THE

PHYSICAL REVIEW

A journal of experimental and theoretical physics established by E. L. Nichols in 1893

SECOND SERIES, VOL. 176, NO. 5 — 25 DECEMBER 1968

Effect of the Anomalous Magnetic Moment of the Electron on the Nonlinear Lagrangian of the Electromagnetic Field

R. F. O'CONNELL

Department of Physics and Astronomy, Louisiana State University, Baton Rouge, Louisiana

(Received 28 June 1968)

When the anomalous magnetic moment of the electron ($g\neq 2$) is considered, by adding a phenomenological Pauli term to the Dirac Hamiltonian, the exact solutions to the Dirac equation can still be found. Using these solutions, and the methods of Heisenberg and Euler and of Weisskopf, we derive, in addition to the usual nonlinear term, a new nonlinear term which is correct to relative order α^2.

I. INTRODUCTION

An exact expression for the corrections to the Lagrangian of the electromagnetic field has been obtained by Heisenberg and Euler[1] and by Weisskopf[2] for fields $\mathbf{F}(\mathbf{E},\mathbf{H})$ satisfying the conditions

$$\hbar/mc|\nabla\mathbf{F}|\ll|\mathbf{F}|,\quad (\hbar/mc^2)|\partial\mathbf{F}/\partial t|\ll|\mathbf{F}|,\qquad(1)$$

i.e., for slowly varying fields. In this analysis, use was made of the exact solution for a pure Dirac particle ($g=2$) in a constant, homogeneous, and arbitrarily strong magnetic field.[2,3] The derivation is also based on the assumption that the addition w' to the classical energy density of the electromagnetic field,

$$w_0=(1/8\pi)(E^2+H^2),\qquad(2)$$

due to the existence of the electron-positron vacuum, is exactly equal to the energy density of the vacuum electrons (i.e., the electrons which fill the negative energy sea postulated by Dirac) minus the potential energy of the electrons in the external electromagnetic field. This led to the conclusion that the addition L_1 to the classical Lagrangian

$$L_0=(1/8\pi)(E^2-H^2)\qquad(3)$$

is equal to the negative of the total energy density of the electron-positron vacuum, w_m, say, in the presence of an external field. Then, since L_1 is only a function of two independent invariants, E^2-H^2 and $(\mathbf{E}\cdot\mathbf{H})^2$, it is sufficient to obtain the value of w_m for particular field configurations. In particular, the magnetic field is taken to be constant and homogeneous along the z axis and the electrostatic field is chosen parallel to it. In this paper, we are interested only in the corrections to the Lagrangian of the magnetic field and thus we take $\mathbf{E}=0$. We will return to the more general case of $\mathbf{E}\neq 0$ in a later publication.

The possible values of the energy of a vacuum electron (neglecting its anomalous magnetic moment) in a constant magnetic field H directed along the z axis are[2–5]

$$\mathcal{E}_{n,s}^{(0)}(p)\equiv E_{n,s}^{(0)}(p)/mc^2 = -[1+p^2+(2n+1+s)(H/H_c)]^{1/2},\qquad(4)$$

where

$$p\equiv p_z/mc,\quad H_c=m^2c^3/e\hbar=4.414\times10^{13}\ \mathrm{G},$$

and where $n=0, 1, 2, \cdots$ is the principal quantum number, $s=\pm1$ is the spin variable, p_z is the momentum of the particle along the z axis, and the superscript on $\mathcal{E}$

[1] W. Heisenberg and H. Euler, Z. Physik 98, 714 (1936).

[2] V. Weisskopf, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. 14, 6 (1936).

[3] A. I. Akhiezer and V. B. Berestetskii [*Quantum Electrodynamics* (John Wiley & Sons, Inc., New York, 1965), Chap. VIII] give an up-to-date treatment of this derivation and we use their notation for the most part. However, as distinct from these authors who take $\alpha=e^2/4\pi=1/137$ (p. 125), we use Gaussian units ($\alpha=e^2/\hbar c=1/137$). In addition, some misprints occurring in this treatment should be noted. For example, the right-hand side of Eq. (54.39) on p. 786 should have the factor $1/(2\pi)^2$ instead of $1/2\pi^2$, the quantities α and β should be interchanged on the right-hand side of the equation following Eq. (54.46) on p. 790, and the e^2 appearing in Eqs. (5.49) and (5.49′) on p. 791 should be omitted.

[4] M. H. Johnson and B. A. Lippmann, Phys. Rev. 76, 828 (1949).

[5] M. H. Johnson and B. A. Lippmann, Phys. Rev. 77, 702 (1950).

refers to the fact that the anomalous magnetic moment has been ignored. By use of Eq. (4) it was found that[2]

$$L_1=-\frac{m^4}{8\pi^2}\int_0^\infty \frac{d\eta}{\eta^3}e^{-\eta}[\eta H^* \coth(\eta H^*)-1-\tfrac{1}{3}\eta^2 H^{*2}], \quad (5)$$

where

$$H^*=H/H_c. \quad (6)$$

It follows that (where $\lambda_c=\hbar/mc$)

$$L_1=\frac{1}{(4\pi)^2}\frac{2}{45}\frac{mc^2}{\lambda_c^3}H^{*4} \quad \text{for} \quad H^*\ll 1 \quad (7)$$

and

$$L_1=\frac{1}{24\pi^2}\frac{mc^2}{\lambda_c^3}H^{*2}\ln H^* \quad \text{for} \quad H^*\gg 1, \quad (8)$$

which represent the weak- and strong-field limits,[3] respectively.

It is our purpose in this communication to obtain the corrections L_2 to L_0+L_1 which result from the fact that the electron has an anomalous magnetic moment (AMM). In a recent brief communication[6] we presented expressions for these corrections; the detailed derivation of these results is presented below.

II. MOTION OF RELATIVISTIC ELECTRON WITH ANOMALOUS MAGNETIC MOMENT IN CONSTANT MAGNETIC FIELD

The anomalous magnetic moment of the electron, which arises from the fact that it can emit or absorb virtual photons, can be taken care of in a phenomenological manner by adding the so-called Pauli anomalous moment interaction term $(\gamma_\mu\gamma_\nu F_{\mu\nu})$ to the usual Dirac Hamiltonian. In the case of a pure magnetic field the Dirac equation may thus be written

$$i\partial\psi/\partial t=[\boldsymbol{\alpha}\cdot(\mathbf{p}+e\mathbf{A})+\gamma_4 m+\mu\gamma_4\boldsymbol{\Sigma}\cdot\mathbf{H}]\psi, \quad (9)$$

where μ is the anomalous magnetic moment. Different values for the energy eigenvalues derived from this equation are quoted in the literature.[7,8] In view of this discrepancy, we have rederived the result (see Appendix A) and we find, in agreement with Ref. 8 (as we have previously reported[6,9]), that the energy eigenvalues of the vacuum electrons are given by

$$\mathcal{E}_{n,s}(p)\equiv E_{n,s}(p)/mc^2=-\{p^2+[(1+(2n+s+1)H^*)^{1/2}+\tfrac{1}{2}saH^*]^2\}^{1/2}, \quad (10)$$

where $a=\mu/\mu_B$ and μ_B is the Bohr magneton. This expression for $\mathcal{E}_{n,s}(p)$ is useful in finding the effect of the anomalous magnetic moment of the electron on many processes, e.g., (a) spontaneous pair production in a strong magnetic field,[10] (b) the magnetic moment of a free-electron gas (from physical grounds, however, it is clear that the effect in this case is rather negligible), and (c) the nonlinear Lagrangian of the electromagnetic field. It is to the latter problem we now direct our attention.

III. EFFECT OF ANOMALOUS MAGNETIC MOMENT OF THE ELECTRON ON NONLINEAR LAGRANGIAN OF THE ELECTROMAGNETIC FIELD

Following Weisskopf,[2] we write the energy density of the vacuum electrons, w_m, say, as follows:

$$w_m=-\frac{H^*}{(2\pi)^2}\frac{mc^2}{\lambda_c^3}\sum_{n=0}^{\infty}\sum_{s=\pm1}\int_{-\infty}^{\infty}\mathcal{E}_{n,s}(p)dp \equiv w_m^{(0)}+\Delta w_m^{(0)}, \quad (11)$$

where $w_m^{(0)}$ refers to the energy density when the AMM is neglected. To facilitate the evaluation w_m, we expand w_m to relative order α^2 and define

$$\Delta w_m^{(0)}\equiv\alpha\Delta w_m^{(1)}+\alpha^2\Delta w_m^{(2)}. \quad (12)$$

It turns out, as we shall see shortly, that the contribution from $\Delta w_m^{(1)}$ to the energy density is divergent and proportional to H^2; this term is included in the unperturbed field energy[3] (renormalization of charge or field intensity). Consequently, the first nonlinear contribution from Eq. (12) is already proportional to α^2 which allows us to restrict ourselves to the lowest order in a, viz., the Schwinger result

$$a=\alpha/2\pi. \quad (13)$$

Thus, to the order required,

$$\mathcal{E}_{n,s}(p)=-[p^2+([1+(2n+s+1)H^*]^{1/2}+s(\alpha/4\pi)H^*)^2]^{1/2}. \quad (14)$$

Expanding this expression to order α^2, we get

$$\mathcal{E}_{n,s}(p)=\mathcal{E}_{n,s}^{(0)}(p)+(\alpha/4\pi)sH^*[\mathcal{E}_{n,s}^{(0)}(0)/\mathcal{E}_{n,s}^{(0)}(p)]-\tfrac{1}{8}(\alpha/2\pi)^2H^{*2}[\mathcal{E}_{n,s}^{(0)}(p)]^{-1}\times\{[\mathcal{E}_{n,s}^{(0)}(0)/\mathcal{E}_{n,s}^{(0)}(p)]^2-1\}. \quad (15)$$

The first term in this expression is the one used by the authors of Refs. 1 and 2, and we will not be concerned about it. To calculate $\Delta w_m^{(1)}$ we substitute the $\alpha/4\pi$ term from Eq. (15) into Eq. (11). Using the fact that

$$\mathcal{E}_{n,-1}^{(0)}(p)=\mathcal{E}_{(n+1),1}^{(0)}(p), \quad (16)$$

[6] R. F. O'Connell, Phys. Letters **27A**, 391 (1968).

[7] Ref. 5, footnote 2, p. 705 obtains the result

$$[\mathcal{E}_{n,s}(p)]^2=[\mathcal{E}_{n,s}^{(0)}(p)]^2+a^2H^{*2}\pm 2aH^*\mathcal{E}_{n,s}^{(0)}(p)\{1-[\mathcal{E}_{n,s}^{(0)}(p)]^{-2}(2n+s+1)H^*\}^{1/2},$$

which is clearly different than the correct expression given by Eq. (10) of the main text.

[8] I. M. Ternov, V. G. Bagrov, and V. Ch. Zhukovskii, Moscow Univ. Bull. **21**, 21 (1966).

[9] R. F. O'Connell, Phys. Rev. Letters **21**, 397 (1968).

[10] We have shown (Ref. 9) that spontaneous pair production may occur for values of H equal to $4\pi\alpha^{-1}H_c\simeq 4\times 10^{16}$ G.

it is easily shown that

$$\sum_{n=0}^{\infty}\sum_{s=\pm1} s\left[\frac{\mathcal{E}_{n,s}^{(0)}(0)}{\mathcal{E}_{n,s}^{(0)}(p)}\right]=-\frac{1}{(1+p^2)^{1/2}}. \tag{17}$$

Thus it follows that

$$\Delta w_m^{(1)}=\frac{e^2H^2}{16\pi^3}\int_{-\infty}^{\infty}\frac{dp}{(1+p^2)^{1/2}}. \tag{18}$$

This is clearly a divergent term. However, we note that it has exactly the same form (proportional to H^2 and some type of divergence) as a term which arises in the zeroth-order calculation[11] and thus may be treated in the same way. In other words, since it is proportional to H^2, it can be included in the unperturbed field energy w_0 simply by a renormalization of the field intensity associated with the renormalization of charge. Thus, similar again to the zeroth-order calculation,[3,11] it should be noted that $\Delta w_m^{(1)}$ does not contain a finite part with the result that the subtraction of this term (in the renormalization procedure) does not lead to a finite contribution to the energy density—the most elegant formulation of this point is to be found in the work of Schwinger,[12] to which we refer the reader for details.

Let us now consider the evaluation of $\Delta w_m^{(2)}$:

$$\begin{aligned}\Delta w_m^{(2)}&=\frac{H^{*3}}{8(2\pi)^4}\frac{mc^2}{\lambda_c^3}\sum_{n=0}^{\infty}\sum_{s=\pm1}\int_{-\infty}^{\infty}[\mathcal{E}_{n,s}^{(0)}(p)]^{-1}\\&\qquad\times\{[\mathcal{E}_{n,s}^{(0)}(0)/\mathcal{E}_{n,s}^{(0)}(p)]^2-1\}dp\\&\equiv\frac{H^{*3}}{8(2\pi)^4}\frac{mc^2}{\lambda_c^3}\int_{-\infty}^{\infty}A(p)dp.\end{aligned} \tag{19}$$

Making use of Eq. (16) again, we get

$$\begin{aligned}A(p)&=\sum_{n=0}^{\infty}\sum_{s=\pm1}[\mathcal{E}_{n,s}^{(0)}(p)]^{-1}\\&\qquad\times\{[\mathcal{E}_{n,s}^{(0)}(0)/\mathcal{E}_{n,s}^{(0)}(p)]^2-1\}\\&=\left(\sum_{n=0}^{\infty}+\sum_{n=1}^{\infty}\right)\mathcal{E}_n^{-1}(p)\left(\frac{\mathcal{E}_n^2(0)}{\mathcal{E}_n^2(p)}-1\right),\end{aligned} \tag{20}$$

where

$$\mathcal{E}_n(p)\equiv(1+p^2+2nH^*)^{1/2}. \tag{21}$$

Now

$$\mathcal{E}_n^{-1}(p)\{[\mathcal{E}_n^2(0)/\mathcal{E}_n^2(p)]-1\}=-\frac{p^2}{(1+p^2+x)^{3/2}}, \tag{22}$$

where

$$x\equiv2nH^*. \tag{23}$$

It is thus obvious that $A(p)$ can be put in the following more transparent form:

$$A(p)=F(0)+2\sum_{n=1}^{\infty}F(x), \tag{24}$$

to which the Euler summation formula (valid for any F) may be applied. Now the latter formula may be written as

$$F(0)+2\sum_{n=1}^{\infty}F(bn)=\frac{2}{b}\left\{\int_0^{\infty}F(z)dz+\sum_{t=1}^{\infty}\frac{B_tb^{2t}}{(2t)!}[F^{(2t-1)}(\infty)-F^{(2t-1)}(0)]\right\}, \tag{25}$$

where B_t are the Bernoulli numbers and $F^{(2t-1)}$ denotes the $(2t-1)$th derivative with respect to z, where $z=bn$. In our particular case, $x\equiv2nH/H_c$ is equal to bn and hence b is equal to $2H^*$. Making use of the fact that

$$[d^{(2t-1)}/dx^{(2t-1)}](p^2+1+x)^{-3/2}=-(2/\sqrt{\pi})\Gamma(2t+\tfrac{1}{2})(p^2+1+x)^{-1/2-2t},$$

it follows that

$$A(p)=-\frac{1}{H^*}\left\{\int_0^{\infty}\frac{p^2dx}{(1+p^2+x)^{3/2}}+\frac{2}{\sqrt{\pi}}\sum_{t=1}^{\infty}\frac{B_tb^{2t}}{(2t)!}\Gamma(2t+\tfrac{1}{2})\frac{p^2}{(p^2+1)^{2t+1/2}}\right\}. \tag{26}$$

Now the integral in Eq. (26) is independent of H and thus when $A(p)$ is substituted in Eq. (19) the result for $\Delta w^{(2)}$ contains this integral multiplied by a factor proportional to H^2 which, as before, is included in the unperturbed field energy and hence neglected in our subsequent discussion. Making use of the formula

$$\int_{-\infty}^{\infty}\frac{p^2dp}{(p^2+1)^q}=(\tfrac{1}{2}\sqrt{\pi})\frac{\Gamma(q-\frac{3}{2})}{\Gamma(q)}\quad\text{for}\quad q>1, \tag{27}$$

we see that

$$\int_{-\infty}^{\infty}A(p)dp=-\frac{1}{H^*}\sum_{t=1}^{\infty}\frac{B_tb^{2t}}{(2t)!}\Gamma(2t-1). \tag{28}$$

Now making use of the fact that

$$\Gamma(2t-1)=\int_0^{\infty}d\eta\, e^{-\eta}\eta^{2t-2}, \tag{29}$$

substituting Eq. (28) in Eq. (19), and replacing t by n, we get

$$\Delta w_m^{(2)}=-\frac{1}{8}\frac{H^{*2}}{(2\pi)^4}\frac{mc^2}{\lambda_c^3}\sum_{n=1}^{\infty}\int_0^{\infty}d\eta\, e^{-\eta}\eta^{2n-2}\frac{B_nb^{2n}}{(2n)!}. \tag{30}$$

[11] See Ref. 3, p. 785, last paragraph.
[12] J. Schwinger, Phys. Rev. 73, 416 (1948); 74, 1439 (1948); 82, 664 (1951).

Making use of the formula

$$\coth x-\frac{1}{x}=\sum_{n=1}^{\infty}\frac{2^{2n}B_n}{(2n)!}x^{2n-1}, \quad (31)$$

we finally get

$$L_2=-\alpha^2\Delta w_m{}^{(2)}=\frac{1}{32\pi^2}\left(\frac{\alpha}{2\pi}\right)^2\frac{mc^2}{\lambda_c{}^3}H^{*2}\times\int_0^\infty\frac{d\eta}{\eta^2}e^{-\eta}[\eta H^*\coth(\eta H^*)-1]. \quad (32)$$

This equation for L_2 represents the desired correction to the nonlinear Lagrangian of the magnetic field which arises from the fact that the electron has an anomalous magnetic moment. It is now of interest to consider the weak- and strong-field limits. We find $(H/H_c\equiv H^*)$

$$L_2=\frac{1}{96\pi^2}\left(\frac{\alpha}{2\pi}\right)^2\frac{mc^2}{\lambda_c{}^3}\left(\frac{H}{H_c}\right)^4 \quad \text{for} \quad H\ll H_c, \quad (33)$$

$$L_2=\frac{1}{32\pi^2}\left(\frac{\alpha}{2\pi}\right)^2\frac{mc^2}{\lambda_c{}^3}\left(\frac{H}{H_c}\right)^2\ln\left(\frac{H}{H_c}\right) \quad \text{for} \quad H\gg H_c. \quad (34)$$

In a future publication we will investigate the implications of our results for such processes as light-light scattering. It is perhaps appropriate to conclude by drawing attention to the discussion of Bethe and Salpeter,[13] who point out inconsistencies in the use of Eq. (9) that may arise for high energies.

ACKNOWLEDGMENTS

Part of this research was accomplished while the author held a National Research Council Senior Research Associateship supported by the National Aeronautics and Space Administration. The author would also like to thank Dr. Robert Jastrow for his hospitality at the Institute for Space Studies, New York City.

APPENDIX A: CALCULATION OF ENERGY EIGENVALUES OF RELATIVISTIC ELECTRON WITH ANOMALOUS MAGNETIC MOMENT IN CONSTANT MAGNETIC FIELD OF ARBITRARY STRENGTH

As discussed in the main text, we use the usual Dirac equation, with the addition of the Pauli anomalous moment interaction term [see Eq. (9)], which we repeat for convenience

$$i\partial\psi/\partial t=[\boldsymbol{\alpha}\cdot(\mathbf{p}+e\mathbf{A})+\gamma_4 m+\mu\gamma_4\boldsymbol{\Sigma}\cdot\mathbf{H}]\psi. \quad (A1)$$

It is also convenient to set $\hbar=c=1$ for the purpose of this derivation. We now choose the following representation for the γ matrices

$$\gamma_i=-i\gamma_4\alpha_i=\begin{pmatrix}0 & -i\sigma_i\\ i\sigma_i & 0\end{pmatrix},\quad \alpha_i=\begin{pmatrix}0 & \sigma_i\\ \sigma_i & 0\end{pmatrix},$$

$$\gamma_4=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \Sigma_i=\begin{pmatrix}\sigma_i & 0\\ 0 & \sigma_i\end{pmatrix},$$

σ_i being the usual Pauli matrices, and we set

$$\Pi_i=-i\nabla_i+eA_i. \quad (A2)$$

The magnetic field is considered to be directed along the z axis and it is convenient to use a cylindrical coordinate system.[7] Similar to the procedure used by Johnson and Lippmann[5] in the case $\mu=0$, we write the solution to Eq. (A1) in the form

$$\psi=\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}, \quad (A3)$$

and we set

$$\psi_{1,3}=\frac{1}{(2\pi)^{1/2}}e^{ip_zz}e^{i(l-1)\phi}f_{1,3}(r), \quad (A4)$$

$$\psi_{2,4}=\frac{1}{(2\pi)^{1/2}}e^{ip_zz}e^{il\phi}f_{2,4}(r), \quad (A5)$$

where p_z is the momentum of the electron in the z direction, and $f_{1,2,3,4}(r)$ are the functions of r which are to be determined. Using the fact that

$$\Pi_x\pm i\Pi_y=-ie^{\pm i\phi}\left(\frac{\partial}{\partial r}\pm\frac{i}{r}\frac{\partial}{\partial\phi}\mp\gamma r\right), \quad (A6)$$

and

$$\gamma_4\Sigma_3\begin{bmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{bmatrix}=\begin{bmatrix}\psi_1\\ -\psi_2\\ -\psi_3\\ \psi_4\end{bmatrix}, \quad (A7)$$

it follows from the above equations that

$$(E-m_1)f_1(r)=-i[\partial/\partial r+l/r+\gamma r]f_4(r)+p_zf_3(r), \quad (A8)$$

$$(E-m_2)f_2(r)=-i[\partial/\partial r-(l-1)/r-\gamma r]f_3(r)-p_zf_4(r), \quad (A9)$$

$$(E-m_1)f_3(r)=-i[\partial/\partial r+l/r+\gamma r]f_2(r)+p_zf_1(r), \quad (A10)$$

$$(E-m_2)f_4(r)=-i[\partial/\partial r-(l-1)/r-\gamma r]f_1(r)-p_zf_2(r), \quad (A11)$$

where

$$\gamma=\tfrac{1}{2}eH=\tfrac{1}{2}m^2(H/H_c),$$
$$m_1=m-\mu H,\quad m_2=m+\mu H,\quad \mu=a\mu_B. \quad (A12)$$

As before, μ_B denotes the Bohr magneton.

---

[13] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms* (Springer-Verlag, Berlin, 1957).

It is now convenient to set $\rho=\gamma r^2$. As a result Eqs. (A9) and (A11) may be rewritten as

$$\left[\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{l^2}{4\rho}-\frac{\rho}{4}-\tfrac{1}{2}(l-1)+B_1\right]f_4(\rho)+B_3f_2(\rho)=0, \quad (A13)$$

$$\left[\rho\frac{\partial^2}{\partial\rho^2}+\frac{\partial}{\partial\rho}-\frac{l^2}{4\rho}-\frac{\rho}{4}-\tfrac{1}{2}(l-1)+B_2\right]f_2(\rho)-B_3f_4(\rho)=0, \quad (A14)$$

where

$$B_1=(1/4\gamma)[E^2-m_1m_2-p_z^2+DE], \quad (A15)$$

$$B_2=(1/4\gamma)[E^2-m_1m_2-p_z^2-DE], \quad (A16)$$

$$B_3=p_z/4\gamma, \quad (A17)$$

and

$$D=m_2-m_1=2\mu H. \quad (A18)$$

The functions f_1 and f_3 obey similar equations but these will not concern us here. To solve these equations, we use the familiar power-series method. Assume

$$f_4(\rho)=e^{-\rho/2}\rho^s\sum_{N=0}^{\infty}C_N\rho^N, \quad (A19)$$

$$f_2(\rho)=e^{-\rho/2}\rho^s\sum_{N=0}^{\infty}D_N\rho^N. \quad (A20)$$

We now substitute Eqs. (A19) and (A20) into Eqs. (A13) and (A14). Equating the coefficients of $e^{-\rho/2}\rho^{N+s-1}$, we find

$$C_{N-1}[B_1-N+\tfrac{1}{2}-\tfrac{1}{2}(l-1)-s]+C_N[(N+s)^2-(\tfrac{1}{2}l)^2]+B_3D_{N-1}=0, \quad (A21)$$

$$D_{N-1}[B_2-N+\tfrac{1}{2}-\tfrac{1}{2}(l-1)-s]+D_N[(N+s)^2-(\tfrac{1}{2}l)^2]-B_3C_{N-1}=0. \quad (A22)$$

Taking $N=0$, it immediately follows that $2s=\pm l$. However, to avoid divergence difficulties for $\rho=0$ it is necessary to discard the $2s=-l$ solution. Thus, henceforth, we take $2s=l$. To obtain a well-behaved wave function we assume, as is usual, that our series terminates at $N=N'$, i.e.,

$$C_{N'+1}=0, \quad D_{N'+1}=0.$$

Now let $N=N'+1$ in Eqs. (A21) and (A22) and we get

$$C_{N'}[B_1-N'-l]+B_3D_{N'}=0, \quad (A23)$$

$$C_{N'}B_3-D_{N'}[B_2-N'-l]=0. \quad (A24)$$

It immediately follows that

$$[B_1-(N'+l)][B_2-(N'+l)]+B_3^2=0. \quad (A25)$$

This equation can now be solved for E, with the result

$$E=\pm m\left\{\left(\frac{p_z}{m}\right)^2+\left[\left(1+2\frac{H}{H_c}(N'+l)\right)^{1/2}\pm\tfrac{1}{2}a\frac{H}{H_c}\right]^2\right\}^{1/2}, \quad (A26)$$

where $N'+l=0, 1, 2, \cdots$. It is now convenient to rewrite this equation in the form

$$E_{n,s,\xi}(p)=\pm m\left\{\left(\frac{p_z}{m}\right)^2+\left[\left(1+\frac{H}{H_c}(2n+s+1)\right)^{1/2}+\tfrac{1}{2}\xi a\frac{H}{H_c}\right]^2\right\}^{1/2}, \quad (A27)$$

where $s=\pm1$ and $\xi=\pm1$. Going now to the nonrelativistic limit leads to the result that

$$E_{n,s,\xi}^{(\mathrm{NR})}(p)=\pm m\left\{1+\frac{1}{2}\left(\frac{p_z}{m}\right)^2+\frac{H}{H_c}\left[(n+\tfrac{1}{2})+\tfrac{1}{2}s\left(1+a\frac{\xi}{s}\right)\right]\right\}, \quad (A28)$$

and thus it follows that

$$\xi=s. \quad (A29)$$

Thus we get our final expression (reinserting the c)

$$E_{n,s}(p)=\pm mc^2\left\{\left(\frac{p_z}{H_c}\right)^2+\left[\left(1+\frac{H}{H_c}(2n+s+1)\right)^{1/2}+\tfrac{1}{2}sa\frac{H}{H_c}\right]^2\right\}^{1/2}, \quad (A30)$$

which corresponds to Eq. (10) in the main text.

Note added in proof. We have pointed out above that our Eq. (10) may be used to find the effect of the AMM of the electron on the magnetic moment of a free-electron gas. Such an investigation has now been carried out[14] and, as expected, the effect on the magnetic moment is rather negligible.

---

[14] H. Y. Chiu, V. Canuto, and L. Fassio-Canuto, Phys. Rev., this issue, **176**, 1438 (1968).