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## Radiating electron: Fluctuations without dissipation in the equation of motion

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(Received 27 March 1997)

The quantum fluctuation-dissipation theorem is generally manifested in the equation of motion through the appearance of a dissipative (frictional) term as well as a fluctuating force. Here it is shown that the equation of motion for a free electron interacting with the radiation field, when expressed in terms of the renormalized (observed) mass, displays fluctuations without dissipation. Despite this apparent contradiction, the equation is consistent with the fluctuation-dissipation theorem. [S1050-2947(98)03704-4]

PACS number(s): 12.20.Ds, 05.30.-d, 05.40.+j, 41.60.-m

One of the cornerstones of statistical mechanics is the fundamental relationship between fluctuations and dissipation. This relationship is generally reflected in the equation of motion for the system variables as it contains both dissipative and fluctuation forces. Here we point out that mass renormalization for a nonrelativistic electron interacting with the radiation field leads to an equation of motion for the free electron which has a fluctuating force but no corresponding dissipative force. The result derives from the universally accepted Hamiltonian of nonrelativistic quantum electrodynamics and incorporates the standard expression relating the bare and renormalized masses.

Fluctuations are due to thermal and quantum effects. In the classical regime, the origin of the relationship between thermal fluctuations and dissipation stems from the work of Einstein, in which he wrote down the relationship between the diffusion constant (which is directly related to the fluctuations) and the mobility of a Brownian particle [1]. Next Nyquist proved that an electrical impedance gives rise to a fluctuating voltage difference between its terminals [2]. This was followed by Onsager's classical regression theorem relating the average decay of fluctuations to the decay law governing the corresponding macroscopic irreversible process [3]. The culmination of these investigations is the celebrated quantum-mechanical fluctuation-dissipation theorem of Callen and Welton [4]. Further insight was provided by the work of Kubo on a linear-response formulation [5] and Mori's demonstration that a macroscopic equation of motion for a many-body system could be written in the form of a generalized quantum Langevin equation [6]. In the quantum Langevin equation, dissipative and fluctuation forces are explicitly displayed.

A heat bath par excellence is the radiation field, because its properties are widely known and universally accepted. In

previous work we derived, in the form of a generalized quantum Langevin equation, an exact equation of motion for a particle with charge  $-e$  and bare mass  $m$  in an arbitrary external potential  $V(x)$  and coupled in dipole approximation to the radiation field. Our starting point was a microscopic Hamiltonian that treated the heat bath as an infinite number of arbitrary independent oscillators (the IO model), for which the equation of motion takes the form [7]

$$m\ddot{x} + \int_{-\infty}^t dt_1 \mu(t-t_1) \dot{x}(t_1) + V'(x) = F(t) + f(t). \quad (1)$$

Here in the left hand side the dot denotes the time derivative, the second term involving the memory function  $\mu(t)$  is the dissipation term, while in the third term  $V'(x) = dV/dx$  with  $V(x)$  the time-independent external potential. It is a consequence of the second law of thermodynamics that  $\tilde{\mu}(z)$ , the Fourier transform of the memory function, must be a positive real function: analytic with no zeros or poles and with real part positive throughout the upper half plane. On the right hand side of Eq. (1),  $F(t)$  is the operator-valued random force, while  $f(t)$  is a  $c$ -number position-independent external force. The random force is Gaussian and characterized by its (symmetric) autocorrelation,

$$\begin{aligned} \frac{1}{2} \langle F(t)F(t') + F(t')F(t) \rangle &= \frac{1}{\pi} \int_0^{\infty} d\omega \operatorname{Re} \{ \tilde{\mu}(\omega + i0^+) \} \hbar \omega \\ &\times \coth \left( \frac{\hbar \omega}{2kT} \right) \cos[\omega(t-t')], \end{aligned} \quad (2)$$

and the nonequal-time commutator

$$[F(t), F(t')] = \frac{2}{i\pi} \int_0^\infty d\omega \operatorname{Re}\{\tilde{\mu}(\omega + i0^+)\} \hbar \omega \sin[\omega(t-t')]. \quad (3)$$

Thus the random force is uniquely determined by the dissipation term. This fact can be shown to be a consequence of the fluctuation-dissipation theorem.

The radiation field is a particular case of this IO model [7,8], with

$$\operatorname{Re}\{\tilde{\mu}(\omega + i0^+)\} = \frac{2e^2}{3c^3} \omega^2 f_k^2, \quad (4)$$

where  $f_k$  is a frequency-dependent form factor (Fourier transform of the charge distribution) which incorporates the electron structure. In the limiting case of a point electron, for which  $f_k=1$ , it can be shown [9] that the equation of motion (1) is the quantum generalization of the well-known Abraham-Lorentz equation, for which the dissipation term takes the form  $-(2e^2/3c^3)d^3x/dt^3$  [10]. However, in this case there is a pole of the generalized susceptibility lying in the upper half plane, thus violating causality and the second law of thermodynamics. This in fact is the underlying cause of the famous problem of runaway solutions of the Abraham-Lorentz equation. We proposed to remedy this defect by choosing [9]

$$f_k^2 = \frac{1}{1 + \omega^2 \tau_e^2}, \quad \tau_e = \frac{2e^2}{3Mc^3} \approx 6 \times 10^{-24} \text{ sec}. \quad (5)$$

Here  $M$  is the renormalized (observed) mass and the choice is in a certain sense maximal, corresponding to bare mass  $m=0$ . For this choice, the equation of motion (1) can be rearranged into the particularly simple form [9]

$$M\ddot{x} + V'_{\text{eff}}(x) = F_{\text{eff}}(t) + f_{\text{eff}}(t), \quad (6)$$

where  $F_{\text{eff}}(t) = F(t) + \tau_e \dot{F}(t)$  and similarly for the other ‘‘effective’’ quantities.

Here we wish first of all to point out that, as can be verified by explicit calculation, the autocorrelation and the nonequal-time commutator of  $F_{\text{eff}}(t)$  are the same as the corresponding results for  $F(t)$  [see Eqs. (2) and (3)] but with  $\operatorname{Re}\{\tilde{\mu}(\omega + i0^+)\} = 2e^2\omega^2/3c^3$ , corresponding to the point electron limit with  $f_k=1$  in Eq. (4). But, since  $F(t)$  and hence  $\dot{F}(t)$  are Gaussian, they are completely determined by the correlation and commutator. That is,  $F_{\text{eff}}(t)$  is exactly the fluctuation force acting on a point electron. Hence we have the remarkable result that

$$F_{\text{eff}}(t) = -eE(t), \quad (7)$$

where  $E(t)$  is the electric field of the free radiation field at the electron.

Next we wish to point out that in the free particle case, where there is no external potential,  $V(x)=0$ , and no external applied field,  $f(t)=0$ , the equation of motion takes the remarkably simple form

$$M\ddot{x} = -eE(t). \quad (8)$$

This stochastic equation is intriguing, since it displays fluctuations (the right hand side is an operator-valued fluctuating force) with no dissipative term on the left hand side. Nevertheless, the fluctuation-dissipation theorem still holds, as we shall now show.

The fluctuation-dissipation theorem states that the velocity autocorrelation function is given by

$$\begin{aligned} & \frac{1}{2} \langle \dot{x}(t)\dot{x}(t') + \dot{x}(t')\dot{x}(t) \rangle \\ &= \frac{\hbar}{\pi} \int_0^\infty d\omega \operatorname{Im}\{\alpha(\omega + i0^+)\} \omega^2 \\ & \quad \times \coth\left(\frac{\hbar\omega}{2kT}\right) \cos[\omega(t-t')], \end{aligned} \quad (9)$$

where  $\alpha(z)$  is the generalized susceptibility (response function) [11]. The generalized susceptibility is defined in terms of the response to a  $c$ -number driving force, so we must consider the equation

$$M\ddot{x} = f(t) + \tau_e \dot{f}(t). \quad (10)$$

From this we readily see that

$$\alpha(z) = -\frac{1 - iz\tau_e}{Mz^2}, \quad (11)$$

and, using the well-known formula  $1/(\omega + i0^+) = \mathcal{P}(1/\omega) - i\pi\delta(\omega)$ , we see that

$$\operatorname{Im}\{\alpha(\omega + i0^+)\} = -\frac{\pi}{M} \delta'(\omega) + \frac{\tau_e}{M} \mathcal{P} \frac{1}{\omega}. \quad (12)$$

With this, we see that the velocity autocorrelation (9) becomes [12]

$$\begin{aligned} & \frac{1}{2} \langle \dot{x}(t)\dot{x}(t') + \dot{x}(t')\dot{x}(t) \rangle \\ &= \frac{kT}{M} + \frac{\tau_e \hbar}{M\pi} \int_0^\infty d\omega \omega \coth\left(\frac{\hbar\omega}{2kT}\right) \cos[\omega(t-t')] \\ &= \frac{kT}{M} + \frac{\tau_e kT}{M} \left\{ 2\delta(t-t') - \frac{\pi kT}{\hbar} \operatorname{csch}^2\left[\frac{\pi kT}{\hbar}(t-t')\right] \right\}. \end{aligned} \quad (13)$$

As might be expected, for  $\tau_e \rightarrow 0$  (zero coupling) the result tends to  $kT/M$ .

The result (13) is consistent with the equation of motion (8), for which the solution may be written as

$$\dot{x}(t) = \dot{x}(-\infty) - e \int_{-\infty}^t dt' E(t'). \quad (14)$$

Here the first term is present since the initial velocity persists for all time because there is no dissipation in the equation of motion. Exactly the result (13) then follows, since there is no correlation between  $\dot{x}(-\infty)$  and the field  $E(t)$  and since

$\frac{1}{2}\langle\dot{x}(-\infty)\dot{x}(-\infty)+\dot{x}(-\infty)\dot{x}(-\infty)\rangle=kT/M$ , while the field correlation is given by Eq. (2) with  $\text{Re}\{\tilde{\mu}(\omega+i0^+)\} = 2e^2\omega^2/3c^3$ .

As a further remark, we note that in the case of an electron in an oscillator potential  $V(x)=\frac{1}{2}Kx^2$ , Eq. (6) takes the form

$$M\ddot{x}+\zeta\dot{x}+Kx=-eE(t), \quad (15)$$

where  $\zeta=K\tau_e$ . The presence of an Ohmic-type dissipation term is now manifest. Incidentally, this term is exactly of the form universally used in practical applications of the Abraham-Lorentz equation to radiation damping problems (where it is derived as a weak coupling approximation to the dissipation term [10]). It is also of interest to note that  $\zeta/\sqrt{KM}$  can be identified as a dimensionless measure of the weak coupling (of the order of  $10^{-7}$  for optical frequencies) of relevance for studies in areas such as quantum optics.

Finally we summarize the key features of the results we have obtained and how they differ from existing results in the literature [10].

(a) For a free particle, in the absence of any external force, our equation of motion does not contain a dissipative term (in contrast to the well-known Abraham-Lorentz equation for which a dissipative term is always present).

(b) Our equation of motion (6) is a simple second-order equation and has well-behaved solutions (in contrast to the runaway solutions of the Abraham-Lorentz third-order equation).

(c) Our equation goes beyond the usual classical results discussed in the literature, in that it is quantum mechanical and necessarily contains the fluctuation force  $F_{\text{eff}}(t)=-eE(t)$ .

(d) A careful distinction is made between the bare mass  $m$  and the renormalized mass  $M$ .

Part of this work was carried out at the Dublin Institute for Advanced Studies. The work of R.F.O.C. was supported in part by the U.S. Army Research Office under Grant No. DAAH04-94-G-0333.

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