Comment on "Dissipative Quantum Dynamics with a Lindblad Functional"

In a recent Letter [1] Gao addressed the question of master equations "... known to violate the positivity requirement of the density operator ...." He began with what we might call a pre-Lindblad equation for a linear oscillator, in a form obtained by many authors, and proposed a modification to put it into Lindblad form [2]. While there have been objections to Gao's proposal [3], we wish to point out here that the equation with which he began is not unique in the sense that unitarily equivalent microscopic Hamiltonians lead to *different forms* of the pre-Lindblad equation. These different forms describe, of course, the same physical system, but there is no reason to give any special significance to any one of them. Moreover, there is a *unique* master equation in Lindblad form which is obtained from these various equations by a well-known prescription [4].

For the system of an oscillator coupled to an oscillator heat bath, one can choose a microscopic Hamiltonian with either coordinate or momentum coupling. (The two forms are familiar in electrodynamics as the "xE" and the "pA" interactions.) The two Hamiltonians are related by a gauge transformation that *does not change the oscillator coordinate* and therefore describe the same system [5]. But the form of the equation for the reduced density matrix one derives from the two microscopic Hamiltonians is different. Thus, if one uses coordinate coupling one obtains the equation (ignoring the energy shift)

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H,\rho] - \frac{\gamma(\omega_0)}{2\hbar}\left\{ i[x, p\rho + \rho p] + m\omega_0 \coth\frac{h\omega_0}{2kT}[x,[x,\rho]]\right\}, \tag{1}$$

where $m\gamma(\omega_0)$ is the Newtonian friction constant and H is the free oscillator Hamiltonian. This is the form of the equation obtained in most previous discussions and the high temperature Ohmic case is that with which Gao begins. On the other hand, if one repeats the derivation with the momentum coupling model [6], one obtains

$$\frac{\partial \tilde{\rho}}{\partial t} = \frac{1}{i\hbar}[H,\tilde{\rho}] - \frac{\gamma(\omega_0)}{4\hbar}\left\{ -i[p, x\tilde{\rho} + \tilde{\rho}x] + \frac{1}{m\omega_0}\coth\frac{\hbar\omega_0}{2kT}[p,[p,\tilde{\rho}]]\right\}. \tag{2}$$

From either of these equations one obtains the same master equation by applying the Wangness-Bloch prescription: go to the interaction representation, discard the terms explicitly oscillating at frequency $2\omega_0$, and return to the Schrödinger representation [4]. The result is

$$\frac{\partial \overline{\rho}}{\partial t} = \frac{1}{i\hbar}[H,\overline{\rho}] - \frac{\gamma(\omega_0)}{4\hbar}\left\{ i([x, p\overline{\rho} + \overline{\rho}p] - [p, x\overline{\rho} + \overline{\rho}x]) + \coth\frac{\hbar\omega_0}{2kT}\left(\frac{1}{m\omega_0}[p,[p,\overline{\rho}]] + m\omega_0[x,[x,\overline{\rho}]]\right)\right\}. \tag{3}$$

Here we have introduced a bar to indicate that $\overline{\rho}$ is the slowly varying mean. It is a simple matter to verify that this has the Lindblad form of the master equation familiar in quantum optics [6,7]. This is the equation sought by Gao and other investigators.

We wish to emphasize that all three of the above equations lead to the same equilibrium state: $\rho_{\mathrm{eq}} = \exp\{-H/kT\}$; i.e., detailed balance is obeyed. The difference is in the *approach* to equilibrium. For the pre-Lindblad equations (1) and (2) this can be through (unphysical) states in which ρ is not positive definite. Note that this form of the equilibrium state holds only if H is the oscillator Hamiltonian.

We are well aware that the Wangness-Bloch prescription requires weak coupling. But Eqs. (1) and (2) as well as master equation (3) are weak coupling results. This is seen in that for all these equations the mean square displacement of the oscillator in the equilibrium state is that of a free oscillator, $\langle x^2\rangle = \frac{\hbar}{2m\omega_0}\coth\frac{\hbar\omega_0}{2kT}$, while the exact result obtained from the fluctuation-dissipation theorem is [5]

$$\langle x^2\rangle = \frac{\hbar}{\pi}\,\mathrm{Im}\int_0^\infty d\omega\,\frac{\coth\frac{\hbar\omega}{2kT}}{-m\omega^2 - i\omega\tilde{\mu}(\omega) + m\omega_0^2}, \tag{4}$$

where $\mathrm{Re}\{\tilde{\mu}(\omega)\} = m\gamma(\omega)$.

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