Exact result for the force autocorrelation in the rotating-wave approximation

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One of the cornerstones of many areas of physics (such as quantum optics, laser theory, and cavity QED) is the rotating-wave approximation (RWA) to the Hamiltonian for a dipole-oscillator coupled to the radiation field. Here it is shown that within the RWA one can obtain explicitly an exact expression for the autocorrelation of the force exerted by the field on the oscillator. This exact result differs from that appearing in much of the standard literature in that the spectrum is not that of white noise and there is no nontrivial dependence on the oscillator frequency.

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The rotating wave approximation (RWA) plays a prominent role in many areas of physics, as for example quantum optics, laser theory and cavity QED. In particular, it is a cornerstone of quantum optics and is discussed in both the old and the more modern textbooks and review articles [1–7]. It concerns the interaction of a reservoir, consisting of an infinite number of oscillators, either with a two-level atom or a cavity mode or charged harmonic oscillator. Since the two latter cases can be treated exactly [8,9], we will concentrate on them, as this will provide us with a clear-cut method for assessing the nature of the results within the RWA.

The Hamiltonian for a harmonic oscillator of frequency ω_0 coupled to a reservoir consisting of a large number of harmonic oscillators may be written in the form [9]

$$H=\hbar\omega_0\left(a^\dagger a+\frac{1}{2}\right)+\sum_j \hbar\omega_j\left(b_j^\dagger b_j+\frac{1}{2}\right)+H_{\mathrm{LC}}+H_{\mathrm{SI}}, \tag{1}$$

with the linear coupling term

$$H_{\mathrm{LC}}=\sum_j (a+a^\dagger)(\lambda_j b_j+\lambda_j^* b_j^\dagger) \tag{2}$$

and the self-interaction term

$$H_{\mathrm{SI}}=\sum_j \frac{|\lambda_j|^2}{\hbar\omega_j}(a+a^\dagger)^2. \tag{3}$$

This is essentially what the authors in Ref. [9] refer to as the independent-oscillator (IO) model and, in particular, it was shown there that a special case of this Hamiltonian is the exact universally accepted Hamiltonian of nonrelativistic quantum electrodynamics.

The rotating wave approximation is obtained by first dropping the self-interaction term and then in the linear coupling term dropping the terms ab_j and $a^\dagger b_j^\dagger$. Then the approximate total Hamiltonian reads

$$H_{\mathrm{RW}}=\hbar\omega_0 a^\dagger a+\sum_j \hbar\omega_j b_j{}^\dagger b_j+\sum_j (\lambda_j a^\dagger b_j+\lambda_j^* a b_j^\dagger). \tag{4}$$

This is the RWA Hamiltonian appearing almost universally in the quantum optics literature. It is used in several different situations [4]: a charged oscillator in the radiation field, a cavity mode interacting with the matter oscillators in the cavity walls, etc. It has been pointed out that this Hamiltonian has a serious defect: there is in general no lower bound on the spectrum [9,10]. Here, however, we ignore this particular shortcoming, since we are primarily interested in the autocorrelation of the reservoir random force, which is the same for H_{RW} as for a repaired Hamiltonian. In the quantum optics literature, this autocorrelation seems everywhere to be assumed to have a white-noise spectrum and to have a nontrivial dependence on the oscillator frequency ω_0. But these features contradict the known results that the spectrum is the universal Planck spectrum of quantum noise, independent of ω_0 [9,11]. What has gone wrong in the RWA analysis? It is our purpose here to reanalyze the discussion of H_{RW}, obtaining exact results and pointing out explicitly where the usual discussion fails.

For H_{RW}, the Heisenberg equations of motion for the dynamical variables take the form

$$\dot{a}(t)+i\omega_0 a(t)=-i\sum_j \frac{\lambda_j}{\hbar}b_j(t),$$

$$\dot{b}_j(t)+i\omega_j b_j(t)=i\frac{\lambda_j^*}{\hbar}a(t). \tag{5}$$

The solution of the equation for $b_j(t)$ is

$$b_j(t)=b_j^h(t)-i\frac{\lambda_j^*}{\hbar}\int_{-\infty}^t dt' e^{-i\omega_j(t-t')}a(t'), \tag{6}$$

where $b_j^h(t)$ is the time-dependent free field operator, the solution of the homogeneous equation. We can write

$$b_j^h(t)=b_j e^{-i\omega_j t}, \tag{7}$$

where b_j is the time-independent free field operator. Putting this solution in the equation for $a(t)$, we can write the resulting equation in the form

$$\dot{a}(t)+i\omega_0 a(t)+\int_{-\infty}^{t}dt' K(t-t')a(t')=B(t), \tag{8}$$

where $K(t)$ is the c-number function given by

$$K(t)=\sum_j \frac{|\lambda_j|^2}{\hbar^2}e^{-i\omega_j t}\theta(t), \tag{9}$$

with $\theta(t)$ the Heaviside function. Also in Eq. (8), $B(t)$ is an operator-valued random force given by

$$B(t)=-i\sum_j \frac{\lambda_j}{\hbar}b_j e^{-i\omega_j t}. \tag{10}$$

It should be emphasized that, apart from the coupling constants λ_j, the random force $B(t)$ depends only upon the *free field* reservoir quantities. The form (8) appears in the quantum optics literature, where it is called the quantum Langevin equation.

The correlation and commutator of the free field operators are

$$\langle b_k^h(t')^\dagger b_j^h(t)\rangle=\frac{e^{-i\omega_j(t-t')}}{e^{\hbar\omega_j/kT}-1}\delta_{jk},$$
$$[b_j^h(t),b_k^h(t')^\dagger]=e^{-i\omega_j(t-t')}\delta_{jk}. \tag{11}$$

With these results, we see that

$$D(t-t')\equiv\langle B^\dagger(t')B(t)\rangle$$
$$=\sum_j \frac{|\lambda_j|^2}{\hbar^2(e^{\hbar\omega_j/kT}-1)}e^{-i\omega_j(t-t')},$$
$$[B(t),B^\dagger(t')]=\sum_j \frac{|\lambda_j|^2}{\hbar^2}e^{-i\omega_j(t-t')}. \tag{12}$$

We wish to emphasize that these are exact general expressions. This form corresponds closely with that appearing in the quantum optics literature. Before proceeding, we write these results in a form more convenient for a general discussion.

The Fourier transform of the memory function (9) is

$$\tilde{K}(z)=\int_0^\infty dt e^{izt}K(t)=-i\sum_j \frac{|\lambda_j|^2}{\hbar^2}\frac{1}{\omega_j-z}, \tag{13}$$

where we have used explicitly the fact that $K(t)$ is zero for negative t. In particular, note that $\tilde{K}(z)$ is analytic in the upper half z plane and that

$$\mathrm{Re}\{K(\omega+i0^+)\}=\pi\sum_j \frac{|\lambda_j|^2}{\hbar^2}\delta(\omega-\omega_j)\theta(\omega). \tag{14}$$

Here we have introduced the Heaviside function to emphasize that $\mathrm{Re}\{\tilde{K}(\omega+i0^+)\}$ vanishes for negative frequencies. With this result, we see from Eq. (9) that we can write the memory function itself in the form

$$K(t)=\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Re}\{K(\omega+i0^+)\}e^{-i\omega t}\theta(t). \tag{15}$$

In the same way, the expressions (12) can be written

$$D(t)=\frac{1}{\pi}\int_0^\infty d\omega\frac{\mathrm{Re}\{\tilde{K}(\omega+i0^+)\}}{e^{\hbar\omega/kT}-1}e^{-i\omega t},$$
$$[B(t),B^\dagger(t')]=\frac{1}{\pi}\int_0^\infty d\omega\,\mathrm{Re}\{\tilde{K}(\omega+i0^+)\}e^{-i\omega(t-t')}. \tag{16}$$

These results correspond to those appearing in the literature, but they are now expressed in a very convenient form in that our basic equations are all expressed in terms of $\mathrm{Re}\{\tilde{K}(\omega+i0^+)\}$. They are general results in the sense that they can be applied to any relevant model simply by selecting the appropriate form of the function $\mathrm{Re}\{\tilde{K}(\omega+i0^+)\}$, subject only to the constraints that it be a real positive function that vanishes for $\omega<0$.

It is at this stage that we depart from the arguments usually given in the quantum optics literature. There it is customary to multiply by a factor $e^{i\omega_0 t}$ and to consider the quantity $D_I(t)\equiv e^{i\omega_0 t}D(t)$ [12]. It is then argued that when this factor is brought inside the integrand the resulting factor $e^{-i(\omega-\omega_0)t}$ is for long times sharply peaked about $\omega=\omega_0$ (so that everything except this factor can be evaluated at $\omega=\omega_0$ and taken outside the integral). The result, it is claimed, is $D_I(t)\rightarrow\mathrm{Re}\{\tilde{K}(\omega_0+i0^+)\}/(e^{\hbar\omega_0}-1)\delta(t)$. This argument, which has persisted in the literature for so long that it is difficult to trace its origins, is simply not correct. One can already see that something is wrong, since one could repeat the argument with ω_0 replaced with any frequency, say, $2\omega_0$. One would then get the same result, but with ω_0 replaced by $2\omega_0$. Clearly there is nothing special about the oscillator frequency in these expressions. We can, however, make the criticism more sharply. In the sense of distributions, $e^{-i(\omega-\omega_0)t}\rightarrow 0$, even at $\omega=\omega_0$. In other words, if one interprets this long time asymptotic form as a time average

$$e^{-i(\omega-\omega_0)t}\rightarrow\lim_{\epsilon\to 0^+}\epsilon\int_0^\infty dt e^{-\epsilon t}e^{-i(\omega-\omega_0)t}$$
$$=\lim_{\epsilon\to 0^+}\frac{\epsilon}{\epsilon+i(\omega-\omega_0)}=0. \tag{17}$$

It is correct that this long time limit is 1 if $\omega=\omega_0$ and zero otherwise. But this is not ''sharply peaked'' in the sense that it gives a finite contribution to an integral over ω. In other words, considered as a distribution this limit is zero. In this connection it will help to quote the Tauberian theorem [13]:

$$\int_0^\infty d\omega f(\omega) e^{-i\omega t} \sim \frac{1}{it} f(0) + \frac{1}{(it)^2} f'(0) + \cdots, \tag{18}$$

which is valid for any ''good'' function $f(\omega)$. The integrand of the expression (16) for $D(t)$ is in general such a function, and multiplication by $e^{i\omega_0 t}$ only multiplies each term by this oscillating factor. The long time behavior of $D(t)$ will therefore necessarily be that of a power law decay. In summary, the usual arguments are not correct and $D(t)$ simply has no nontrivial dependence on ω_0 [and $D_I(t)$ differs only by the factor $e^{i\omega_0 t}$]. We will now demonstrate this explicitly with a pair of examples: an ''Ohmic'' model and the coupling to the radiation field.

The model which is treated perhaps most often in the literature corresponds to choosing $\tilde{K}(\omega) = \gamma/2$, independent of frequency. Here there is a technical problem, since such a choice contradicts the requirement noted in Eq. (14) that $\tilde{K}(\omega)$ must vanish for negative frequencies. We accordingly choose

$$K(\omega) = \frac{\gamma}{2}\theta(\omega). \tag{19}$$

Putting this in the expression (15), we find

$$\begin{aligned} K(t) &= \frac{\gamma}{2\pi}\int_0^\infty d\omega e^{-i\omega t}\theta(t) \\ &= \frac{\gamma}{2}\left\{\delta(t) + \frac{1}{i\pi} P\frac{1}{t}\right\}\theta(t), \end{aligned} \tag{20}$$

where P denotes the principal value. We are aware that such a product of distributions is not well defined, but that will not trouble us here since we merely want to point out that this choice leads to a memory function that is not real and, moreover, that has a long time tail power law decay consistent with the Tauberian theorem (18).

Next, consider the expression (16) for $D(t)$. With the expression (19) for $\tilde{K}(\omega)$, we obtain the integral formula

$$D(t) = \frac{\gamma}{2\pi}\int_0^\infty d\omega \frac{e^{-i\omega t}}{e^{\hbar\omega/kT} - 1}. \tag{21}$$

But this integral is divergent at zero frequency, therefore this ''Ohmic'' model gives a physically meaningless result for the correlation. Note that this divergence arises from the behavior of $\tilde{K}(\omega)$ at zero frequency; it is not associated with some high- frequency cutoff. In fact this is a difficulty with the RWA Hamiltonian and the procedure leading to the basic equation (8). There is no such difficulty in a corresponding discussion of the exact Hamiltonian (1) [9].

As a second illustration involving a more physical model, we consider the the example of a charged oscillator interacting with the radiation field (blackbody reservoir). Here there is a choice of the form of the exact Hamiltonian from which we drop terms to make the rotating wave approximation: the momentum or ''$\mathbf{p}\cdot\mathbf{A}$'' coupling or the dipole or ''$\mathbf{r}\cdot\mathbf{E}$'' coupling, which give different forms of the RWA Hamiltonian, although the exact Hamiltonians are equivalent [9]. Here we will use the momentum coupling, which following the procedure described above leads to the expression

$$\mathrm{Re}\{K(\omega + i0^+)\} = \frac{e^2\omega_0\omega}{3mc^3}. \tag{22}$$

With this in the expression (16) for $D(t)$, we obtain the integral formula

$$D(t) = \frac{e^2\omega_0}{3\pi mc^3}\int_0^\infty d\omega \frac{\omega e^{-i\omega t}}{e^{\hbar\omega/kT} - 1}. \tag{23}$$

This integral is convergent and can be integrated exactly, with the result expressed in terms of the generalized zeta function of Riemann [14]. However, what is important for our discussion is that the *spectrum* is independent of the oscillator frequency. Indeed, the only dependence of this expression on the oscillator frequency is the factor of ω_0, which occurs only because the expression for the oscillator momentum in terms of creation and annihilation operators, $p = -i\sqrt{m\hbar\omega_0/2}(a - a^\dagger)$, involves a factor of $\sqrt{\omega_0}$.

Actually, from the point of view of observations, it is perhaps better to consider rather than $D(t)$ the correlation

$$\begin{aligned} C(t-t') &= \frac{1}{2}\langle B^\dagger(t')B(t) + B^\dagger(t)B(t')\rangle \\ &= \mathrm{Re}\{D(t-t')\}, \end{aligned} \tag{24}$$

which is real and related to the force-force correlation for the oscillator. This correlation can be expressed in terms of elementary functions [15,16]

$$\begin{aligned} C(t) = \frac{e^2\omega_0}{6\pi mc^3}\bigg\{ &\frac{1}{t^2} - \frac{\pi^2k^2T^2}{\hbar^2}\operatorname{csch}^2(\pi kTt/\hbar) \\ &+ \frac{2\pi kT}{\hbar}\delta(t)\bigg\}. \end{aligned} \tag{25}$$

Again, we emphasize that this correlation is, in form, independent of ω_0. In addition, we emphasize that this and the corresponding expression (23) for $D(t)$ are exact, with no restrictions on the strength of the coupling. The coupling strength is measured by the square of the electron charge and appears only as a multiplying factor. Therefore, the correlation and its power spectrum are, in form, also independent of the coupling strength. Note finally that the long time t^{-2} dependence is consistent with the Tauberian theorem (18).

More generally, the *force correlation*, in particular its power spectrum, must be *independent of the potential* [9]. Indeed it must be the same as that for the free electron. In other words, quantum noise is *universal*, its spectrum corresponds to that of Planck (with zero-point oscillations when appropriate) not that of white noise. In summary, in the present context the power spectrum of the correlation resulting from the RWA Hamiltonian is not that of white noise and, apart from a trivial scale factor, is independent of the oscillator frequency ω_0, in sharp disagreement with expressions appearing in the literature.

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[1] W. H. Louisell, *Quantum Statistical Properties of Radiation* (Wiley, New York 1973).

[2] M. Sargent, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics* (Addison-Wesley, Reading, MA, 1974).

[3] P. Meystre and M. Sargent III, *Elements of Quantum Optics* (Springer, Berlin 1991).

[4] C. Cohen-Tannoudji, J. Dupont-Roc, and G. Grynberg, *Atom-Photon Interactions*(Wiley, New York 1992).

[5] D. F. Walls and G. J. Milburn, *Quantum Optics* (Springer, Berlin, 1994).

[6] M. Freyberger, K. Vogel, W. Schleich, and R. F. O'Connell, in *Atomic, Molecular, and Optical Physics Handbook* (AIP Press, Woodbury, NY, 1996).

[7] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University Press, Cambridge, 1995).

[8] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. Lett. **55**, 2273 (1985).

[9] G. W. Ford, J. T. Lewis, and R. F. O'Connell, Phys. Rev. A **37**, 4419 (1988).

[10] G. W. Ford and R. F. O'Connell, Physica A **243**, 377 (1997).

[11] G. W. Ford and R. F. O'Connell, Phys. Lett. A **224**, 22 (1996).

[12] This is sometimes said to correspond to going to the interaction representation. We would, however, remind the reader that $D(t)$ is a physical expectation value, which is necessarily independent of representation.

[13] G. Doetsch, *Introduction to the Theory and Application of the Laplace Transform* (Springer-Verlag, Berlin, 1974).

[14] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1940).

[15] Bateman manuscript project, *Tables of Integral Transforms*, edited by E. Erdélyi (McGraw-Hill, New York, 1954).

[16] G. W. Ford and R. F. O'Connell, Nature (London) **380**, 113 (1996).