Exact solution for charge solitons in two coupled one-dimensional arrays of small tunnel junctions

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An exact solution for the potential profile of two coupled one-dimensional arrays each with m junctions with equal junction capacitances C, equal stray capacitances C_0, and equal coupling capacitances C_c is presented. Analytical expressions for the total free energy as well as the injection threshold voltage V_t of various charge solitons (an electron-hole pair, a single charge soliton, or a combined electron-hole pair and a single charge soliton) are derived. For $m=3$, we find that in the weak coupling regime ($C_c \ll C$) single electron transport dominates, in contrast with existing results [D. V. Averin *et al.*, Phys. Rev. Lett. **66**, 2818 (1991)], and other differences are also found in the $C_c \gg C$ limit. [S0163-1829(96)00727-8]

Delsing[1] proposed that coupled arrays of small tunnel junctions may be turned into a quantum current transformer, where a small well-defined current could be transformed to a larger (and more usable) current. These systems typically consist of two parallel one-dimensional (1D) arrays (see Fig. 1), where each electrode of one array is coupled to an electrode in the other array with a capacitor C_c. The key physics underlying these systems is the *Coulomb blockade* effect[2] for which in a single small tunnel junction, having capacitance C, such that the charging energy $e^2/2C$ exceeds the characteristic energy k_BT of thermal fluctuations, it is found that a suppression of single charge tunneling dramatically reduces the current at voltages $V<e/2C$. Averin, Korotkov, and Nazarov[2] (AKN) have studied theoretically the *electron-hole pair* transport for the case of three junctions in series. However, their study is restricted to $C_0=0$ (see Fig. 1), and they did not systematically study the Gibbs free energy F, the key quantity required for the determination of the tunneling rate in the multiple junction configuration. It is therefore of great interest to study the general behavior of F in the presence of multiple junctions, and to see the effects of the stray capacitances, the coupling strength, and the number of junctions in the two coupled 1D arrays. Here, we present an exact solution to the electrostatic problem of the two coupled 1D arrays, and study the charge profiles of various charge solitons. In particular, we show that the electrostatic problem can be formulated quite generally from the expression for F instead of the charge conservation equations for each of the individual islands. Of course F is also needed to determine the tunneling condition. Application of our formalism to the particular case studied by AKN leads to results in conflict with AKN, as we shall see.

The system with which we are concerned is illustrated in Fig. 1, where two 1D arrays each with m small tunnel junctions (in Ref. 3 what we now call m was called N), with equal junction capacitances C and equal stray capacitances C_0, are coupled to each other by the coupling capacitance C_c. The bias potentials of the two edges of the upper (first) array are, respectively, Φ_0 and Φ_m, while that of the lower (second) array are φ_0 and φ_m. In Fig. 1 we depict the case of symmetric bias voltage, i.e., $\Phi_0=V/2$, $\Phi_m=-V/2$, and $\varphi_0=\varphi_m=0$, but our initial formalism allows for other choices. Also, the tunnel resistance R of each junction is assumed to be the same and $R \gg h/e^2$, which ensures that the wave function of an excess electron on an island is localized there. We denote the potential on each of the individual $m-1$ islands of the first and the second array by the column vectors $\underline{\Phi}=\{\Phi_1,\Phi_2,\ldots,\Phi_{m-1}\}^T$ and $\underline{\varphi}=\{\varphi_1,\varphi_2,\ldots,\varphi_{m-1}\}^T$, respectively. In addition, the number of excess electrons on each of the individual $m-1$ islands of the first and the second array is denoted by the column vectors $\underline{N}=\{N_1,N_2,\ldots,N_{m-1}\}^T$ and $\underline{n}=\{n_1,n_2,\ldots,n_{m-1}\}^T$, respectively.

In general, the Gibbs free energy for the biased two coupled 1D arrays contains two terms, the electrostatic energy E_s and the work done due to the charge redistribution associated with the change of the charge profile $\{\underline{N},\underline{n}\}$ on the islands, which can be written as

$$F=E_s+W, \tag{1}$$

where the work done due to the charge redistribution associated with the change of the charge profile $\{\underline{N},\underline{n}\}$ is given by

$$W=-\sum_{i=1}^{m}[\phi_i(Q_i^s+Q_i^c)+\varphi_i(q_i^s+q_i^c)]-\sum_{i=1}^{m-1}(V_iQ_i+v_iq_i), \tag{2}$$

where (Q_i^s,Q_i^c,Q_i) and (q_i^s,q_i^c,q_i) are the charges on the stray capacitor, the coupling capacitor, and the junction be-

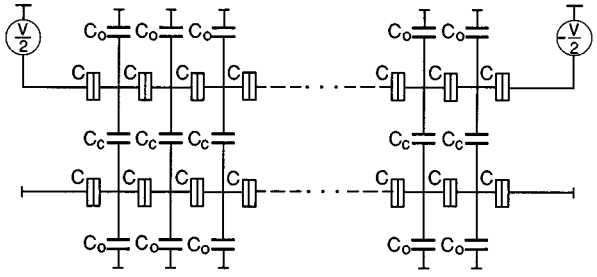

FIG. 1. Two coupled 1D arrays each with m small tunnel junctions, with equal junction capacitances C, equal stray capacitances C_0, and coupling capacitances C_c. The bias voltages of the first array are $\Phi_0=V/2$, $\Phi_m=-V/2$, and that of the second array are $\varphi_0=\varphi_m=0$.

tween the upper and lower arrays, respectively. Also, the local voltage for the upper array is defined as $V_i=\Phi_{i-1}-\Phi_i$, and that of the lower array is $v_i=\varphi_{i-1}-\varphi_i$. In addition, the electrostatic energy in (1) is defined as

$$E_s=E_1+E_2+E_{12}-e\sum_{i=0}^{m}(N_i\phi_i+n_i\varphi_i), \tag{3}$$

with

$$E_1=\frac{C_0}{2}\sum_{j=1}^{m-1}\Phi_j^2+\frac{C}{2}\sum_{j=1}^{m}[\Phi_j-\Phi_{j-1}]^2, \tag{4}$$

$$E_2=\frac{C_0}{2}\sum_{j=1}^{m-1}\varphi_j^2+\frac{C}{2}\sum_{j=1}^{m}[\varphi_j-\varphi_{j-1}]^2, \tag{5}$$

$$E_{12}=\frac{C_c}{2}\sum_{j=1}^{m-1}[\Phi_j-\varphi_j]^2, \tag{6}$$

representing the charging energy for the first array, the second array, and the coupling between the two arrays, respectively. We note that the use of (1) has at least two advantages. First, the important process of inner junction charge transfer, i.e., charge transfer between two neighboring islands without any other transfer on the other islands, can now be studied in detail. The other advantage is that it is now possible to obtain the electrostatic equations for the system directly from (1), which we discuss in the following.

Our goal here is to find out the exact explicit form of (1) at the fixed potential profile $\{\Phi,\varphi\}$ and changeable island charge number $\{N,n\}$. For this purpose, we take the derivative of (1) with respect to the potentials Φ_i and φ_i ($i=1,2,\ldots,m-1$), after which we get $2(m-1)$ equilibrium conditions. These equilibrium conditions form $2(m-1)$ linear equations for the potential profile $\{\Phi,\varphi\}$ and the island charge number $\{N,n\}$, and they can be conveniently put into a simple matrix form,

$$\begin{pmatrix}\bar{\bar{M}} & \alpha\bar{\bar{1}}\\ \alpha\bar{\bar{1}} & \bar{\bar{M}}\end{pmatrix}\begin{pmatrix}\overline{\Phi}\\ \overline{\varphi}\end{pmatrix}=\frac{e}{C}\begin{pmatrix}\overline{N}\\ \overline{n}\end{pmatrix}, \tag{7}$$

where the coupling constant $\alpha=C_c/C$, and we have used a double bar for the matrix: $\bar{\bar{1}}$ is an $m-1$ by $m-1$ unit matrix and $\bar{\bar{M}}$ is an $m-1$ by $m-1$ symmetric tridiagonal matrix, having the same diagonal elements $D=-2-C_0/C-C_c/C$, and the same off-diagonal elements 1. In addition, the first and last elements of N and n in (7) are understood to be $N_1-C\Phi_0/e$ and $N_{m-1}-C\Phi_m/e$, $n_1-C\varphi_0/e$ and $n_1-C\varphi_m/e$, respectively, to accommodate the effects of the bias voltages. We note that Eq. (7) is also known as the electrostatic equation for the system, the 1D version of which has been obtained by the consideration of the charge conservation law.[3]

We find that (7) can be solved analytically, and the result is

$$\begin{pmatrix}\overline{\Phi}\\ \overline{\varphi}\end{pmatrix}=-\frac{e}{C}\begin{pmatrix}\bar{\bar{A}} & \bar{\bar{B}}\\ \bar{\bar{B}} & \bar{\bar{A}}\end{pmatrix}\begin{pmatrix}\overline{N}\\ \overline{n}\end{pmatrix}, \tag{8}$$

where the elements of the symmetric matrix $\bar{\bar{A}}$ and $\bar{\bar{B}}$ are given by, respectively,

$$A_{ij}=\tfrac{1}{2}[R_{ij}(\lambda_+)+R_{ij}(\lambda_-)], \tag{9}$$

$$B_{ij}=\tfrac{1}{2}[R_{ij}(\lambda_+)-R_{ij}(\lambda_-)], \tag{10}$$

with $\lambda_\pm$ defined by

$$-2\cosh\lambda_\pm=D\pm\alpha, \tag{11}$$

and

$$R_{ij}(\lambda_\pm)=\frac{\cosh(m-|j-i|)\lambda_\pm-\cosh(m-i-j)\lambda_\pm}{2\sinh\lambda_\pm\sinh m\lambda_\pm}. \tag{12}$$

We note that in (9)–(12), one can check that in general $\lambda_+<\lambda_-$ and $R_{ij}(\lambda_+)>R_{ij}(\lambda_-)>0$, so that $A_{ij}>B_{ij}>0$. Also, one observes that the value of λ_+ as defined by (11) is independent of the value of C_c. In the weak coupling limit ($C_c\to0$), B_{ij} tends to zero ($\lambda_-\to\lambda_+$) and the two arrays become decoupled.

Equation (8), supplemented by (9)–(12), is a key result of this paper. Once a charge profile $\{N,n\}$ is known, we can use (8) to determine the potential profile $\{\Phi,\varphi\}$ and the corresponding Gibbs free energy. After some algebra, we obtain from (1), (7), and (8),

$$F=E_0+\frac{e^2}{2C}\sum_{i,j=1}^{m-1}\{N_iA_{ij}N_j+n_iA_{ij}n_j+n_iB_{ij}N_j+N_iB_{ij}n_j\}$$
$$-\phi_0Q_0-\phi_mQ_m-\varphi_0q_0-\varphi_mq_m, \tag{13}$$

where E_0 is a quantity independent of the charge profile $\{N,n\}$, and

$$Q_0=N_0e+C(\phi_0-\phi_1),\quad Q_m=N_me+C(\phi_m-\phi_{m-1}), \tag{14}$$

$$q_0=n_0e+C(\varphi_0-\varphi_1),\quad q_m=n_me+C(\varphi_m-\varphi_{m-1}). \tag{15}$$

Equation (13) is a general expression for the Gibbs free energy F of two coupled 1D arrays with changeable charge $e\{N,n\}$ on the islands, at the fixed bias voltage $\{\Phi_0,\Phi_m,\varphi_0,\varphi_m\}$ and the fixed potential profile $\{\Phi,\varphi\}$ on the islands. By means of (13), one can directly study the dynamics of the single electron tunneling by calculating the change of ΔF due to some charge transfer event. To be definite, here we discuss the case $\{N_k,N_{k'},n_k,n_k\}$ to $\{N_k',N_{k'}',n_k',n_{k'}'\}$, i.e., the charge transfer between two islands k and k', while the charges on the other islands are unchanged. We denote the net transferred charge as Q (Q can be electron-hole, single electron, or the combined electron-hole and single electron case, which will be discussed later). Thus, we obtain from (13) the $\Delta F^Q(k,k')$ due to the charge transfer $\{N_k,N_{k'},n_k,n_{k'}\}$ to $\{N_k',N_{k'}',n_k',n_{k'}'\}$,

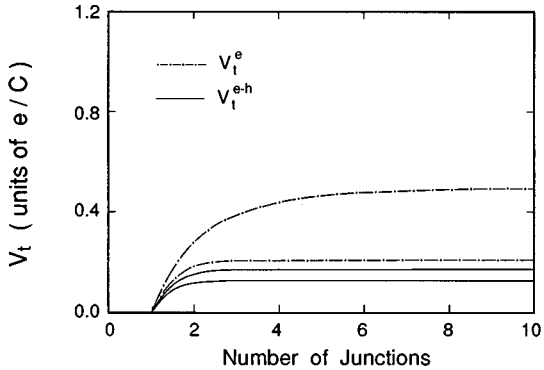

FIG. 2. Threshold voltage V_t (in units of e/C) for injecting a charge soliton (e-h for an electron-hole pair and e for single charge) into the first island of two coupled 1D arrays each with m small junctions as a function of the number of junctions m, at $C_c/C=1.0$ and $C_0/C=1$ (top curve) and 100 (bottom curve). Also, C_c, C, and C_0 are the coupling capacitances, junction capacitances, and stray capacitances, respectively.

$$\Delta F^{Q}(k,k') \equiv F(\{N',n'\}) - F(\{N,n\}) = \Delta E^{Q}(k,k') + W^{Q}(k,k'), \tag{16}$$

where the detailed form of the change of the charging energy $\Delta E^{Q}(k,k')$ and the work done $W^{Q}(k,k')$ in (16) can be worked out directly from (13).

One can now determine the tunneling threshold voltage by means of (16). At $T=0$, the tunneling of charge Q from the k to the k' island (restricting our discussion to the case of $k'=k\pm1$) is energy favorable when $\Delta F^{Q}(k,k')$ of (16) is less than zero. Thus, the threshold energy V_t for the injection of a charge soliton from the kth island onto the neighboring k' island can be obtained by equating $\Delta F^{Q}(k,k')=0$. Here we study three cases of particular interest: (i) the *electron-hole pair* (e-h) case, where an electron is transferred in the upper array from the k island to the k' island, and at the same time in the lower array an electron from the k' island to the k island is transferred, i.e., $N'_{k'}-N_{k'}=1$, $n'_{k'}-n_{k'}=-1$, $N'_k-n_k=-1$, $n'_k-n_k=1$; (ii) the *single charge soliton* (e) case, where an electron is transferred from the k island to the k' island in the upper array, i.e., $N'_{k'}-N_{k'}=1$, $n'_{k'}-n_{k'}=0$, $N'_k-N_k=-1$, $n'_k-n_k=0$; (iii) the *combined* (e-h,e) case, where, in addition to (i) a single electron is transferred from k' to k'', with the results $N'_{k''}-N_{k''}=1$, $N'_{k'}-N_{k'}=0$, $n'_{k'}-n_{k'}=-1$, $N'_k-N_k=-1$, $n'_k-n_k=1$. It turns out that the final form of V_t depends on the setup of the bias voltage. In the case of symmetrical bias voltage scheme depicted in Fig. 1, we obtain from (13) and (16),

$$V_{t,s}^{e\text{-}h}(k,k') = \frac{2e}{C}\,\frac{R^{-}_{k'k'}-R^{-}_{kk}}{\delta_{k,0}+\delta_{k',m}-R^{-}_{1k'}+R^{-}_{1k}-R^{-}_{k,m-1}+R^{-}_{k',m-1}}, \tag{17}$$

$$V_{t,s}^{e}(k,k') = \frac{e}{C}\,\frac{A_{k'k'}-A_{kk}}{\delta_{k,0}+\delta_{k',m}-A_{1k'}+A_{1k}-A_{k,m-1}+A_{k',m-1}}, \tag{18}$$

$$V_{t,s}^{e\text{-}h;e}(k;k',k'') = \frac{e}{C}\,\frac{A_{k'k'}+A_{k''k''}-2R^{-}_{kk}-2B_{k'k''}}{\delta_{k,0}+\delta_{k'',m}-A_{1k''}+B_{1k'}-B_{k',m-1}+A_{k'',m-1}+R^{-}_{1k}-R^{-}_{k,m-1}}, \tag{19}$$

where A_{ij}, $R^{\pm}_{i,j}\equiv R_{i,j}(\lambda_{\pm})$ are defined by (9) and (12), respectively. Equations (17)–(19) are interesting results. First, by using them one immediately finds that $dV_t/dm>0$ and $dV_t/dk<0$. This implies two facts: (i) for fixed values of C, C_0, and C_c, an array of larger m will generally have larger V_t; (ii) once a charge soliton is injected into the array, it will have no difficulty traveling through (with increasing k) the array. These features of the threshold voltage V_t are further illustrated in Fig. 2, where we plot V_t versus the number of junctions, for the $\alpha=1.0$ and $k=0$ case, for two coupled 1D arrays with $C_0/C=1$ and 100.

We now apply the above general analytical results to the $m=3$ system, which was previously studied by AKN (with the restriction of $C_0=0$).[2] As with AKN, we choose a symmetrical bias voltage $\Phi_0=V/2$, $\Phi_m=-V/2$, and $\varphi_0=\varphi_m=0$. In this case, with the help of (9)–(11), Eq. (8) can be written explicitly as

$$\begin{pmatrix}\Phi_1\\ \Phi_2\\ \varphi_1\\ \varphi_2\end{pmatrix} = -\frac{e}{2C}\begin{pmatrix} R^{+}_{11}+R^{-}_{11} & R^{+}_{12}+R^{-}_{12} & R^{+}_{11}-R^{-}_{11} & R^{+}_{12}-R^{-}_{12}\\ R^{+}_{12}+R^{-}_{12} & R^{+}_{22}+R^{-}_{22} & R^{+}_{12}-R^{-}_{12} & R^{+}_{22}-R^{-}_{22}\\ R^{+}_{11}-R^{-}_{11} & R^{+}_{12}-R^{-}_{12} & R^{+}_{11}+R^{-}_{11} & R^{+}_{12}+R^{-}_{12}\\ R^{+}_{12}-R^{-}_{12} & R^{+}_{22}-R^{-}_{22} & R^{+}_{12}+R^{-}_{12} & R^{+}_{22}+R^{-}_{22}\end{pmatrix}\begin{pmatrix} N_1-CV/2e\\ N_2+CV/2e\\ n_1\\ n_2\end{pmatrix}, \tag{20}$$

where we have introduced the notation $R^{\pm}_{ij}\equiv R_{ij}(\lambda_{\pm})$, and $R_{ij}(\lambda_{\pm})$ is defined by (12). In the absence of extra charges in any of the islands ($N_1=N_2=n_1=n_2=0$), by using (20) one can easily find the voltage across each of the junctions (V_i for the ith junction in the upper array and U_i for the ith junction in the lower array) in two limiting cases of $C_c(=\alpha C)$:

$$V_1=V_2=V_2=\tfrac{1}{3}V,\quad U_1=U_2=U_3=0,\quad \alpha\ll 1 \tag{21a}$$

$$V_1=V_3=\tfrac{5}{12}V,\quad V_2=\tfrac{1}{6}V,\quad U_1=U_3=-\tfrac{1}{12}V,\quad U_2=\tfrac{1}{6}V,\quad \alpha\gg 1. \tag{21b}$$

Equation (21a) shows that in the $\alpha\ll1$ region, the voltage drops uniformly along the horizontal direction of the two arrays. On the other hand, (21b) tells us that in the $\alpha\gg1$ region, the voltage drop across the middle junction is much smaller than that across the two edge junctions. This latter fact suggests that in studying hte present system in the $\alpha\gg1$ region, one must study carefully the transitions at both the edge and middle junctions.

The tunneling of charge Q from the k to the k' island of the arrays is energy favorable at $T=0$ when $\Delta F^{Q}(k,k')$ of (16) is less than zero. Thus, the threshold energy V_t for the injection of a charge soliton from the kth island onto the k'

island can be obtained by setting $\Delta F^{Q}(k,k')=0$. Applying this principle to (16) [supplemented by (13)], we obtain

$$V_t^{e\text{-}h}(0,1)=\frac{2e}{C}\,\frac{R_{1,1}^{-}}{1-R_{1,1}^{-}+R_{1,2}^{-}}, \tag{22}$$

$$V_t^{e}(0,1)=\frac{e}{C}\,\frac{A_{1,1}}{1-A_{1,1}+A_{1,2}}, \tag{23}$$

$$V_t^{e\text{-}h;e}(0;1,2)=\frac{e}{C}\,\frac{A_{2,2}+A_{1,1}-2B_{1,2}}{1+B_{1,1}-A_{1,2}+A_{2,2}-B_{1,2}}, \tag{24}$$

where A_{ij} and $R_{i,j}^{\pm}\equiv R_{i,j}(\lambda_{\pm})$ are defined by (9) and (12), respectively. Note that in the $\alpha\ll 1$ region, from (9)–(12) one observes that $R_{i,j}^{+}$, $R_{i,j}^{-}$, A_{ij} become the same, and (22)–(24) have similar 1D behavior. Another feature of (22)–(24) is that from these expressions one can directly check that $V_t^{Q}(0,1)>V_t^{Q}(1,2)$ so that the actual threshold voltage value of the system is $V_t^{Q}=V_t^{Q}(0,1)$. When $\alpha\gg 1$, from (9) and (12) it is easy to observe that $R_{i,j}^{-}\to 0$, while $A_{i,j}$ approaches $\frac{1}{2}R_{i,j}^{+}$. In this case, it is straightforward to show that (22)–(24) become, respectively,

$$V_t^{e\text{-}h}=\frac{e}{C_c}, \tag{25}$$

$$V_t^{e}=\frac{e}{C}\,\frac{m-1}{m+2}\quad \text{for } C_0\ll C;\quad V_t^{e}=\frac{e}{2C_0}\quad \text{for } C_0\gg C, \tag{26}$$

$$V_t^{e\text{-}h;e}=\frac{e}{2C}\,\frac{m-1}{m+1}\quad \text{for } C_0\ll C;\quad V_t^{e\text{-}h;e}=\frac{e}{C_0}\quad \text{for } C_0\gg C. \tag{27}$$

To further illustrate our results (25)–(27), in Fig. 3. we plot V (in units of e/C_c) versus C_c/C for two coupled 1D arrays each with three junctions ($m=3$) and $C_0=0$. There are two major differences with Fig. 3 of Ref. 2: (i) In the case of strong coupling ($\alpha\gg 1$) between the two 1D arrays, our results show that $V_t^{e}=2e/5C$ [given by (26) at $m=3$] and $V_t^{e\text{-}h;e}=e/4C$ [given by (27) at $m=3$]. The former result is at variance with that of Ref. 2, where it is stated that $V_t^{e}=e/2C$. (ii) In the weak coupling region ($\alpha<1$), our results, enlarged in the inset of Fig. 3, indicate that V_t^{e} has a smaller value than that of $V_t^{e\text{-}h;e}$ and $V_t^{e\text{-}h}$, in contrast with the result obtained in Ref. 2. In other words, for $\alpha<1$ ($C_c<C$), we conclude that single electron transport dominates whereas AKN conclude that the combined electron-hole and single electron transport dominates. Our conclusion is physically appealing since, for $C_c<C$, the two arrays are practically decoupled and one recovers the single electron tunneling behavior of a 1D array. In addition to Fig. 3, our results (25)–(27) can also be used to study the effect of C_0 on the two coupled 1D arrays. In general, the threshold voltage V_t^{e} strongly depends on the total number m of junctions and is very sensitive to the value of C_0; the larger the values of C_0, the smaller V_t^{e} is. In the infinite m limit, one has $V_t^{e}=e/C$, which is larger than the corresponding value for finite m. Also, in order to delineate more precisely the main competing processes, we did not include multiple exciton effects, which would simply superimpose multiple steps on the borderline between the excitonic and the combined regions.[2]

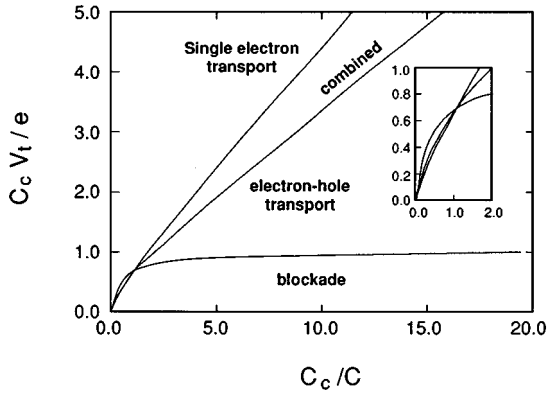

FIG. 3. $T=0$ charge transport mechanism for two coupled (coupling capacitances C_c) 1D arrays each with three junctions ($m=3$) and no stray capacitors ($C_0=0$), and symmetrized bias voltage V. Only charge solitons (single electron, an electron-hole, and a combined electron-hole and single electron) are considered. Shown in the inset is an enlarged part of the figure in the small C_c region.

In summary, in this paper we have presented an exact solution (8) for the potential profiles of two biased coupled 1D arrays of m junctions with equal junction capacitance, equal stray capacitance, and equal coupling capacitance. Based on (8), we have obtained an analytical Gibbs free energy expression (13), as well as the threshold voltages V_t (17)–(19) for various charge solitons. Our study shows that V_t changes dramatically in the region of $\alpha<1$, and becomes insensitive to the value of C when $\alpha>1$. In contrast with the results of Ref. 2, for $m=3$, we find that (a) in the weak coupling regime single electron transport dominates and (b) in the strong coupling and the $C_0\to 0$ limit, we obtain a different V_t for the single electron case. These differences are highlighted in Fig. 3, which should be contrasted with Fig. 3 of Ref. 2. We conclude that, in studying charge transport in a system of 1D arrays with a finite number of small junctions, it is necessary to treat the electrostatic problem exactly. The quantitative behavior identified in this paper should provide useful information pertaining to future experiments, in particular for the quantum current transformer proposed by Delsing.[1]

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