the z direction with angular frequency ω_P,

$$U(t)=\exp\left\{-\tfrac{1}{2}\mathrm{i}\omega_P t\sigma_z\right\}. \tag{5.10}$$

Noting that

$$U^{\dagger}(t)\sigma_{\pm}U(t)=\mathrm{e}^{\pm\mathrm{i}\omega_P t}\sigma_{\pm}\ , \tag{5.11}$$

we see that the master equation for $\bar{\rho}'=U^{\dagger}\bar{\rho}U$ takes the form

$$\frac{\partial\bar{\rho}'}{\partial t}=-\mathrm{i}\left[\omega_R\sigma_x-\delta\omega\sigma_z,\bar{\rho}'\right]+N\frac{\gamma}{2}\left(\left[\sigma_+,\sigma_-\bar{\rho}'\right]+\left[\bar{\rho}'\sigma_+,\sigma_-\right]\right)+(N+1)\frac{\gamma}{2}\left(\left[\sigma_-,\sigma_+\bar{\rho}'\right]+\left[\bar{\rho}'\sigma_-,\sigma_+\right]\right), \tag{5.12}$$

where $\delta\omega=\omega_P-\omega_0$ is the detuning. The solution of this equation can be expressed in the form

$$\bar{\rho}'=\tfrac{1}{2}(1+\boldsymbol{\sigma}\cdot\boldsymbol{P}'), \tag{5.13}$$

where $\boldsymbol{P}'$, the polarization vector, is a solution of the Bloch equation,

$$\frac{\mathrm{d}\boldsymbol{P}'}{\mathrm{d}t}=\left(\omega_R\hat{\boldsymbol{x}}-\delta\omega\hat{\boldsymbol{z}}\right)\times\boldsymbol{P}'-\frac{\Gamma}{2}\left(\boldsymbol{P}'+\hat{\boldsymbol{z}}\cdot\boldsymbol{P}'\hat{\boldsymbol{z}}\right)-\gamma\hat{\boldsymbol{z}}. \tag{5.14}$$

Here to shorten the writing we have introduced

$$\Gamma=(2N+1)\gamma=\gamma\coth\frac{\hbar\omega_0}{2kT}. \tag{5.15}$$

The steady state solution of this equation is obtained by setting the right hand side equal to zero. The result is

$$\left\langle\boldsymbol{P}'\right\rangle_{\mathrm{S}}=\frac{\omega_R\delta\omega\hat{\boldsymbol{x}}+\frac{\Gamma}{2}\omega_R\hat{\boldsymbol{y}}-\left(\delta\omega^2+\frac{\Gamma^2}{4}\right)\hat{\boldsymbol{z}}}{\frac{1}{2}\omega_R^2+\delta\omega^2+\frac{\Gamma^2}{4}}\frac{\gamma}{\Gamma}. \tag{5.16}$$

This corresponds to the steady-state reduced density matrix

$$\bar{\varrho}'_{\mathrm{S}}=\tfrac{1}{2}\left(1+\boldsymbol{\sigma}\cdot\left\langle\boldsymbol{P}'\right\rangle_{\mathrm{S}}\right). \tag{5.17}$$

We emphasize that this steady-state density matrix is time-independent.

We now express the Lax formula in terms of the quantities in the rotating frame. Inserting the expression (5.9) for the density matrix in the formula (5.7) for the correlation, then cyclically permuting the factors and using (5.11) we find

$$g(t_2-t_1)=\mathrm{e}^{-\mathrm{i}\omega_P t_2}\mathrm{Tr}\left\{\bar{\rho}'(t_2)\sigma_-\right\}, \tag{5.18}$$

while the initial condition (5.8) becomes

$$\bar{\rho}'(t_1)=\mathrm{e}^{\mathrm{i}\omega_P t_1}\bar{\varrho}'_{\mathrm{S}}\sigma_+\ . \tag{5.19}$$

Using the expression (5.17) for $\bar{\varrho}'_{\mathrm{S}}$, this initial condition becomes

$$\bar{\rho}'(t_1)=\mathrm{e}^{\mathrm{i}\omega_P t_1}\tfrac{1}{2}\left\langle P'_x+\mathrm{i}P'_y\right\rangle_{\mathrm{S}}\tfrac{1}{2}\left(1+\boldsymbol{\sigma}\cdot\left[\frac{1+\left\langle P'_z\right\rangle_{\mathrm{S}}}{\left\langle P'_x+\mathrm{i}P'_y\right\rangle_{\mathrm{S}}}(\hat{\boldsymbol{x}}+\mathrm{i}\hat{\boldsymbol{y}})-\hat{\boldsymbol{z}}\right]\right). \tag{5.20}$$

Aside from a normalizing factor, this is a density matrix of the form (5.13) with

$$\boldsymbol{P}'(t_1) = \frac{1+\langle P'_z\rangle_{\rm S}}{\langle P'_x + {\rm i}P'_y\rangle_{\rm S}}(\hat{\boldsymbol{x}} + {\rm i}\hat{\boldsymbol{y}}) - \hat{\boldsymbol{z}}. \tag{5.21}$$

Therefore, the density matrix at time t_2 is of the form

$$\bar{\rho}'(t_2) = {\rm e}^{{\rm i}\omega_P t_1}\tfrac{1}{2}\langle P'_x + {\rm i}P'_y\rangle_{\rm S}\tfrac{1}{2}\left[1 + \boldsymbol{\sigma}\cdot\boldsymbol{P}'(t_2)\right], \tag{5.22}$$

where $\boldsymbol{P}'(t_2)$ is the solution of the Bloch Eq. (5.14) with the initial condition (5.21). Putting this in the expression (5.18) for the correlation, we see that

$$g(t_2 - t_1) = {\rm e}^{-{\rm i}\omega_P(t_2-t_1)}\tfrac{1}{2}\langle P'_x + {\rm i}P'_y\rangle_{\rm S} \times \tfrac{1}{2}(\hat{\boldsymbol{x}} - {\rm i}\hat{\boldsymbol{y}})\cdot\boldsymbol{P}'(t_2). \tag{5.23}$$

The time-dependent solution of the Bloch equation is obtained by the method of Laplace transforms. Introduce

$$\boldsymbol{P}'_s = \int_{t_1}^{\infty} {\rm d}t_2 {\rm e}^{-s(t_2-t_1)}\boldsymbol{P}'(t_2). \tag{5.24}$$

Then the Bloch Eq. (5.14) becomes

$$s\boldsymbol{P}'_s - (\omega_R\hat{\boldsymbol{x}} - \delta\omega\hat{\boldsymbol{z}})\times\boldsymbol{P}'_s + \frac{\Gamma}{2}(\boldsymbol{P}'_s + \hat{\boldsymbol{z}}\cdot\boldsymbol{P}'_s\hat{\boldsymbol{z}}) = \boldsymbol{P}'(t_1) - \frac{\gamma}{s}\hat{\boldsymbol{z}}. \tag{5.25}$$

The solution can be expressed in matrix form,

$$\begin{pmatrix} \hat{\boldsymbol{x}}\cdot\boldsymbol{P}'_s \\ \hat{\boldsymbol{y}}\cdot\boldsymbol{P}'_s \\ \hat{\boldsymbol{z}}\cdot\boldsymbol{P}'_s \end{pmatrix} = \begin{pmatrix} \dfrac{\left(s+\frac{\Gamma}{2}\right)(s+\Gamma)+\omega_R^2}{D(s)} & \dfrac{(s+\Gamma)\delta\omega}{D(s)} & -\dfrac{\omega_R\delta\omega}{D(s)} \\ -\dfrac{(s+\Gamma)\delta\omega}{D(s)} & \dfrac{\left(s+\frac{\Gamma}{2}\right)(s+\Gamma)}{D(s)} & -\dfrac{\left(s+\frac{\Gamma}{2}\right)\omega_R}{D(s)} \\ -\dfrac{\omega_R\delta\omega}{D(s)} & \dfrac{\left(s+\frac{\Gamma}{2}\right)\omega_R}{D(s)} & \dfrac{\left(s+\frac{\Gamma}{2}\right)^2+\delta\omega^2}{D(s)} \end{pmatrix} \times \begin{pmatrix} \hat{\boldsymbol{x}}\cdot\boldsymbol{P}'(t_1) \\ \hat{\boldsymbol{y}}\cdot\boldsymbol{P}'(t_1) \\ \hat{\boldsymbol{z}}\cdot\boldsymbol{P}'(t_1) - \dfrac{\gamma}{s} \end{pmatrix}, \tag{5.26}$$

where

$$D(s) = (s+\Gamma)\left[\left(s+\frac{\Gamma}{2}\right)^2 + \delta\omega^2\right] + \left(s+\frac{\Gamma}{2}\right)\omega_R^2. \tag{5.27}$$

Now, forming the Laplace transform of the correlation function (5.23) we see that

$$g_s = \tfrac{1}{2}\langle P'_x + {\rm i}P'_y\rangle_{\rm S}\tfrac{1}{2}(\hat{\boldsymbol{x}} - {\rm i}\hat{\boldsymbol{y}})\cdot\boldsymbol{P}'_{s+{\rm i}\omega_P}. \tag{5.28}$$

Using the solution (5.26) with the initial state (5.21), with the expressions (5.16) for $\langle\boldsymbol{P}'\rangle_{\rm S}$, we see that

$$g_s = \frac{\frac{\Gamma}{2}\omega_R^2 + (\Gamma-\gamma)\left(\delta\omega^2 + \frac{\Gamma^2}{4}\right)}{2\Gamma\left(\frac{1}{2}\omega_R^2 + \delta\omega^2 + \frac{\Gamma^2}{4}\right)} \times \frac{(s+{\rm i}\omega_P+\Gamma)\left(s+{\rm i}\omega_P+{\rm i}\delta\omega+\frac{\Gamma}{2}\right)+\frac{1}{2}\omega_R^2}{D(s+{\rm i}\omega_P)} + \frac{\frac{\gamma}{2}\omega_R^2\left(\frac{\Gamma}{2}-{\rm i}\delta\omega\right)}{2\Gamma\left(\frac{1}{2}\omega_R^2 + \delta\omega^2 + \frac{\Gamma^2}{4}\right)} \times \frac{(s+{\rm i}\omega_P+\gamma)\left(s+{\rm i}\omega_P+{\rm i}\delta\omega+\frac{\Gamma}{2}\right)}{(s+{\rm i}\omega_P)D(s+{\rm i}\omega_P)}. \tag{5.29}$$

This is a rather complicated formula, but it simplifies considerably in a number of important limits, which we consider below. First, however, we recall that the quantity of interest is the Fourier transform of the correlation function, $\tilde{g}(\omega)$, appearing in (5.2). Since $g(-t) = g(t)^*$ we see that this is related to the Laplace transform through the simple formula

$$\tilde{g}(\omega) = 2\,{\rm Re}\{g_{-{\rm i}\omega+0^+}\}. \tag{5.30}$$

Zero temperature. The first limit of interest, which will serve as a check on our general formula, is that appearing in the textbooks, the case of zero temperature. This corresponds to putting $\Gamma = \gamma$ in (5.29).

Then, if we specialize to the case of resonant driving, setting $\omega_P = \omega_0$ so $\delta\omega = 0$, we find

$$\tilde{g}(\omega) = \frac{\omega_R^2}{2(2\omega_R^2+\gamma^2)}\left\{\frac{4\pi\gamma^2}{2\omega_R^2+\gamma^2}\delta(\omega-\omega_0) + \frac{\gamma}{(\omega-\omega_0)^2+\frac{\gamma^2}{4}} + \gamma\frac{4\omega_R^2-\gamma^2-(\omega-\omega_0)^2}{\left[(\omega-\omega_0)^2-\omega_R^2-\frac{\gamma^2}{2}\right]^2+\left(\frac{3\gamma}{2}\right)^2(\omega-\omega_0)^2}\right\}. \tag{5.31}$$

This formula corresponds to that given in Eq. (10.5.27) of Scully and Zubairy [6]. In the strong driving limit, $\omega_R \gg \gamma$, this shows the well known three-peaked structure

$$\tilde{g}(\omega) \simeq \frac{1}{4}\frac{\gamma}{(\omega-\omega_0)^2+\frac{\gamma^2}{4}} + \frac{1}{4}\frac{\frac{3\gamma}{4}}{(\omega-\omega_0-\omega_R)^2+\left(\frac{3\gamma}{4}\right)^2} + \frac{1}{4}\frac{\frac{3\gamma}{4}}{(\omega-\omega_0+\omega_R)^2+\left(\frac{3\gamma}{4}\right)^2}. \tag{5.32}$$

High temperature. Next, we consider the high temperature limit, where we assume $kT \gg \hbar\omega_0$. From the expression (5.15) for Γ, we see that this implies $\Gamma \simeq \frac{kT}{\hbar\omega_0}\gamma \gg \gamma$. With this and, again, specializing to the case of resonant driving, setting $\delta\omega = 0$, we find

$$\tilde{g}(\omega) = \frac{1}{4}\frac{\Gamma}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}} + \frac{1}{4} \times \frac{\Gamma\left[(\omega-\omega_0)^2+2\omega_R^2+\Gamma^2\right]}{\left[(\omega-\omega_0)^2-\omega_R^2-\frac{\Gamma^2}{2}\right]^2+\left(\frac{3\Gamma}{2}\right)^2(\omega-\omega_0)^2}. \tag{5.33}$$

In the strong driving limit this shows the same three-peaked structure observed in the zero temperature case, but with Γ replacing γ,

$$\tilde{g}(\omega) = \frac{1}{4}\frac{\Gamma}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}} + \frac{1}{4}\frac{\frac{3\Gamma}{4}}{(\omega-\omega_0-\omega_R)^2+\left(\frac{3\Gamma}{4}\right)^2} + \frac{1}{4}\frac{\frac{3\Gamma}{4}}{(\omega-\omega_0+\omega_R)^2+\left(\frac{3\Gamma}{4}\right)^2}. \tag{5.34}$$

Zero driving. The final limit of interest is that of zero driving, corresponding to fluctuations about the equilibrium state. Setting $\omega_R = 0$ in (5.29) we see that

$$g_s = \frac{\Gamma-\gamma}{2\Gamma}\frac{1}{s+\mathrm{i}\omega_0+\frac{\Gamma}{2}}, \tag{5.35}$$

This corresponds to

$$\tilde{g}(\omega) = \frac{1}{2}\frac{\Gamma-\gamma}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}} = N(\omega_0)\frac{\gamma}{(\omega-\omega_0)^2+\frac{\Gamma^2}{4}}, \tag{5.36}$$

where we have used the fact that $\Gamma = [2N(\omega_0)+1]\gamma$. Comparing this with the corresponding result (3.15) for the oscillator, it should be clear that, since the Planck factor $N(\omega)$ is evaluated at the resonance frequency $\omega = \omega_0$, this is a weak coupling result. Further insight can be gained by forming the inverse Fourier transform to obtain,

$$g(t) = \frac{1}{\mathrm{e}^{\hbar\omega_0/kT}+1}\mathrm{e}^{(-\mathrm{i}\omega_0-\frac{\Gamma}{2})t}. \tag{5.37}$$

For the equal-time correlation, setting $t = 0$, we find

$$g(0) = \langle\sigma_+\sigma_-\rangle = \tfrac{1}{2}\left(1+\langle\sigma_z\rangle\right) = \frac{1}{\mathrm{e}^{\hbar\omega_0/kT}+1}. \tag{5.38}$$

At zero temperature this result reduces to zero, i.e., there are no fluctuations of σ_z about its minimum value of $-\frac{1}{2}$. But since σ_z does not commute with the Hamiltonian of the system of atom interacting with the radiation field, it *necessarily* must fluctuate. The result (5.38) is therefore a weak coupling (even zero coupling) result.

6. Conclusions

As was appreciated by Lax in his original paper [4], his formula or, rather, the master equation implied in the Lax formula involves the weak coupling approximation. We have seen that there is a stronger approximation implied by the formula: it applies only for frequencies near a resonance frequency. But it is exactly in the frequency dependence of, say, the pre-factor in the exact result (3.18) that the quantum phenomena lie. This frequency dependence is missed in the results obtained with the Lax formula. It is for this reason that there is no quantum regression theorem.

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