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Master Equation for an Oscillator Coupled to the Electromagnetic Field

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The macroscopic description of a quantum oscillator with linear passive dissipation is formulated in terms of a master equation for the reduced density matrix. The procedure used is based on the asymptotic methods of nonlinear dynamics, which enables one to obtain an expression for the general term in the weak coupling expansion. For the special example of a charged oscillator interacting with the electromagnetic field, an explicit form of the master equation through third order in this expansion is obtained. This form differs from that generally obtained using the rotating wave approximation in that there is no electromagnetic (Lamb) shift and that an explicit expression is given for the decay rate.

I. INTRODUCTION

In the field of quantum optics, a central topic is the subject of master equations. It is discussed in all the modern textbooks [1–5] and review articles [6] in the area, with special emphasis being placed on the coupling of the radiation field to either the two-level atom or the harmonic oscillator. In every case the discussion involves the so-called rotating-wave Hamiltonian. But this approximate Hamiltonian has serious defects [7], the most serious being that as an operator it has no lower bound on its spectrum. This has the consequence that, strictly speaking, there is no equilibrium state and, therefore, no fluctuation-dissipation theorem. On the other hand, to the level of approximation generally used (weak coupling and only through second order) this defect does not lead to unphysical results. However, the approximation does, as we shall see, lead to an incorrect formula for the level shifts.

The master equation is a macroscopic equation for the density matrix of an atomic system (in our case the oscillator) coupled to a thermal reservoir. In the physically interesting case of coupling to the electromagnetic field, the reservoir is the radiation in a blackbody cavity. The master equation corresponds to a reduced description of the approach to equilibrium, in the sense that it does not involve the

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reservoir variables, except through the parameters of the equation, nor does it contain an explicit time dependence. The general form of such a master equation, satisfying conditions of positivity and boundedness, has been given by Lindblad [8].

Our primary purpose here will be to give a consistent derivation of the master equation for a charged oscillator coupled to the radiation field, without making any approximations to the Hamiltonian. We will be able to discuss the mass renormalization, generally ignored in previous discussions, and the resulting level shifts. The methods we use, based on the asymptotic methods of nonlinear dynamics, will enable us to give a succinct general derivation of the master equation, applicable to an oscillator in a general dissipative environment, of which particular examples are the radiation field and the Ohmic environment. We will also be able to obtain an expression for the general term in the weak coupling expansion, which can be used to go beyond the usual second order approximation. We obtain explicit expressions through third order.

II. THE MICROSCOPIC HAMILTONIAN FOR AN OSCILLATOR COUPLED TO A FIELD

The system consisting of an oscillator coupled to a reservoir giving rise to linear passive dissipation can be described by a Hamiltonian of the form

$$H = \frac{(p+A)^2}{2m} + \frac{1}{2}Kx^2 + H_R, \tag{2.1}$$

where m is the oscillator mass and K the force constant. Here,

$$H_R = \sum_j \left(\frac{1}{2m_j} p_j^2 + \frac{1}{2} m_j \omega_j^2 q_j^2 \right) \tag{2.2}$$

is the Hamiltonian for the reservoir and the coupling of the oscillator to the reservoir is described by the field operator

$$A = \sum_j m_j \omega_j q_j. \tag{2.3}$$

With the canonical commutation relations,

$$[x, p] = ih, \qquad [q_j, p_k] = ih\delta_{jk}, \tag{2.4}$$

this defines the system. Note that, since each term is a positive operator, the Hamiltonian (2.1) is a positive operator, with zero a lower bound on its spectrum. This Hamiltonian, sometimes called a velocity-coupling model, is in fact very general. It is unitarily equivalent to a wide class of oscillator models of the reservoir

[7]. In particular, it is shown in Ref. [7] to be equivalent to what is there called the independent oscillator model. Here we use the velocity-coupling form, since it will be convenient for the discussion of mass renormalization when we consider the case of a blackbody radiation reservoir.

The Hamiltonian (2.1) corresponds to a very general microscopic description of an oscillator with linear passive dissipation. To see what is meant by this statement, consider the macroscopic description of the system in terms of the quantum Langevin equation. This is a Heisenberg equation of motion for the time-dependent position operator $x(t)$, of the form

$$m\ddot{x} + \int_{-\infty}^{t} dt' \mu(t-t')\, \dot{x}(t') + Kx = F(t). \tag{2.5}$$

Here $F(t)$ is an operator-valued random force and $\mu(t)$ is the memory function. As a consequence of the second law of thermodynamics, it can be shown that the Fourier transform of the memory function,

$$\tilde{\mu}(z) \equiv \int_0^{\infty} dt e^{izt} \mu(t), \qquad Im\{z\} > 0, \tag{2.6}$$

must be what is called a positive real function, analytic and with real part positive in the entire upper half plane [7]. A well known theorem (the Stieltjes inversion theorem) tells us that such a positive real function is determined by the real part of its boundary value on the real axis, $Re\{\tilde{\mu}(\omega + i0^+)\}$. We call this the spectral distribution. For the quantum Langevin equation, the spectral distribution determines not only the memory function but also, as a consequence of the fluctuation-dissipation theorem, the correlation and commutator of the random force. Thus, the spectral distribution completely characterizes the macroscopic description of the system, as given by the quantum Langevin equation. Finally, we note that if the quantum Langevin equation is to describe an approach to a unique equilibrium state for a general oscillator, the spectral distribution, $Re\{\tilde{\mu}(\omega + i0^+)\}$, must not vanish on any finite interval of the real axis. The point here being that if there is such a "gap" in the spectrum, an oscillator whose natural frequency is in the gap cannot decay but will persist in an arbitrary excited state. This phenomenon has become of some interest recently under the name "cavity QED" [9].

It can be shown that the microscopic Hamiltonian (2.1) leads to the following expression for the spectral distribution [7],

$$Re\{\tilde{\mu}(\omega + i0^+)\} = \frac{\pi}{2} \sum_j m_j \omega_j^2 [\delta(\omega - \omega_j) + \delta(\omega + \omega_j)]. \tag{2.7}$$

We note first of all that this is a positive distribution and, moreover, that by a suitable choice of the frequencies and masses of the reservoir-oscillators one can in this way represent the most general spectral distribution. It is in this sense that we claim that the Hamiltonian (2.1) is general. If, however, the spectral distribution is

not to vanish in any finite interval, the number of reservoir oscillators must be infinite and their frequencies must be distributed over all real values. Note in particular that the spectral distribution depends only on the parameters of the reservoir and is independent of the oscillator parameters: the mass m and the spring constant K.

III. THE VON NEUMANN EQUATION

The time development of the density matrix operator for the system is determined by the von Neumann equation,

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H, \rho]. \tag{3.1}$$

Our aim will be to derive from this equation a macroscopic description in terms of a reduced density matrix, depending only on the oscillator variables x and p. The picture here is that in the distant past the oscillator is decoupled from the reservoir, for example by fixing it to a very large mass. The reservoir will then be in equilibrium at temperature T, described by the equilibrium density matrix

$$\rho_R = \frac{\exp(-H_R/kT)}{Tr_R\{\exp(-H_R/kT)\}}. \tag{3.2}$$

Here Tr_R indicates the trace over a complete set of states in the Hilbert space of the reservoir Hamiltonian (2.2), sometimes called the partial trace. The initial density matrix for the von Neumann equation (3.1) will then be of the form of a product,

$$\rho(0) = \tilde{\rho}(0)\, \rho_R, \tag{3.3}$$

where $\tilde{\rho}(0)$is the partial trace of $\rho(0)$. The system will then evolve according to the von Neumann equation, with this initial value of $\rho(t)$. We can then form the partial trace of the solution,

$$\tilde{\rho}(t) = Tr_R\{\rho(t)\}. \tag{3.4}$$

We identify this as the reduced density matrix of the system.

Our first goal will be to find an equation for this reduced density matrix. Only for long times can this equation be independent of our assumption about the initial state. We therefore shall use an asymptotic method, valid for long times, to determine the equation. To do this we assume the coupling is weak and begin by writing the Hamiltonian (2.1) in the form

$$H = H_O + H_R + \varepsilon H_I, \tag{3.5}$$

where H_O is the Hamiltonian of the free oscillator,

$$H_O = \frac{p^2}{2m} + \frac{1}{2}Kx^2, \tag{3.6}$$

while H_R is the reservoir Hamiltonian (2.2) and the interaction Hamiltonian is

$$\varepsilon H_I = \frac{pA}{m} + \frac{A^2}{2m}. \tag{3.7}$$

Here, for convenience in keeping track of the various orders, we have introduced a parameter ε, which in the weak coupling approximation will be considered small.

Next, since the coupling is weak, we introduce the density matrix in the interaction representation,

$$\sigma = e^{i(H_O + H_R)t/h} \rho e^{-i(H_O + H_R)t/h}. \tag{3.8}$$

Using the von Neumann equation (3.1), we obtain the equation for this density matrix,

$$\frac{\partial \sigma}{\partial t} = \frac{\varepsilon}{ih}[H_I(t), \sigma]. \tag{3.9}$$

Here, $H_I(t)$ is the interaction Hamiltonian carried forward a time t with the free motion,

$$\begin{aligned} \varepsilon H_I(t) &= e^{i(H_O + H_R)t/h} \varepsilon H_I e^{-i(H_O + H_R)t/h} \\ &= \frac{p(t)A(t)}{m} + \frac{A^2(t)}{2m}. \end{aligned} \tag{3.10}$$

For the field $A(t)$ we have the explicit expression

$$\begin{aligned} A(t) &= e^{iH_R t/h} A e^{-iH_R t/h} \\ &= \sum_j \{m_j \omega_j q_j \cos(\omega_j t) + p_j \sin(\omega_j t)\}, \end{aligned} \tag{3.11}$$

while the free motion of the oscillator is given by the explicit expressions

$$\begin{aligned} x(t) &= e^{iH_O t/h} x e^{-iH_O t/h} = x \cos(\omega_0 t) + p \sin(\omega_0 t)/m\omega_0 \\ p(t) &= e^{iH_O t/h} p e^{-iH_O t/h} = p \cos(\omega_0 t) - xm\omega_0 \sin(\omega_0 t). \end{aligned} \tag{3.12}$$

where ω_0 is the free oscillation frequency of the oscillator, given by

$$\omega_0 = \sqrt{K/m}. \tag{3.13}$$

We now apply an asymptotic method based on the method of averaging (see Appendix B) to obtain from the Eq. (3.9) an equation for the partial trace,

$$\tilde{\sigma}(t) = Tr_R\{\sigma(t)\}. \tag{3.14}$$

The idea is to use the weak coupling assumption to argue that at any instant of time, not just for the initial time, the oscillator and the reservoir will be approximately decoupled. That is, we expect that $\sigma \cong \tilde{\sigma}\rho_R$. Moreover, since the right hand side of the Eq. (3.9) for σ is small, we expect that the partial trace will be slowly varying. We are therefore led to seek a solution in the form

$$\sigma = \tilde{\sigma}\rho_R + \varepsilon F^{(1)}(\tilde{\sigma}, t) + \varepsilon^2 F^{(2)}(\tilde{\sigma}, t) + \cdots. \tag{3.15}$$

Here the successive terms correspond to small amplitude fluctuations of the density matrix about $\tilde{\sigma}\rho_R$, its slowly varying mean. For this reason, and for consistency with the definition (3.14), we must require

$$Tr_R\{F^{(n)}(\tilde{\sigma}, t)\} = 0, \qquad n = 1, 2, \ldots. \tag{3.16}$$

The reduced density matrix must satisfy an equation of the form

$$\frac{\partial \tilde{\sigma}}{\partial t} = \varepsilon B^{(1)}(\tilde{\sigma}, t) + \varepsilon^2 B^{(2)}(\tilde{\sigma}, t) + \cdots. \tag{3.17}$$

Here, since the reduced density matrix $\tilde{\sigma}$ is the partial trace of the density matrix σ, the right hand side must be independent of the reservoir operators. That is,

$$Tr_R\{B^{(n)}(\tilde{\sigma}, t)\,\rho_R\} = B^{(n)}(\tilde{\sigma}, t). \tag{3.18}$$

To determine the terms in these expansions, we insert the expansion (3.15) in the von Neumann equation (3.9) and use the expansion (3.17) for $\partial\tilde{\sigma}/\partial t$. Equating the coefficients of equal powers of ε on either side of the equation, we get the sequence of equations:

$$\begin{aligned}
&\rho_R B^{(1)} + \frac{\partial F^{(1)}}{\partial t} = \frac{1}{ih}[H_I(t), \tilde{\sigma}\rho_R],\\
&\rho_R B^{(2)} + \frac{\partial F^{(2)}}{\partial t} = \frac{1}{ih}[H_I(t), F^{(1)}(\tilde{\sigma}, t)] - F^{(1)}[B^{(1)}(\tilde{\sigma}, t), t],\\
&\qquad \cdots,\\
&\rho_R B^{(n)} + \frac{\partial F^{(n)}}{\partial t} = \frac{1}{ih}[H_I(t), F^{(n-1)}] - \sum_{m=1}^{n-1} F^{(n-m)}(B^{(m)}, t),\\
&\qquad \cdots.
\end{aligned} \tag{3.19}$$

Each of the equations in this sequence is of the same form, with the right hand side of each known in terms of the solution of the earlier equations. The solution of the first order equation is therefore typical and we discuss its solution in some detail.

The two operators, $B^{(1)}(\tilde{\sigma}, t)$ and $F^{(1)}(\tilde{\sigma}, t)$, must both be determined from the same first order equation (3.19). The point is that the former is determined as a condition of existence of the solution for the latter. Thus, if $F^{(1)}(\tilde{\sigma}, t)$ is to satisfy the condition (3.16), $B^{(1)}(\tilde{\sigma}, t)$ must be chosen to be equal to the partial trace of the right hand side,

$$B^{(1)}(\tilde{\sigma}, t) = Tr_R \left\{ \frac{1}{i\hbar} [H_I(t), \tilde{\sigma}\rho_R] \right\}. \tag{3.20}$$

With this choice, one can integrate the first order equation to find

$$F^{(1)}(\tilde{\sigma}, t) = \frac{1}{i\hbar} \int_{-\infty}^{t} dt_1 ([H_I(t_1), \tilde{\sigma}\rho_R] - Tr_R\{[H_I(t_1), \tilde{\sigma}\rho_R]\}\ \rho_R). \tag{3.21}$$

Here, the choice of the lower limit of integration is consistent with our assumption that in the distant past the oscillator is decoupled from the reservoir.

With these results for the first order terms, the second order equation (3.19) becomes

$$\begin{aligned} \rho_R B^{(2)} + \frac{\partial F^{(2)}}{\partial t} = \frac{1}{(i\hbar)^2} \int_{-\infty}^{t} dt_1 ([H_I(t), [H_I(t_1), \tilde{\sigma}\rho_R]] \\ - [H_I(t), Tr_R\{[H_I(t_1), \tilde{\sigma}\rho_R]\}\ \rho_R] \\ - [H_I(t_1), Tr_R\{[H_I(t), \tilde{\sigma}\rho_R]\}\ \rho_R] \\ + Tr_R\{[H_I(t_1), Tr_R\{[H_I(t), \tilde{\sigma}\rho_R]\}\ \rho_R]\}\ \rho_R). \end{aligned} \tag{3.22}$$

Forming the partial trace, we find

$$\begin{aligned} B^{(2)}(\tilde{\sigma}, t_1) = \frac{1}{(i\hbar)^2} \int_{-\infty}^{t_1} dt_2 (Tr_R\{[H_I(t_1), [H_I(t_2), \tilde{\sigma}\rho_R]]\} \\ - Tr_R\{[H_I(t_1), Tr_R\{[H_I(t_2), \tilde{\sigma}\rho_R]\}\ \rho_R]\}). \end{aligned} \tag{3.23}$$

With this expression for $B^{(2)}$, one can integrate the second order equation to obtain an expression for $F^{(2)}$. We don't exhibit this expression, since it is rather long and we will not be needing it in the further discussion. In Appendix A, we give a formula for constructing the general term $B^{(n)}$.

IV. EXPLICIT FORM OF THE TERMS

From the form (3.10) for $H_I(t)$, we see that the reservoir variables appear only through the field operator $A(t)$. It will be convenient to introduce a shorthand notation for the thermal expectation of products of this operator. Thus, we write

$$\begin{aligned} \langle 1\rangle &\equiv Tr\{\rho_R A(t_1)\},\\ \langle 12\rangle &\equiv Tr\{\rho_R A(t_1)\, A(t_2)\},\\ \langle 123\rangle &\equiv Tr\{\rho_R A(t_1)\, A(t_2)\, A(t_3)\},\\ &\text{etc.} \end{aligned} \tag{4.1}$$

The field operator $A(t)$ is a Gaussian operator, which implies that the thermal expectation of an odd number of factors of A vanishes and that of an even number of factors is the sum of products of pair expectations. That is,

$$\begin{aligned} \langle 1\rangle &= \langle 123\rangle = 0,\\ \langle 1234\rangle &= \langle 12\rangle\langle 34\rangle + \langle 13\rangle\langle 24\rangle + \langle 14\rangle\langle 23\rangle. \end{aligned} \tag{4.2}$$

These are the results we shall be using.

To obtain an explicit expression for the pair expectation, we use the expression (3.11) for the field operator $A(t)$, and the well known results

$$\begin{aligned} \langle p_j p_k\rangle &= m_j^2\omega_j^2\langle q_j q_k\rangle = \frac{m_j\hbar\omega_j}{2}\coth(\hbar\omega_j/2kT)\,\delta_{jk}\\ \langle q_j p_k\rangle &= -\langle p_j q_k\rangle = \frac{i\hbar}{2}\delta_{jk}. \end{aligned} \tag{4.3}$$

The result is

$$\langle 12\rangle = \sum_j \frac{m_j\hbar\omega_j}{2}\left\{\coth\left(\frac{\hbar\omega_j}{2kT}\right)\cos[\omega_j(t_1-t_2)] - i\sin[\omega_j(t_1-t_2)]\right\}. \tag{4.4}$$

If we use the expression (2.7) for the spectral distribution $Re\{\tilde{\mu}(\omega+i0^+)\}$, we can write this in the form

$$\begin{aligned} \langle 12\rangle = \frac{\hbar}{\pi}\int_0^\infty d\omega\, &\frac{Re\{\tilde{\mu}(\omega+i0^+)\}}{\omega}\\ &\times\left\{\coth\left(\frac{\hbar\omega}{2kT}\right)\cos[\omega(t_1-t_2)] - i\sin[\omega(t_1-t_2)]\right\}. \end{aligned} \tag{4.5}$$

We now use these results to obtain explicit expressions for $B^{(1)}$ and $B^{(2)}$. Consider first the expression (3.20) for $B^{(1)}$, which can be written

$$B^{(1)}(\tilde{\sigma}, t_1) = \frac{1}{ih}[Tr_R\{H_I(t_1)\,\rho_R\}, \tilde{\sigma}], \tag{4.6}$$

where we have used the fact that $\tilde{\sigma}$ and ρ_R commute, since they are operators in different Hilbert spaces. Using the expression (3.10) for $H_I(t)$ and the above notation for the thermal expectation of products of the A's, we see that

$$Tr_R\{\varepsilon H_I(t_1)\,\rho_R\} = p(t_1)\langle 1\rangle + \langle 11\rangle = \langle 11\rangle, \tag{4.7}$$

where we have used the Gaussian property (4.2). But $\langle 11\rangle$ is a time-independent c-number, which commutes with $\tilde{\sigma}$. Therefore we see that

$$B^{(1)}(\tilde{\sigma}, t_1) = 0. \tag{4.8}$$

Note that the term $\langle 11\rangle$ above comes from the A^2 term in the interaction Hamiltonian (3.7), which does not contribute to this order.

Next, consider the expression (3.23) for $B^{(2)}$. From the above argument applied to $B^{(1)}$, we see that the second term within the integral vanishes and we are left with the expression

$$B^{(2)}(\tilde{\sigma}, t_1) = \frac{1}{(ih)^2}\int_{-\infty}^{t_1} dt_2\, Tr_R\{[H_I(t_1), [H_I(t_2), \tilde{\sigma}\rho_R]]\}. \tag{4.9}$$

Next, again using the expression (3.10) for $H_I(t)$, and expanding out the double commutator, we find

$$Tr_R\{[\varepsilon H_I(t_1), [\varepsilon H_I(t_2), \tilde{\sigma}\rho_R]]\} = m^{-2}(\langle 12\rangle[p(t_1), p(t_2)\tilde{\sigma}] + \langle 12\rangle^*[\tilde{\sigma}p(t_2), p(t_1)]). \tag{4.10}$$

In obtaining this result it is necessary to keep in mind that the order of operators must be preserved. Within the trace one can cyclically permute the field operators, but not those of the oscillator. Note further that $\langle 12\rangle$ is a function only of $t_1 - t_2$ and that

$$\langle 21\rangle = \langle 12\rangle^*. \tag{4.11}$$

Finally, we remark that, as was the case in first order, the A^2 term in the interaction Hamiltonian (3.7) does not contribute in second order.

Collecting these results, we can write the Eq. (3.17) for the reduced density matrix in the interaction representation through second order in the form

$$\frac{\partial\tilde{\sigma}}{\partial t_1} = -\frac{1}{h^2m^2}\int_{-\infty}^{t_1} dt_2(\langle 12\rangle[p(t_1), p(t_2)\,\tilde{\sigma}] + \langle 12\rangle^*[\tilde{\sigma}p(t_2), p(t_1)]). \tag{4.12}$$

In this equation, we can use the explicit form (3.12) for the time-dependent operators to write

$$p(t_2) = p(t_1)\cos[\omega_0(t_1-t_2)] + m\omega_0 x(t_1)\sin[\omega_0(t_1-t_2)]. \tag{4.13}$$

The equation can therefore be written in the form

$$\begin{aligned}\frac{\partial\tilde{\sigma}}{\partial t} = &-\frac{1}{2mh\omega_0}([p(t), \{Cp(t)+m\omega_0 Sx(t)\}\ \tilde{\sigma}]\\ &+[\tilde{\sigma}\{C^*p(t)+m\omega_0 S^*x(t)\}, p(t)]),\end{aligned} \tag{4.14}$$

where C and S are complex constants (c-numbers) defined by

$$\begin{aligned}C &= \frac{2\omega_0}{hm}\int_{-\infty}^{t_1} dt_2 \langle 12\rangle \cos[\omega_0(t_1-t_2)],\\ S &= \frac{2\omega_0}{hm}\int_{-\infty}^{t_1} dt_2 \langle 12\rangle \sin[\omega_0(t_1-t_2)].\end{aligned} \tag{4.15}$$

These integrals can be performed with the aid of the well known standard integrals,

$$\int_0^\infty dt\cos(\omega t) = \pi\delta(\omega), \qquad \int_0^\infty dt\sin(\omega t) = P\frac{1}{\omega}, \tag{4.16}$$

where P stands for principal value. The results are

$$Re\{C\} = -\coth\left(\frac{\hbar\omega_0}{2kT}\right) Im\{S\}, \qquad Im\{S\} = -\frac{1}{m} Re\{\tilde{\mu}(\omega_0 + i0^+)\}, \tag{4.17}$$

while $Im\{C\}$ and $Re\{S\}$ are given by the integrals

$$Im\{C\} = \frac{2\omega_0}{\pi m} P\int_0^\infty d\omega \frac{Re\{\tilde{\mu}(\omega+i0^+)\}}{\omega_0^2-\omega^2}, \tag{4.18}$$

$$Re\{S\} = \frac{2\omega_0^2}{\pi m} P\int_0^\infty d\omega \frac{Re\{\tilde{\mu}(\omega+i0^+)\}}{\omega(\omega_0^2-\omega^2)}\coth\left(\frac{\hbar\omega}{2kT}\right). \tag{4.19}$$

Note that if $Re\{\tilde{\mu}(\omega+i0^+)\}$ does not vanish at $\omega=0$, $Re\{S\}$ will be infra-red divergent. Thus, in what is called the Ohmic model, where $Re\{\tilde{\mu}(\omega+i0^+)\} = \zeta$, the frequency-independent friction constant, this divergence occurs.

V. THE BLACKBODY RESERVOIR

Consider now a charged oscillator coupled to the radiation field at temperature T, the blackbody reservoir. For this case the spectral distribution takes the form [10]

$$Re\{\tilde{\mu}(\omega_0 + i0^+)\} = \frac{2e^2\omega^2\Omega^2}{3c^3(\omega^2+\Omega^2)}, \tag{5.1}$$

where Ω is a large cutoff frequency. Note that this spectral distribution vanishes at $\omega = 0$ (corresponding to the fact that a uniformly moving charge does not radiate) so there is no infra-red divergence of the integral (4.19) for $Re\{S\}$. On the other hand, the integral (4.18) for $Im\{C\}$ has an ultra-violet divergence, growing linearly in the cutoff frequency as that frequency goes to infinity. This corresponds to the phenomenon of mass renormalization. To see this, we first separate the divergence by writing the integral in the form

$$Im\{C\} = \frac{4e^2\omega_0}{3\pi mc^3} P\int_0^\infty d\omega \left(-\frac{\Omega^2}{\omega^2+\Omega^2} + \frac{\omega_0^2\Omega^2}{(\omega_0^2-\omega^2)(\omega^2+\Omega^2)}\right). \tag{5.2}$$

The integral of the first term is

$$\int_0^\infty d\omega \frac{\Omega^2}{\omega^2+\Omega^2} = \frac{\pi}{2}\Omega, \tag{5.3}$$

while we can take the limit $\Omega \to \infty$ in the integral of the second term. This remaining integral vanishes,

$$P\int_0^\infty d\omega \frac{1}{\omega_0^2-\omega^2} = 0. \tag{5.4}$$

The result is that we can write

$$Im\{C\} = -\omega_0 \frac{\delta m}{m}, \tag{5.5}$$

where δm is given by

$$\delta m = \frac{2e^2\Omega}{3c^3}. \tag{5.6}$$

The effect of this divergent term in the Eq. (4.14) for the reduced density matrix in the interaction representation can be seen by keeping only this term in C and dropping S. The equation then takes the form

$$\frac{\partial\tilde{\sigma}}{\partial t} = \frac{1}{ih}\left[-\frac{\delta m}{2m^2} p^2(t), \tilde{\sigma}\right] + \cdots, \tag{5.7}$$

where the neglected terms are finite, or at least less divergent. But, through second order, this is exactly what would result if instead of the Hamiltonian (3.6) for the oscillator one had

$$H_O = \frac{p^2}{2(m+\delta m)} + \frac{1}{2} Kx^2. \tag{5.8}$$

That is, the divergent term in (5.2) can be removed by a mass renormalization where $m+\delta m$ is interpreted as the renormalized (observed) mass [11]. The result is that, through second order, we can simply drop the divergent first term in the expression (5.5) for $Im\{C\}$ and interpret the mass m as it appears in our equation as the renormalized mass. The expression (5.6) for the mass shift is exactly that obtained by Bethe in his celebrated nonrelativistic calculation of the Lamb shift [12]. Note in particular that the mass renormalization (5.6) is independent of the external potential, which is why we have made the separation (5.2) of divergent and finite terms in $Im\{C\}$.

We conclude, therefore, that in the second order Eq. (4.14), we can drop $Im\{C\}$ and in the remaining terms take the limit of large cutoff. In this limit,

$$Im\{S\} = -\gamma, \tag{5.9}$$

where γ is the decay rate, given by

$$\gamma = \omega_0^2 \tau_e, \qquad \tau_e = 2e^2/3mc^3 \cong 6 \times 10^{-24} \text{ sec.} \tag{5.10}$$

(It is also worth noting that the ratio γ/ω_0, which is a measure of the weak coupling constant, is of order 10^{-7} for optical frequencies.) Also, from (4.17) we see that

$$Re\{C\} = \gamma \coth(\hbar\omega_0/2kT). \tag{5.11}$$

As we have remarked above, there is no infra-red divergence of the integral (4.19) for $Re\{S\}$, which can be written in the form

$$Re\{S\} = \gamma I, \tag{5.12}$$

where I is the integral

$$I = \frac{2}{\pi} P \int_0^\infty d\omega \, \frac{\Omega^2 \omega \coth(\hbar\omega/2kT)}{(\omega_0^2 - \omega^2)(\omega^2 + \Omega^2)}. \tag{5.13}$$

For large Ω this integral grows like $\log \Omega$, but we will not discuss it further since, as we shall see, it will not contribute to the master equation.

If we put these expressions for C and S in the Eq. (4.14) we obtain the explicit form of the equation (through second order) for the reduced density matrix in the interaction representation. This density matrix is an operator in the Hilbert space of the oscillator and only the oscillator operators x and p appear in the equation. The radiation field appears only through the parameter γ and the integral I.

However, this equation is not yet the master equation. To obtain that equation another step is required, and we show that in the next section.

VI. THE MASTER EQUATION

The right hand side of the second order Eq. (4.14) contains explicitly time-dependent terms, oscillating at frequency $2\omega_0$. These terms will give rise in the solution $\tilde{\sigma}(t)$ to small amplitude oscillations about a slowly varying mean motion. The equation for this mean density matrix, which we denote by $\bar{\sigma}$, is obtained by averaging the right hand side over time (see Appendix B). In performing this time-average it will be convenient to introduce raising and lowering operators in the standard form

$$a = \frac{m\omega_0 x + ip}{\sqrt{2mh\omega_0}}, \qquad a^\dagger = \frac{m\omega_0 x - ip}{\sqrt{2mh\omega_0}}. \tag{6.1}$$

Then, using the expressions (3.12) for $x(t)$ and $p(t)$, we see that

$$\begin{aligned} x(t) &= \sqrt{\frac{h}{2m\omega_0}}\,(ae^{-i\omega_0 t} + a^\dagger e^{i\omega_0 t}), \\ p(t) &= -i\sqrt{\frac{mh\omega_0}{2}}\,(ae^{-i\omega_0 t} - a^\dagger e^{i\omega_0 t}). \end{aligned} \tag{6.2}$$

With these, the second order Eq. (4.14) becomes

$$\begin{aligned} \frac{\partial\tilde{\sigma}}{\partial t} = &-\frac{1}{4}\{(C+iS)[a^\dagger, a\tilde{\sigma}] + (C-iS)[a, a^\dagger\tilde{\sigma}] \\ &+ (C^*-iS^*)[\tilde{\sigma}a^\dagger, a] + (C^*+iS^*)[\tilde{\sigma}a, a^\dagger]\} \\ &+\frac{1}{4}\{(C+iS)[a, a\tilde{\sigma}] + (C^*+iS^*)[\tilde{\sigma}a, a]\}\, e^{-2i\omega_0 t} \\ &+\frac{1}{4}\{(C^*-iS^*)[\tilde{\sigma}a^\dagger, a^\dagger] + (C-iS)[a^\dagger, a^\dagger\tilde{\sigma}]\}\, e^{2i\omega_0 t}. \end{aligned} \tag{6.3}$$

In this form the effect of the time average is clear: we simply discard the terms oscillating as $e^{\pm 2i\omega_0 t}$, with the result

$$\begin{aligned} \frac{\partial\bar{\sigma}}{\partial t} = &\frac{1}{2i}\,Im\{C\}\left[\left(a^\dagger a + \frac{1}{2}\right), \bar{\sigma}\right] \\ &-\frac{1}{4}\,(Re\{C\} - Im\{S\})([a^\dagger, a\bar{\sigma}] + [\bar{\sigma}a^\dagger, a]) \\ &-\frac{1}{4}\,(Re\{C\} + Im\{S\})([a, a^\dagger\bar{\sigma}] + [\bar{\sigma}a, a^\dagger]), \end{aligned} \tag{6.4}$$

where we have used the canonical commutation relation $[a, a^\dagger] = 1$. Note that, as we have remarked earlier, $Re\{S\}$ does not contribute to the master equation to this order.

We now return to the Schrödinger representation, writing

$$\bar{\rho} = e^{-iH_O t/\hbar} \bar{\sigma} e^{iH_O t/\hbar}, \tag{6.5}$$

where H_O is the free oscillator Hamiltonian (3.6) with, in the electromagnetic case, m now interpreted as the renormalized mass. Using (6.1) this can be written

$$H_O = \hbar\omega_0 (a^\dagger a + \tfrac{1}{2}). \tag{6.6}$$

With this and using the expressions (4.17) for $Im\{S\}$ and $Re\{C\}$, the master equation in the Schrödinger representation can be written in the form

$$\begin{aligned} \frac{\partial \bar{\rho}}{\partial t} = {} & \frac{1}{i\hbar} [H_O + \Delta H_O, \bar{\rho}] \\ & -\frac{1}{2} \gamma(\omega_0)\, N(\omega_0) ([a, a^\dagger \bar{\rho}] + [\bar{\rho} a, a^\dagger]) \\ & -\frac{1}{2} \gamma(\omega_0) \{N(\omega_0) + 1\} ([a^\dagger, a\bar{\rho}] + [\bar{\rho} a^\dagger, a]). \end{aligned} \tag{6.7}$$

Here, to conform with the customary notation, we have introduced the frequency-dependent decay rate,

$$\gamma(\omega) = \frac{1}{m} Re\{\tilde{\mu}(\omega + i0^+)\}, \tag{6.8}$$

and the mean quantum number of a free oscillator with natural frequency ω in equilibrium at temperature T,

$$N(\omega) = \frac{1}{e^{\hbar\omega/kT} - 1}. \tag{6.9}$$

The operator ΔH_O corresponds to a shift in the oscillator frequency, $\Delta\omega_0 = Im\{C\}/2$. That is, from (4.18),

$$\Delta H_O = \hbar \Delta\omega_0 \left(a^\dagger a + \frac{1}{2} \right), \qquad \Delta\omega_0 = \frac{\omega_0}{\pi m} P \int_0^\infty d\omega \, \frac{Re\{\tilde{\mu}(\omega + i0^+)\}}{\omega_0^2 - \omega^2}. \tag{6.10}$$

We should perhaps emphasize that the second-order master Eq. (6.7), which is our principal result, has been obtained without restriction as to the form of the spectral distribution. It is therefore the general form for any linear passive dissipation. It also conforms to the general form for a master equation obtained by Lindblad [8]. This is also the form of the master equation generally obtained using the rotating

wave approximation (RWA) [1], excepting that we now have explicit expressions for the decay rate $\gamma(\omega_0)$ and the frequency shift $\Delta\omega_0$ in terms of the spectral distribution $Re\{\tilde{\mu}(\omega + i0^+)\}$.

The operator form (6.7) for the master equation is that appearing most often in the quantum optics literature. Equivalent forms which appear are in terms of the coordinate space matrix element or the Wigner function. The coordinate space matrix element is defined in terms of the operator $\bar{\rho}$ by

$$\rho(x_1, x_2; t) \equiv \langle x_1 | \bar{\rho} | x_2 \rangle = \int_{-\infty}^{\infty} dx\, \delta(x - x_1)\, \bar{\rho}\delta(x - x_2). \tag{6.11}$$

In terms of this, the Wigner function is given by

$$W(q, p; t) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} du e^{iup/\hbar} \rho\left(q - \frac{u}{2}, q + \frac{u}{2}; t\right). \tag{6.12}$$

In terms of $\rho(x_1, x_2; t)$, the master Eq. (6.7) takes the form

$$\begin{aligned}\frac{\partial \rho}{\partial t} = {} & i\left(1 + \frac{\Delta\omega_0}{\omega_0}\right)\frac{\hbar}{2m}\left(\frac{\partial^2 \rho}{\partial x_1^2} - \frac{\partial^2 \rho}{\partial x_2^2}\right) - i\left(1 + \frac{\Delta\omega_0}{\omega_0}\right)\frac{m\omega_0^2}{2\hbar}(x_1^2 - x_2^2)\,\rho \\ & + \frac{1}{2}\gamma(\omega_0)\left\{\rho + x_1\frac{\partial \rho}{\partial x_2} + x_2\frac{\partial \rho}{\partial x_1}\right\} \\ & - \gamma(\omega_0)\left\{N(\omega_0) + \frac{1}{2}\right\}\left\{\frac{m\omega_0}{2\hbar}(x_1 - x_2)^2\,\rho - \frac{\hbar}{2m\omega_0}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 \rho\right\},\end{aligned} \tag{6.13}$$

while in terms of the Wigner function this becomes

$$\begin{aligned}\frac{\partial W}{\partial t} = {} & -\left(1 + \frac{\Delta\omega_0}{\omega_0}\right)\frac{p}{m}\frac{\partial W}{\partial q} + m\left(1 + \frac{\Delta\omega_0}{\omega_0}\right)\omega_0^2 q\frac{\partial W}{\partial p} \\ & + \frac{1}{2}\gamma(\omega_0)\left\{\frac{\partial qW}{\partial q} + \frac{\partial pW}{\partial p}\right\} \\ & + \gamma(\omega_0)\left\{N(\omega_0) + \frac{1}{2}\right\}\left\{\frac{\hbar}{2m\omega_0}\frac{\partial^2 W}{\partial q^2} + \frac{mh\omega_0}{2}\frac{\partial^2 W}{\partial p^2}\right\}.\end{aligned} \tag{6.14}$$

For the electromagnetic case, after mass renormalization, there is no level shift, $\Delta\omega_0 = 0$, and we have the explicit form (5.10) for the decay rate, $\gamma(\omega_0) = \omega_0^2\tau_e$. Note that the coupling to the radiation field only affects the oscillator Hamiltonian (3.6) through the mass renormalization. This has the effect of reinterpreting the mass parameter in the kinetic energy but leaves the parameter K, the spring constant in the potential energy unchanged. Thus, when expressed in terms of the renormalized or physical mass, the oscillator energy level spacings are unchanged. This is consistent with the nonrelativistic calculation of the Lamb shift [12], which vanishes for the oscillator.

In the case of Ohmic dissipation the spectral distribution is a constant, ζ, the frequency-independent friction constant, i.e., $Re\{\tilde{\mu}(\omega + i0^+)\} = \zeta$. In that case, $\gamma(\omega_0) = \zeta/m$ and, as we see from (5.4), there is again no frequency shift, $\Delta\omega_0 = 0$. This is in disagreement with the RWA result [1].

VII. THE THIRD ORDER TERM

We now consider the third order contribution to the master equation, given by (A.4). Since the contributions of the terms corresponding to arrangements containing a block consisting of a single integer vanish, this can be written

$$B^{(3)}(\tilde{\sigma} t_1) = \frac{1}{(i\hbar)^3} \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \{123\}, \tag{7.1}$$

where $\{123\}$ stands for the expression given in (A.2). From the form (3.10) of $H_I(t)$ and the properties (4.1) and (4.2) of the partial trace, we see that the A^2 term in H_I must occur exactly once in each of the three factors in this expression. Using the fact that one can cyclically permute the factors of A within the partial trace, this quantity can be written

$$\begin{aligned}\{123\} = \frac{1}{2m^3} \{ &(\langle 1223\rangle - \langle 2213\rangle)[p(1), p(3)\,\tilde{\sigma}] \\ &-(\langle 3122\rangle - \langle 3221\rangle)[p(1), \tilde{\sigma} p(3)] \\ &+(\langle 1233\rangle - \langle 3312\rangle)[p(1), p(2)\tilde{\sigma}] \\ &-(\langle 2133\rangle - \langle 3321\rangle)[p(1), \tilde{\sigma} p(2)]\},\end{aligned} \tag{7.2}$$

where the quantities $\langle 1234\rangle$ are the trace of products of the A's, defined in (4.1). Expanding the expectation of products of four factors according to (4.2), one can write

$$\begin{aligned}\{123\} = \frac{1}{m^3} \{ &(\langle 12\rangle - \langle 21\rangle)(\langle 23 > [p(1), p(3)\,\tilde{\sigma}] - \langle 32 > [p(1), \tilde{\sigma} p(3)]) \\ &+(\langle 13\rangle\langle 23\rangle - \langle 31\rangle\langle 32\rangle)[p(1), [p(2), \tilde{\sigma}]]\}.\end{aligned} \tag{7.3}$$

Putting this in (7.1) and using the expression (4.13) for $p(2)$ and $p(3)$, we can write

$$\begin{aligned}B^{(3)}(\tilde{\sigma}, t) = -\frac{1}{2mh\omega_0} \{ &[p(t), \{(I_1 + I_3)\, p(t) + (I_2 + I_4)\, m\omega_0 x(t)\}\, \tilde{\sigma}] \\ &+[\tilde{\sigma}\{(I_1^* + I_3)\, p(t) + (I_2^* + I_4)\, m\omega_0 x(t)\}, p(t)]\},\end{aligned} \tag{7.4}$$

where the I_j are the integrals

$$
\begin{aligned}
I_1 &= \frac{4\omega_0}{(hm)^2}\int_0^\infty dt_{12}\int_0^\infty dt_{23}\, Im\{\langle 12\rangle\}\langle 23\rangle\, \cos(\omega_0 t_{13}),\\
I_2 &= \frac{4\omega_0}{(hm)^2}\int_0^\infty dt_{12}\int_0^\infty dt_{23}\, Im\{\langle 12\rangle\}\langle 23\rangle\, \sin(\omega_0 t_{13}),\\
I_3 &= \frac{4\omega_0}{(hm)^2}\int_0^\infty dt_{12}\int_0^\infty dt_{23}\, Im\{\langle 13\rangle\langle 23\rangle\}\, \cos(\omega_0 t_{12}),\\
I_4 &= \frac{4\omega_0}{(hm)^2}\int_0^\infty dt_{12}\int_0^\infty dt_{23}\, Im\{\langle 13\rangle\langle 23\rangle\}\, \sin(\omega_0 t_{12}).
\end{aligned}
\tag{7.5}
$$

Note that I_3 and I_4 are real.

Before proceeding to the evaluation of these integrals, we note that this form for third order term is the same as that for the second order given by the right hand side of the Eq. (4.14) for $\tilde{\sigma}$. Therefore, through third order that equation still applies, providing that we make the replacement

$$
C \to C + I_1 + I_3, \qquad S \to S + I_2 + I_4. \tag{7.6}
$$

Thus, *the contribution of the third order term consists of higher order corrections to the parameters of the second order equation.* To see the form of these corrections, we turn now to the evaluation of the integrals (7.5).

The first two of the integrals (7.5) are simple: we use the addition formula for the trigonometric functions and the definitions (4.15) of the integrals C and S to obtain

$$
\begin{aligned}
I_1 &= \frac{1}{\omega_0}\,(Im\{C\}\; C - Im\{S\}\; S),\\
I_2 &= \frac{1}{\omega_0}\,(Im\{S\}\; C + Im\{C\}\; S\}.
\end{aligned}
\tag{7.7}
$$

The remaining integrals are more complicated. Consider first I_3. Note that we can write (4.5) in the form

$$
\langle 12\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, P(\omega)\, e^{-i\omega t}, \tag{7.8}
$$

where

$$
P(\omega) = h\,\frac{Re\{\tilde{\mu}(\omega + i0^+)\}}{\omega}\left[\coth\left(\frac{h\omega}{2kT}\right) + 1\right]. \tag{7.9}
$$

With this, the integral I_3 can be written

$$I_3 = -\frac{\omega_0}{\pi^2 m^2 h^2} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \, P(\omega_1) \, P(\omega_2)$$

$$\times \int_0^{\infty} dt_{12} \int_0^{\infty} dt_{23} \sin(\omega_1 t_{13} + \omega_2 t_{23}) \cos(\omega_0 t_{12}). \tag{7.10}$$

Since $t_{13} = t_{12} + t_{23}$, using the addition formula for the sine function reduces the last two integrations to a product of a pair of simple trigonometric integrals that can be performed using the standard integrals (4.16). The result is

$$\int_0^{\infty} dt_{12} \int_0^{\infty} dt_{23} \sin(\omega_1 t_{13} + \omega_2 t_{23}) \cos(\omega_0 t_{12})$$

$$= \pi \delta(\omega_1 + \omega_2) \frac{\omega_1}{\omega_1^2 - \omega_0^2} + \frac{\pi}{2} \frac{\delta(\omega_1 - \omega_0) + \delta(\omega_1 + \omega_0)}{\omega_1 + \omega_2}. \tag{7.11}$$

When this result is put in (7.10), the first term gives an integral of an odd function over an even interval, which vanishes. The remaining term gives

$$I_3 = -\frac{\omega_0}{2\pi m^2 h^2} \left\{ P(\omega_0) \int_{-\infty}^{\infty} d\omega \, \frac{P(\omega)}{\omega + \omega_0} + P(-\omega_0) \int_{-\infty}^{\infty} d\omega \, \frac{P(\omega)}{\omega - \omega_0} \right\}. \tag{7.12}$$

Comparing this with the results (4.17) to (4.19), we see that this can be written

$$I_3 = \frac{1}{\omega_0} \left(Im\{C\} \; Re\{C\} + Im\{S\} \; Re\{S\} \right). \tag{7.13}$$

In a similar way, we can show that the integral I_4 can be written

$$I_4 = \frac{\omega_0^2}{2\pi^2 m^2 h^2} \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2 \, \frac{P(\omega_1) \, P(\omega_2)}{\omega_1 + \omega_2} \left\{ \frac{1}{\omega_1^2 - \omega_0^2} + \frac{1}{\omega_2^2 - \omega_0^2} \right\}. \tag{7.14}$$

We leave this integral in this form since it will not contribute to the master equation.

For the electromagnetic case of the blackbody reservoir, we have the results, obtained in Sect. 5,

$$C(\omega_0) = [\, 2N(\omega_0) + 1 \,] \, \gamma - i\omega_0 \frac{\delta m}{m}, \qquad S(\omega_0) = \gamma I - i\gamma. \tag{7.15}$$

Using these results in the expressions (7.7) and (7.13) for the integrals and making in (4.14) the replacement (7.6), we see that through third order, the Eq. (3.17) for $\tilde{\sigma}$ can be written

$$\frac{\partial\tilde{\sigma}}{\partial t}=\frac{1}{ih}\left[\left(-\frac{\delta m}{2m^2}+\frac{\delta m^2}{2m^3}-\frac{\gamma^2}{2m\omega_0^2}\right)p^2(t),\,\tilde{\sigma}\right]$$

$$-\frac{N(\omega_0)}{mh\omega_0}\gamma\left(1-2\frac{\delta m}{m}\right)[\,p(t),[\,p(t),\tilde{\sigma}\,]\,]-\frac{1}{2mh\omega_0}\gamma\left(1\text{-}2\frac{\delta m}{m}\right)$$

$$\times([\,p(t),\{\,p(t)-im\omega_0 x(t)\}\,\tilde{\sigma}\,]+[\,\tilde{\sigma}\{\,p(t)+im\omega_0 x(t)\},p(t)\,]\,)$$

$$-\frac{1}{2h}\left\{\gamma\left(1-\frac{\delta m}{m}\right)I-[\,2N(\omega_0)+1\,]\frac{\gamma^2}{\omega_0}+I_4\right\}[\,p(t),[\,x(t),\tilde{\sigma}\,]\,]. \qquad (7.16)$$

Here in the first term we recognize the expansion through second order in the mass shift of the Hamiltonian (5.8), with a small additional term. The next two terms are the same as the corresponding terms in (4.14) excepting that $\gamma\to\gamma(1-2\delta m/m)$. But, using the definition (5.10),

$$\gamma\left(1-2\frac{\delta m}{m}\right)\cong\frac{2e^2}{3c^3}K(m+\delta m)^{-2}, \qquad (7.17)$$

where K is the force constant. Thus, this can also be interpreted as due to the mass renormalization. Finally, the last term, with its complicated prefactor, does not contribute to the master equation. Thus, including the third order term, we still obtain the master Eq. (6.7) but with the parameters interpreted in terms of the renormalized mass through higher order.

As a final remark, we should recall that here we have evaluated the terms in the limit of very large cutoff, $\Omega\to\infty$. But, in the context of the quantum Langevin equation it has been learned that for consistency with the second law of thermodynamics the cutoff Ω must be chosen to have the large, but finite value $\Omega=\tau_e^{-1}$ [7]. It is not quite clear how to do this within the approximations we have made in deriving the master equation, but it will give corrections of order γ^2, and so such terms in the master equation, i.e., the term of order γ^2 in the first term on the right of (7.16), should perhaps not be taken seriously.

VIII. CONCLUDING REMARKS

First of all, we recapitulate the steps leading to the master equation. In Sect. 2 we introduced the system of an oscillator coupled to a reservoir giving rise to linear passive dissipation. For such a system, the dissipation is completely characterized by the spectral distribution, $Re\{\tilde{\mu}(\omega+i0^+\}$. The momentum coupling Hamiltonian (2.1) can realize the most general case, the relation being given by (2.7). The next step is described in Sect. 3, where we first introduce the density matrix, ρ, and its partial trace, $\tilde{\rho}$, defined by (3.4). We then assume the coupling is weak and in (3.8) introduce the interaction representation, where we denote the density matrix by σ

and its partial trace by $\tilde{\sigma}$. For the case of weak coupling, the solution of the von Neumann Eq. (3.9) for $\sigma(t)$ is expressed in terms of the expansion (3.15), where the successive terms represent small amplitude fluctuations of σ about its mean, $\tilde{\sigma}\rho_R$. In turn, $\tilde{\sigma}$ is the solution of Eq. (3.17), which is of the form of an expansion in successive orders of the coupling. Consistency of this double expansion, (3.15) and (3.17), with the von Neumann equation leads to the sequence of Eqs. (3.19) for the determination of the coefficients. We then showed how to solve these equations to obtain expressions for the coefficients and, in Appendix A, give a prescription for constructing the general term in the Eq. (3.17) for $\tilde{\sigma}$. The evaluation of these terms is discussed in Sect. 4, where the chief result is the second order Eq. (4.14) for $\tilde{\sigma}$. This equation involves two complex parameters, which are expressed in terms of the spectral distribution by (4.17) to (4.19). In Sect. 5 we discuss this equation for the special case of coupling to the radiation field, where the spectral distribution is given by (5.1). There we show that the ultra-violet divergence occurring in the equation for $\tilde{\sigma}$ can be removed by a mass renormalization, which is exactly the mass renormalization of nonrelativistic QED. The Eq. (4.14) for the partial trace is not the master equation. In Sect. 6 we again use an asymptotic method to express $\tilde{\sigma}$ in terms of its slowly varying mean, $\bar{\sigma}$, about which there are small amplitude oscillations. The resulting equation for $\bar{\sigma}$, when transformed back to the Schrödinger representation, is the master Eq. (6.7) for $\bar{\rho}$. Finally, in Sect. 7 we construct the third order term, which is found to leave the form of the master equation unchanged, but gives rise to higher order corrections to the coefficients. For the electromagnetic case, these corrections are consistent with mass renormalization.

We should emphasize that the Eq. (4.14) for the partial trace does not correspond to the master equation. In particular that equation is not of Lindblad form. One is led to the master equation only when the slowly varying mean is separated from the partial trace, as we did in Sect. 6 by imposing a time average on (4.14). Only then does the master equation take the general Lindblad form. The necessity for this final time-average in the derivation has been recognized since the very earliest discussions [13].

It is remarkable that in the quantum description one can obtain the master equation only by a detailed consideration of a microscopic model, in our case described by the Hamiltonian (2.1). This despite the fact that in the quantum Langevin equation we have a complete macroscopic description of the system without, incidentally, the weak coupling assumption. The situation is to be contrasted with that in the classical description, where the analog of the master equation is the Fokker–Planck equation. The classical Langevin equation and the Fokker–Planck equation are completely equivalent macroscopic descriptions of the system; one can derive the one from the other, at least for the case of a Gaussian Markov process [14]. This last is perhaps the key to understanding the difficulty, since in the Markov (Ohmic) case the random force in the classical Langevin equation is delta-correlated and that is exactly what allows the derivation of the Fokker–Planck equation. But the random force in the quantum Langevin equation is never delta-correlated [7].

We should remark upon some recent exact derivations of an evolution equation for the reduced density matrix for the oscillator [15–17]. In fact, the equation obtained by these authors corresponds to our Eq. (4.14) for the partial trace $\tilde{\sigma}$, transformed to the Schrödinger representation and written in terms of the coordinate space matrix elements. As we have noted, this is not a master equation, but if one makes the additional step described in Sect. 6 involving the time average one arrives at a master equation of exactly the form of our Eq. (6.13). On the other hand, the expressions for the parameters $\gamma(\omega_0)$ and $\Delta\omega_0$ are different. This is due to the fact that these authors use a linear coupling model, which would correspond in our interaction Hamiltonian (3.7) to replacing $pA/m \to -x\dot{A}$ and dropping the A^2 term. This linear coupling model is seriously flawed by the neglect of quadratic terms. The flaw can be repaired, but the repair is not unique [7]. Thus, we would agree that the methods used by these authors will lead to a correct master equation, in agreement with the one we have obtained, when the physically consistent Hamiltonian (2.1) is used, when the additional step is taken to remove the rapidly varying time-dependent terms, and when mass renormalization is incorporated when appropriate.

APPENDIX A: CONSTRUCTION OF THE GENERAL TERM $B^{(N)}$

We give here the general rule for constructing the higher order expressions in the Eq. (3.17) for the reduced density matrix. In order to do so, we introduce a shorthand notation. We write the first and second order terms (3.20) and (3.23) in the form

$$\begin{aligned} B^{(1)}(\tilde{\sigma}, t_1) &= \frac{1}{ih}\{1\}, \\ B^{(2)}(\tilde{\sigma}, t_1) &= \frac{1}{(ih)^2}\int_{-\infty}^{t_1} dt_2(\{12\} - \{1:2\}). \end{aligned} \tag{A1}$$

Here, the braces stand for an overall trace with respect to the reservoir variables. Within the braces, a factor $H_I(t_j)$ is represented by an integer j and a multiple commutator of these factors is understood. Finally, a colon stands for a partial trace of the factors to the right followed by multiplication by ρ_R. For example,

$$\begin{aligned} \{123\} &= Tr_R\{[H_I(t_1), [H_I(t_2), [H_I(t_3), \tilde{\sigma}\rho_R]]\} \\ \{12:34\} &= Tr_R\{[H_I(t_1), [H_I(t_2), Tr_R\{[H_I(t_3), [H_I(t_4), \tilde{\sigma}\rho_R]]\}\ \rho_R]]\}. \end{aligned} \tag{A2}$$

With this notation we can state the general rule:

$$B^{(n)}(\sigma, t_1) = (ih)^{-n}\int_{-\infty}^{t_1} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n \sum (-)^c\{1 \cdots\}, \tag{A3}$$

where the sum is over all arrangements of the integers 1, 2, ..., n into nonempty blocks separated by colons, with c the number of colons. The first block must begin with the integer 1 and the order within each block must be ascending, otherwise all arrangements are allowed. In third order this gives

$$B^{(3)}(\tilde{\sigma}, t_1) = \frac{1}{(ih)^3} \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 (\{123\} - \{12:3\} - \{13:2\} - \{1:23\} + \{1:2:3\} + \{1:3:2\}). \tag{A4}$$

Finally, in fourth order we get

$$\begin{aligned} B^{(4)}(\tilde{\sigma}, t_1) = \frac{1}{(ih)^4} \int_{-\infty}^{t_1} dt_2 \int_{-\infty}^{t_2} dt_3 \int_{-\infty}^{t_3} dt_4 (&\{1234\} - \{1:234\} \\ &- \{12:34\} - \{13:24\} - \{14:23\} - \{123:4\} - \{124:3\} - \{134:2\} \\ &+ \{12:3:4\} + \{12:4:3\} + \{13:2:4\} + \{13:4:2\} + \{14:2:3\} \\ &+ \{14:3:2\} + \{1:2:34\} + \{1:3:24\} + \{1:4:23\} + \{1:23:4\} \\ &+ \{1:24:3\} + \{1:34:2\} - \{1:2:3:4\} - \{1:2:4:3\} \\ &- \{1:3:2:4\} - \{1:3:4:2\} - \{1:4:2:3\} - \{1:4:3:2\}). \end{aligned} \tag{A5}$$

For the interaction Hamiltonian (3.7) appearing in the velocity-coupling Hamiltonian (2.1), we showed in (4.7) that the partial trace is a c-number and, therefore that $B^{(1)}$ vanishes identically. A further consequence is that in the expansion (A.3) of $B^{(n)}$ all the terms corresponding to an arrangement containing a block consisting of a single integer vanish.

APPENDIX B: THE METHOD OF AVERAGING

Here we briefly outline the Method of Averaging of Krylov and Bogoliubov [18–20]. This is an asymptotic scheme for obtaining the long time behavior of the solution of certain nonlinear differential equations. For simplicity, we shall restrict our discussion to the case of a single dependent variable, although it should be clear that this is not a serious restriction.

Consider a differential equation of the form

$$\frac{dy}{dt} = \varepsilon F(y, t), \tag{B1}$$

where ε is a small parameter. The solution of this equation will have a time dependence on two scales, one the scale of dy/dt, which is slow if ε is sufficiently small, the other the scale of the explicit time dependence of $F(y, t)$, which is independent of ε. It is clear that the solution, $y(t)$, will be slowly varying (since its rate

of change is on the slow scale) but will have small amplitude rapid oscillations. We accordingly seek a solution in the form

$$y = \bar{y} + \varepsilon G^{(1)}(\bar{y}, t) + \varepsilon^2 G^{(2)}(\bar{y}, t) + \cdots, \tag{B2}$$

where $\bar{y}$ is the solution of a differential equation of the form

$$\frac{d\bar{y}}{dt} = \varepsilon \Omega^{(1)}(\bar{y}) + \varepsilon^2 \Omega^{(2)}(\bar{y}) + \cdots. \tag{B3}$$

Here the relation (B.2) can be considered to be a time-dependent transformation from y to a new dependent variable $\bar{y}$, with the $G^{(n)}$ describing the rapid oscillations of y about $\bar{y}$, which is to be slowly varying. We emphasize that in the differential Eq. (B.3) the $\Omega^{(n)}$ are to have no explicit time dependence, since $\bar{y}$ is to vary only on the slow time scale.

Inserting the expansion (B.2) in the differential equation (B.1), using the chain rule and the Eq. (B.3) for $d\bar{y}/dt$, we equate the coefficients of equal powers of ε on either side of the equation. The result is the following sequence of equations for the successive determination of the $G^{(n)}$ and $\Omega^{(n)}$.

$$\begin{aligned} &\Omega^{(1)}(\bar{y}) + \frac{\partial G^{(1)}(\bar{y}, t)}{\partial t} = F(\bar{y}, t), \\ &\Omega^{(2)}(\bar{y}) + \frac{\partial G^{(2)}(\bar{y}, t)}{\partial t} = \frac{\partial F(\bar{y}, t)}{\partial \bar{y}} - \frac{\partial G^{(1)}(\bar{y}, t)}{\partial \bar{y}} \Omega^{(1)}(\bar{y}), \\ &\qquad \cdots, \text{ etc.} \end{aligned} \tag{B4}$$

These equations are all of the same form, with the right hand side of each expressed in terms of the solutions of the previous equations in the sequence. The solution of the first order equation is therefore typical.

The first order Eq. (B.4) contains two unknowns: $\Omega^{(1)}(\bar{y})$ and $G^{(1)}(\bar{y}, t)$. The idea is that the first of these is to be determined as a condition of existence of the solution for the second. Thus, since $G^{(1)}(\bar{y}, t)$ is to correspond to small amplitude oscillations of y about $\bar{y}$, it must not increase linearly with time. This will be the case only if we chose $\Omega^{(1)}(\bar{y})$ to cancel the non-oscillating part of the right hand side of the equation. That is, we must chose $\Omega^{(1)}(\bar{y})$ to be the time average of the right hand side,

$$\Omega^{(1)}(\bar{y}) = \lim_{T \to \infty} \frac{1}{T} \int_0^T dt\, F(\bar{y}, t). \tag{B5}$$

With this, we can integrate to find $G^{(1)}(\bar{y}, t)$.

In the same way, one can go on to solve the higher order equations. The point here is that this is a consistent successive approximation scheme that can be carried to higher orders.

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